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A Note on the Global Attractivity of a Discrete Model of Nicholson's Blowflies*

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In this paper, we further study the global attractivity of the positive equilibrium of the discrete Nicholson's blowflies model

$$N_{n+1} - N_n = -\delta N_n + pN_{n-k}\mathrm{e}^{-aN_{n-k}}, \quad n = 0, 1, 2, \ldots.$$

We obtain a new criterion for the positive equilibrium N^* to be a global attractor, which improve the corresponding results obtained by So and Yu (*J. Math. Anal. Appl.* 193 (1995), 233–244).

Keywords: Attractivity, Positive equilibrium, Discrete Nicholson's blowflies model

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I. INTRODUCTION

The delay difference equation

$$N_{n+1} - N_n = -\delta N_n + pN_{n-k}\mathrm{e}^{-aN_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

is a discrete analogue of the delay differential equation

$$N'(t) = -\delta N(t) + pN(t-\tau)\mathrm{e}^{-aN(t-\tau)}, \quad t \geq 0,$$

which has been used in describing the dynamics of Nicholson's blowflies [2,4–6].

By the biology consideration, we assume that $\delta \in (0, 1)$, $p, a \in (0, +\infty)$, and $k \in N = \{0, 1, 2, \ldots\}$. The initial condition is

$$N_j = \varphi_j \geq 0, \quad j \in \{-k, -k+1, \ldots, 0\}, \tag{2}$$

and $\varphi_j > 0$, for some $j \in \{-k, -k+1, \ldots, 0\}$.

By a solution of (1) and (2) we mean a sequence $\{N_n\}$ which satisfies (1) for $n = 0, 1, 2, \ldots$ as well as the initial condition (2). Clearly, the unique solution $\{N_n\}$ of the above initial value problem is positive for all large n [1].

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If $p > \delta$, then Eq. (1) has a unique positive equilibrium N^* and

$$N^* = \frac{1}{a}\ln\left(\frac{p}{\delta}\right). \tag{3}$$

The global attractivity of N^* was studied by Kocic and Lada [3] and So and Yu [1] respectively.

The recent result is the following [1].

THEOREM A *Assume that $p > \delta$ and that*

$$[(1-\delta)^{-k-1} - 1]\ln\left(\frac{p}{\delta}\right) \le 1. \tag{4}$$

Then any nontrival solution N_n of (1) *and* (2) *satisfies*

$$\lim_{n\to\infty} N_n = N^*.$$

In this note, our purpose is to improve condition (4). Exactly speaking, we will show some conditions for the global attractivity of N^* when (4) does not hold. Our results are discrete analogues of the results in [2].

To prove our main results, we need some known results.

LEMMA 1 [1] *Let $\{N_n\}$ be a solution of* (1) *and* (2). *Then*

$$\limsup_{n\to\infty} N_n \le \frac{p}{ae\delta}. \tag{5}$$

As in [2], *the following system of inequalities*

$$\begin{cases} y + \ln(1 + (y/aN^*)) \le M(\mathrm{e}^{-x} - 1), \\ x + \ln(1 + (x/aN^*)) \ge M(\mathrm{e}^{-y} - 1), \end{cases} \tag{6}$$

play an important role in our analysis, where $M = aN^*[(1-\delta)^{-k-1} - 1] = [(1-\delta)^{-k-1} - 1]\ln(p/\delta)$.

Let

$$D = \{(x, y) \colon\ -aN^* < x \le 0 \le y < \infty\}. \tag{7}$$

LEMMA 2 [2] *If one of the following conditions holds:*

(i) $M \le 1$;
(ii) $M < 1 + (1/aN^*)$ *and* $aN^* \ge (\sqrt{5} - 1)/2$;
(iii) $M \le 1 + (1/aN^*)$ *and* $aN^* > (\sqrt{1 + 4\sqrt{3}} - 1)/2$,

then (6) *has a unique solution $x = y = 0$ in D.*

II. MAIN RESULTS

The following theorem provides a new sufficient condition for the equilibrium $N^* = (1/a)\ln(p/\delta)$ to be a global attractor.

THEOREM 1 *Assume that $p > \delta$ and the assumption in Lemma* 2 *holds. Then any nontrivial solution $\{N_n\}$ of* (1) *and* (2) *satisfies*

$$\lim_{n\to\infty} N_n = N^*.$$

Proof Let

$$N_n = N^* + \frac{1}{a}x_n.$$

Then $\{x_n\}$ is a solution of the equation

$$x_{n+1} - x_n + \delta x_n + a\delta N^*(1 - \mathrm{e}^{-x_{n-k}}) - \delta x_{n-k}\mathrm{e}^{-x_{n-k}} = 0, \quad n = 0, 1, 2, \ldots. \tag{8}$$

Since $N_n > 0$ for all large n, it follows that $x_n > -aN^*$ for all large n.

To prove this theorem, it is sufficient to prove $\lim_{n\to\infty} x_n = 0$. Lemma 1 implies that $\{x_n\}$ is bounded above. Let

$$\mu = \limsup_{n\to\infty} x_n \quad \text{and} \quad \lambda = \liminf_{n\to\infty} x_n. \tag{9}$$

Then $-aN^* \le \lambda \le \mu < \infty$. We claim that $\lambda = \mu = 0$. For the case $\{x_n\}$ is eventually nonnegative or eventually nonpositive, this has been proved in the proof of Theorem 2 in [3]. Therefore it is sufficient to consider the case that $\{x_n\}$ is an oscillatory solution of (8).

Our purpose is to prove that $\lambda = \mu = 0$ under the assumptions. There are four possible cases:

(1) $\lambda = \mu = 0$;
(2) $\mu > 0$ and $\lambda = 0$;

(3) $\mu = 0$ and $\lambda < 0$;
(4) $\mu > 0$ and $\lambda < 0$.

The cases 2 and 3 can be considered to be special cases of case 4. Now we consider case 4.

In this case, there exists a sequence $\{n_i\}$ of positive integers such that

$$k < n_1 < n_2 < \cdots < n_i < n_{i+1} \to \infty \quad \text{as } i \to \infty.$$

$$x_{n_i} < 0 \quad \text{and} \quad x_{n_i+1} \geq 0, \quad \text{for } i = 1, 2, \ldots,$$

and for each $i = 1, 2, \ldots$, the terms of the finite sequence x_j for $n_i < j < n_{i+1}$ assume both positive and negative values. Let m_i and M_i be integers in (n_i, n_{i+1}) such that for $i = 1, 2, \ldots$

$$x_{M_i} = \max\{x_j : \ n_i < j < n_{i+1}\},$$

and

$$x_{m_i} = \min\{x_j : \ n_i < j < n_{i+1}\}.$$

We can assume without loss of generality that for $i = 1, 2, \ldots$

$$x_{M_i} > 0, \quad x_{M_i} - x_{M_i-1} \geq 0 \quad \text{and} \quad \limsup_{i\to\infty} x_{M_i} = \mu > 0,$$

while

$$x_{m_i} < 0, \quad x_{m_i} - x_{m_i-1} \leq 0 \quad \text{and} \quad \liminf_{i\to\infty} x_{m_i} = \lambda < 0.$$

Then there exist subsequence $\{q_i\}$ of $\{m_i\}$ and subsequence $\{Q_i\}$ of $\{M_i\}$ such that

$$x_{Q_i} > 0, \quad x_{Q_i} - x_{Q_i-1} \geq 0 \quad \text{and} \quad \lim_{i\to\infty} x_{Q_i} = \mu > 0, \tag{10}$$

while

$$x_{q_i} < 0, \quad x_{q_i} - x_{q_i-1} \leq 0 \quad \text{and} \quad \lim_{i\to\infty} x_{q_i} = \lambda < 0. \tag{11}$$

It follows from (8) and (10) that

$$x_{Q_i-1} + aN^* \leq [x_{Q_i-k-1} + aN^*]\mathrm{e}^{-x_{Q_i-k-1}},$$

thus

$$\begin{aligned} x_{Q_i} + aN^* &= (1-\delta)(x_{Q_i-1} + aN^*) \\ &\quad + \delta(x_{Q_i-k-1} + aN^*)\mathrm{e}^{-x_{Q_i-k-1}} \\ &\leq (1-\delta)(x_{Q_i-k-1} + aN^*)\mathrm{e}^{-x_{Q_i-k-1}} \\ &\quad + \delta(x_{Q_i-k-1} + aN^*)\mathrm{e}^{-x_{Q_i-k-1}} \\ &= (x_{Q_i-k-1} + aN^*)\mathrm{e}^{-x_{Q_i-k-1}} \end{aligned}$$

that is

$$x_{Q_i} + aN^* \leq (x_{Q_i-k-1} + aN^*)\mathrm{e}^{-x_{Q_i-k-1}}. \tag{12}$$

Now let us prove

$$x_{Q_i-k-1} < 0, \tag{13}$$

assume the contrary, then $x_{Q_i-k-1} = 0$ or $x_{Q_i-k-1} > 0$. If $x_{Q_i-k-1} = 0$, then $x_{Q_i} \leq 0$, which contradicts (10). If $x_{Q_i-k-1} > 0$, then $x_{Q_i-k-1} > x_{Q_i}$, thus

$$\liminf_{i\to\infty} x_{Q_i-k-1} \geq \liminf_{i\to\infty} x_{Q_i} = \mu,$$

on the other hand, we have

$$\limsup_{i\to\infty} x_{Q_i-k-1} \leq \limsup_{i\to\infty} x_{M_i} = \mu,$$

so we get

$$\lim_{i\to\infty} x_{Q_i-k-1} = \mu, \tag{14}$$

then taking the limit in (12), we obtain

$$\mu + aN^* \leq (\mu + aN^*)\mathrm{e}^{-\mu},$$

which implies $\mu \leq 0$ that contradicts (10), so (13) holds.

From (12) and (13), we have

$$x_{Q_i} + aN^* < aN^*\mathrm{e}^{-x_{Q_i-k-1}},$$

therefore

$$x_{Q_i-k-1} < -\ln\Big(1+\frac{x_{Q_i}}{aN^*}\Big). \tag{15}$$

For given $\varepsilon > 0$, by (9), there exists a positive integer n^* such that

$$\lambda - \varepsilon < x_n < \mu + \varepsilon, \quad \text{for } n \geq n^* - k,$$

this induce $x_{n-k}e^{-x_{n-k}} < \mu + \varepsilon$, for $n \geq n^*$.

Rewriting Eq. (8) into the following form:

$$\begin{aligned}(1-\delta)^{-n-1}x_{n+1} - (1-\delta)^{-n}x_n \\ + a\delta N^*(1-\delta)^{-n-1}(1-e^{-x_{n-k}}) \\ - \delta(1-\delta)^{-n-1}x_{n-k}e^{-x_{n-k}} = 0.\end{aligned} \tag{16}$$

Now summing (16) up from $n = Q_i - k - 1$ (assuming $Q_i - k - 1 \geq n^*$) to $n = Q_i - 1$. we have

$$\begin{aligned}(1-\delta)^{-Q_i}x_{Q_i} &= (1-\delta)^{-Q_i+k+1}x_{Q_i-k-1} - a\delta N^* \\ &\quad \times \sum_{n=Q_i-k-1}^{Q_i-1}(1-\delta)^{-n-1}(1-e^{-x_{n-k}}) \\ &\quad + \delta \sum_{n=Q_i-k-1}^{Q_i-1}(1-\delta)^{-n-1}x_{n-k}e^{-x_{n-k}} \\ &< (1-\delta)^{-Q_i+k+1}x_{Q_i-k-1} + a\delta N^* \\ &\quad \times \sum_{n=Q_i-k-1}^{Q_i-1}(1-\delta)^{-n-1}(e^{-\lambda+\varepsilon}-1) \\ &\quad + \delta \sum_{n=Q_i-k-1}^{Q_i-1}(1-\delta)^{-n-1}(\mu+\varepsilon) \\ &= (1-\delta)^{-Q_i+k+1}x_{Q_i-k-1} \\ &\quad + [(\mu+\varepsilon) + aN^*(e^{-\lambda+\varepsilon}-1)] \\ &\quad \times (1-\delta)^{-Q_i}[1-(1-\delta)^{k+1}].\end{aligned}$$

Substituting (15) into the above inequality, we get

$$\begin{aligned}(1-\delta)^{-Q_i}x_{Q_i} &< -(1-\delta)^{-Q_i+k+1}\ln\Big(1+\frac{x_{Q_i}}{aN^*}\Big) \\ &\quad + [(\mu+\varepsilon) + aN^*(e^{-\lambda+\varepsilon}-1)] \\ &\quad \times (1-\delta)^{-Q_i}[1-(1-\delta)^{k+1}],\end{aligned}$$

and

$$\begin{aligned}x_{Q_i} + (1-\delta)^{k+1}\ln\Big(1+\frac{x_{Q_i}}{aN^*}\Big) \\ < [(\mu+\varepsilon) + aN^*(e^{-\lambda+\varepsilon}-1)][1-(1-\delta)^{k+1}],\end{aligned}$$

let $i \to \infty$, $\varepsilon \to 0$, we get

$$\begin{aligned}\mu + (1-\delta)^{k+1}\ln\Big(1+\frac{\mu}{aN^*}\Big) \\ \leq [\mu + aN^*(e^{-\lambda}-1)][1-(1-\delta)^{k+1}].\end{aligned}$$

We rewrite the above inequality:

$$\mu + \ln\Big(1+\frac{\mu}{aN^*}\Big) \leq M(e^{-\lambda}-1). \tag{17}$$

In a similar way, we have

$$\lambda + \ln\Big(1+\frac{\lambda}{aN^*}\Big) \geq M(e^{-\mu}-1). \tag{18}$$

Then we establish the following system of inequalities:

$$\begin{cases}\mu + \ln(1+(\mu/aN^*)) \leq M(e^{-\lambda}-1), \\ \lambda + \ln(1+(\lambda/aN^*)) \geq M(e^{-\mu}-1).\end{cases} \tag{19}$$

For case 2, the system of inequalities corresponding to (19) is

$$\begin{cases}\mu + \ln(1+(\mu/aN^*)) \leq M(e^{-\lambda}-1), \\ \lambda = 0.\end{cases} \tag{20}$$

It is obvious that (20) holds iff $\lambda = \mu = 0$.

For case 3, the system of inequalities corresponding to (19) is

$$\begin{cases}\mu = 0, \\ \lambda + \ln(1+(\lambda/aN^*)) \geq M(e^{-\mu}-1).\end{cases} \tag{21}$$

Similarly, (21) holds iff $\lambda = \mu = 0$.

Thus it will suffice to consider case 4, for (19) in case 4, by Lemma 2, we get $\lambda = \mu = 0$. So the proof is complete.

Remark 1 In cases 2 and 3 in Theorem 1, we add some reasonable conditions to aN^*. We know

$$M = aN^*[(1-\delta)^{-k-1} - 1] \leq 1 + \frac{1}{aN^*},$$

on the right side of which there is nothing to do with δ and k. While $1+(1/aN^*) \to \infty$ as $aN^* \to 0+$, properly choosing the values of $[(1-\delta)^{-k-1} - 1]$, we can let M equal or infinitely tend to the value of $1+(1/aN^*)$, then M can be changed to arbitrarily large. Obviously this is not reasonable.

Remark 2 Theorem 4.1 in [1] only applies to the case $M \leq 1$, while Theorem 1 in this paper not only applies to $M \leq 1$ but also to $M > 1$. So the results in this paper improve those in [1].

Example Consider the delay difference equation

$$N_{n+1} - N_n = -\tfrac{1}{4}N_n + \tfrac{1}{4}e^{(\sqrt{5}-1)/2} N_{n-3} e^{-2N_{n-3}}, \quad (22)$$

then we can calculate

$$aN^* = \frac{\sqrt{5}-1}{2} \quad \text{and} \quad [(1-\delta)^{-k-1} - 1] = \frac{175}{81},$$

thus,

$$M \approx 1.335 \quad \text{and} \quad 1 + \frac{1}{aN^*} = \frac{\sqrt{5}+3}{2} \approx 2.618.$$

The conditions in Theorem 1 are satisfied. Thus

$$N^* = \frac{\sqrt{5}-1}{4}$$

is a global attractor or (22). But Theorem 4.1 in [1] cannot apply to this case.

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