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Intergenerational Equity and Dynamic Duality Principles

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The concept of intergenerational equity concerning intertemporal paths of consumption and capital accumulation is introduced and the analysis of the dynamic processes of capital accumulation and changes in environmental quality that are intergenerationally equitable is developed. The analysis is based upon the dynamic duality principles, as originally developed by Koopmans and Uzawa, and later extended to the case involving environmental quality.

A time-path of consumption and capital accumulation is defined intergenerationally equitable when it is dynamically efficient and, at the same time, the imputed price of each kind of capital, either private capital, or social overhead capital, is identical over time. The existence of an intergenerationally equitable time-path of consumption and capital accumulation is guaranteed when the processes of various kind of capital are subject to the Penrose effect, exhibiting the law of diminishing marginal rates of investment capital accumulation.

Keywords: Intergenerational equity; Imputed prices; Dynamic duality principles for intertemporal preference and capital accumulation; Environmental quality; The Penrose effect; Global warming

INTERGENERATIONAL EQUITY

The analysis is primarily conducted within the conceptual framework of the dynamic analysis of environmental quality, as was developed by Mäler in his classic work, Mäler (1974), with particular emphasis on the circumstances where the irreversibility of the processes of capital accumulation occurs due to the presence of the Penrose effect, as was originally introduced by Uzawa (1968) in the context of macro-economic analysis. The analysis is based upon the formula concerning the system of imputed prices associated with the time-path of consumption that is dynamically optimum, where the presence of the Penrose effect implies the diminishing marginal rate of investment in private capital and social overhead capital upon the rate at which capital is accumulated. The dynamically optimum time-path of consumption is characterized by the proportionality of two systems of imputed prices, one associated with the given intertemporal preference ordering and another with the process of capital accumulation of private capital and social overhead capital.

The basic premises of the dynamic analysis of capital accumulation and changes in environmental quality are that the intertemporal preference ordering of the society in question is given independently of the technological conditions and processes of capital accumulation, as typically expressed by the utility integral of the Ramsey–Koopmans–Cass type:

$$U(x) = \int_0^\infty u(x_t)\mathrm{e}^{-\delta t}\,\mathrm{d}t, \quad x = (x_t)$$

where $u_t = u(x_t)$ is the utility function expressing the instantaneous level of utility at time t and δ is the utility rate of discount. Utility function $u(\cdot)$ and utility rate of discount δ are assumed to be both constants.

The dynamically optimum time-path of consumption and capital accumulation is obtained in terms of the imputed prices of various kind of capital, as in detail discussed in Uzawa (2000). Suppose there is only one kind of capital either private capital or social overhead capital, we denote by k_t the stock of capital and by x_t, A_t consumption and investment at time t, respectively. Then,

$$x_t + A_t = f(k_t)$$

where $f(\cdot)$ is the production function, to be assumed independent of time t. The rate of capital accumulation at time t, $\dot{k}_t$ is then given by

$$\dot{k}_t = \phi(A_t, k_t)$$

where $\phi(A_t, k_t)$ is the Penrose function, expressing the relation between the rate of capital accumulation $\dot{k}_t$ with investment A_t and the stock of capital k_t. The Penrose

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effect is expressed by the conditions that

$$\phi_A(A,k) > 0,\ \phi_k(A,k) > 0,\ \phi_{kk}(A,k) < 0,\ \phi_{AA}(A,k) < 0$$

It may remind us that all variables are positive

$$x_t, k_t, A_t > 0 \quad (\forall t \geqq 0)$$

The imputed price ψ_t of capital accumulation at time t is defined as the discounted present value of the marginal increase in the output in future due to a marginal increase in the level of investment at time t; that is,

$$\psi_t = \int_t^{\infty} m_\tau \mathrm{e}^{-\delta(\tau - t)}\,\mathrm{d}t$$

where

$$m_\tau = (r\phi_A + \phi_k)_\tau$$

is the marginal product value of investment evaluated at time τ.

If we differentiate both sides of the differential equations above with respect to time t, we obtain the following Euler–Lagrange differential equation:

$$\frac{\dot{\psi}_t}{\psi} = \delta - (r\phi_A + \phi_k)$$

where $r = f'(k)$, ϕ_A, ϕ_k are all evaluated at time t.

The imputed real income at time t is given by

$$H_t = u(x_t) + \psi_t \phi(A_t, k_t)$$

The dynamically optimum path of consumption and capital accumulation (x_t,k_t) is obtained if, imputed real income H_t, (5), is maximized subject to the feasibility constraints and the following transversality conditions are satisfied.

$$\lim_{t \to +\infty} \psi_t k_t = 0$$

We have, in particular, that

$$u'(x_t) = \psi_t \phi_A$$

The utility integral of the Ramsey–Koopmans–Cass type indicated above presumes that not only is the utility rate of discount δ constant, but also the instantaneous level of utility at each time t is aggregated with equal weight. Indeed, it is a special case of the more general form of intertemporal preference where the utility integral is given by

$$U(x) = \int_0^{\infty} \varpi_t u(x_t) \mathrm{e}^{-\delta t}\,\mathrm{d}t, \quad x = (x_t)$$

where ϖ_t is a positive weight associated with the instantaneous level of utility at each time t, which are *a priori* given. The utility integral of the Ramsey–Koopmans–Cass type is the special case when

$$\varpi_t = 1 \quad (\forall t \geqq 0)$$

This assumption gives the impression as if the intergenerational equity prevails concerning the utility levels of all future generations and the current generation. However, the stock of capital, in particular that of social overhead capital, differs between the generations, so do the technological conditions and institutional arrangements in the economy. The resulting levels of the utility, accordingly, are not necessarily equal between the generations, so that the intergenerational equity may not prevail if intergenerational distribution of real income is explicitly taken into account.

A time-path of consumption and capital accumulation is defined intergenerationally equitable when it is dynamically efficient and, at the same time, the imputed price of each kind of capital, either private capital or social overhead capital, is identical over time. In other words, a time-path of consumption is intergenerationally equitable if all future generations face with the imputed prices of various kind of capital that are equal to those of the current generation faces. The existence of an intergenerationally equitable time-path of consumption and capital accumulation is guaranteed when the processes of various kind of capital are subject to the Penrose effect, exhibiting the law of diminishing marginal rates of investment capital accumulation.

In the context of global warming, under certain qualifying constraints on the utility functions, it is shown that, the processes of capital accumulation is intergenerationally equitable if, and only if, a proportional carbon tax scheme is implemented with respect to the emission of carbon dioxide and other greenhouse gases within the institutional framework of competitive markets.

INTERTEMPORAL PREFERENCES

To give a proper perspective to the concept of intergenerational equity as introduced above, we recapitulate the endogenous theory of time preference, focusing upon the role of imputed prices with regards to the dynamic duality principles, as extensively developed in Uzawa (1998, 2000).

An intertemporal preference ordering is a binary relation $>$ defined over the set of all conceivable time-paths of consumption Ω. Each time-path of consumption in the set Ω, $x = (x_t)$, is a vector-valued function x_t defined for all t, $0 \geqq t < +\infty$. It is assumed that, for each time t, x_t is positive and piecewise continuously differentiable. The intertemporal preference ordering $>$ is assumed to be irreflexible, transitive, monotone, continuous, convex, and time-invariant.

We consider a special class of intertemporal preference orderings, for each of which the instantaneous utility $u_t = u(x_t)$ at each time t is well defined as a function of the vector of consumption x_t at time t. It is assumed that the utility function $u_t = u(x_t)$ is positive valued, continuously twice-differentiable, and strictly concave:

$$u(x_t) > 0, \quad u'(x_t) > 0, \quad u''(x_t) < 0, \text{ for all} \quad x_t > 0$$

It is also assumed that the intertemporal preference ordering with which we are concerned may be represented by a certain utility functional, $U = U(u)$, $u = (u_t)$, such that

$$x \gtrsim x' \text{ if, and only if,} \quad U(u) \geqq U(u'), \quad u = (u(x_t)),$$
$$u' = (u(x_t'))$$

Let us now consider a time-path of instantaneous utilities $u = (u_t)$, and denote by U_t the utility functional for the truncated time-path at time t, ${}^t u = (u_{t+\tau})$, of the time-path of instantaneous utilities $u = (u_\tau)$; that is,

$$U_t = U({}^t u), \quad t \geqq 0$$

The utility functional $U = U(u)$, $u = (u_t)$, is assumed to satisfy the following differential equation:

$$\dot{U}_t = \beta(u_t, U_t) - u_t$$

where the function $\beta(u_t, U_t)$ specifies the way by which future utilities are discounted to the present.

It is assumed that the discounting function $\beta(u,U)$ is defined for all $(u, U) > (0,0)$, continuously twice differentiable, and satisfies the following conditions:

$$\beta = \beta(u, U) > 0, \text{ for all} \quad (u, U) > (0,0)$$

$$\beta(0, U) = \beta(u, 0) = 0, \text{ for all} \quad (u, U) \geqq (0,0)$$

$$\beta_u < 0, \quad \beta_U > 0, \text{ for all} \quad (u, U) > (0,0)$$

and $\beta(u,U)$ is convex and strictly quasi-convex in the sense that

$$\beta_{uu} > 0, \quad \beta_{UU} > 0, \quad \beta_{uu}\beta_{UU} - \beta_{uU}^2 \geqq 0,$$
$$\text{for all} \quad (u, U) > (0,0)$$

For any given time-path of utilities, ${}^0 u = (u_t)$ the value of the utility functional $V = U({}^0 u)$ may be obtained by the following procedure:

$V = U({}^0 u)$ if, any only if, the solution path (U_t) to the basic differential equation (1) with initial condition $U_0 = V$ satisfies the following condition:

$$\int_0^\infty u_t \, e^{-\hat{\Delta}_t} \, dt = V$$

where $\hat{\Delta}_t$ is the accumulated average rate of discount:

$$\hat{\Delta}_t = \int_0^t \delta(u_\tau, U_\tau) \, d\tau, \quad \delta(u, U) = \frac{\beta(u, U)}{U}$$

In Uzawa (1974), it is proved that, for any time-path of utilities $u = (u_t)$, where $u_t > 0$, for all $t \geqq 0$, the value of utility functional $V = U(u)$ is uniquely determined. That is, $V = U(u)$ is equal to the minimum value of initial condition $U_0 = V$ for which the solution (U_t) to differential equation (1) exists and is positive for all t, $0 \leqq t < +\infty$.

It can also be shown that the above condition is satisfied if, and only if,

$$\lim_{t \to \infty} U_t \, e^{-\hat{\Delta}_t} = 0$$

which in turn implies that

$$\lim_{t \to \infty} U_t \, e^{-\Delta_t} = 0$$

where Δ_t is the accumulated marginal rate of discount:

$$\Delta_t = \int_0^t \beta_U(u_\tau, U_\tau) \, d\tau$$

It may be noted that, because of the convexity of $\beta(u,U)$,

$$0 < \delta(u, U) < \beta_U(u, U)$$

It is now possible to obtain an explicit formula for the system of efficient prices for time-path of utilities, $u = (u_t)$, to be denoted by $p(u) = (p_t(u))$:

$$p_t(u) = (1 - \beta_U(u_t, U_t)) e^{-\Delta_t}$$

where (U_t) is the solution path to the basic differential equation associated with the time-path $u = (u_t)$ and Δ_t is the accumulated marginal rate of discount.

To illustrate the concept of efficient prices, suppose there exist two time-paths of utilities, $u^0 = (u_t^0)$ and $u^1 = (u_t^1)$, which are indifferent in terms of the given intertemporal preference ordering:

$$u^0 \sim u^1$$

Let $u(\theta) = (u_t(\theta))$ be any smooth curve connecting u^0 and u^1 along the indifferent surface; namely, $u(\theta)$ is defined for all θ, $0 < \theta < 1$, positive valued, continuously, piecewise continuously twice differentiable with respect to θ, and

$$u(\theta) \sim u^0 \sim u^1, \text{ for all} \quad 0 \leqq \theta \leqq 1$$

where $u(0) = u^0$, $u(1) = u^1$. Then, the system of efficient prices $p(u) = (p_t(u))$ satisfies the following conditions:

$$\int_0^{\infty} p_t(u(\theta))u_t'(\theta)\,\mathrm{d}t = 0, \quad \int_0^{\infty} p_t(u(\theta))u_t''(\theta)\,\mathrm{d}t \geqq 0,$$

$$(0 \leqq \theta \leqq 1)$$

Analogous to the static situation, the system of efficient prices $p(u) = (p_t(u))$ plays the role of a separating hyperplane to the indifferent surface. That is, if a given time-path of utilities $u^0 = (u_0^t)$ minimizes the expenditures evaluated at a certain given price system $p = (p_t)$, $p_t > 0$:

$$p \cdot u = \int_0^{\infty} p_t u_t\,\mathrm{d}t$$

among all time-paths of utilities which are indifferent with or preferred to the given time-path $u^0 = (u_t^0)$, then we have

$$p_t = \ell p_t(u^0), \text{ for all } \quad t \geqq 0$$

with a positive constant ℓ.

In contrast, if the proportionality condition above is satisfied, together with the transversality condition:

$$\lim_{t \to +\infty} p_t u_t^0 = 0$$

then u^0 minimizes the expenditures among all time-paths of utilities that are indifferent with or preferred to u^0.

The relationships of our approach to the classic Euler–Lagrange method are transparent. Let us denote

$$\xi_t = \frac{1}{p_t}\mathrm{e}^{-\Delta_t}$$

and differentiate it logarithmically with respect to t, to obtain

$$\frac{\dot{\xi}}{\xi} = \beta_U - \frac{\dot{p}}{p}, \quad 1 - \beta_u = \xi$$

The basic differential equations, together with the transversality condition:

$$\lim_{t \to +\infty} \xi_t U_t = 0$$

correspond to the system of the Euler–Lagrange equations in the standard method of the calculus of variations. We may note that the initial condition for the basic differential equation, V, is equal to the value of the utility functional.

CAPITAL ACCUMULATION AND EFFICIENT PRICES

In the previous section, a simple formula for the system of efficient prices for a class of intertemporal preference ordering is obtained. A similar analysis may be carried out for the intertemporal processes of production involving the accumulation of capital. It will become a prerequisite to the analysis of a dynamically optimum allocation of scarce resources involving social overhead capital in general and the natural environment in particular, and we devote the present section to describing in detail the steps by which the formula for the system of efficient prices for the intertemporal processes of production is derived.

We denote by k_t the stock of capital at each time t, and assume that the stock of capital is the only factor of production relevant in the process of production. The production function is denoted by $f(k)$, which is defined for all non-negative $k \geqq 0$, non-negative-valued, continuously twice differentiable, and satisfies the following conditions:

$$f(k) > 0, \quad f'(k) > 0, \quad f''(k) < 0,$$

$$\text{for all} \quad k > 0 \quad f(0) = 0, \quad f(+\infty) = +\infty :$$

$$f'(0) = \infty, \quad f'(+\infty) = 0$$

It is assumed that there is only one type of goods, and output and capital are identical. When output is invested as capital, it will become impossible to extract output from the stock of capital without the process of production.

The dynamic process of capital accumulation is described by the following differential equation:

$$\dot{k}_t = f(k_t) - x_t, \quad k_0 = K$$

where x_t is the level of consumption at time t, and K is the initial stock of capita. In the basic differential equation, it is required that k_t remains non-negative for all $t \geqq 0$. Hence, for a given positive time-path of consumption, $x = (x_t)$, there exists the minimum value of the initial stock of capital, K^0, for which the positive solution k_t to the basic differential equation exists for all $t \geqq 0$. Such an K^0 may be denoted by $K^0(x)$ to emphasize the dependency upon the given time-path of consumption $x = (x_t)$.

A time-path of consumption $x = (x_t)$ is termed feasible with respect to the stock of capital K, if the solution (k_t) to the basic differential equation, with initial condition $k_0 = K$ exists for all $t \geqq 0$. Thus, $K_0 = K_0(x)$ defined above gives us the minimum stock of capital with respect to which the given time-path consumption $x = (x_t)$ is feasible. Such an $K^0(x)$ may be occasionally called the stock of capital associated with $x = (x_t)$.

Proposition 1 Let $x = (x_t)$ be a given time-path of consumption, where $x_t > 0$, for all $t \geqq 0$. A stock of capital K is the associated stock of capital with respect to x; that is, $K = K^0(x)$, if, and only if,

$$\lim_{t \to +\infty} k_t\,\mathrm{e}^{-\nabla_t} = 0$$

where ∇_t is the accumulated marginal rate of discount:

$$\nabla_t = \int_0^t f'(k_\tau)\,\mathrm{d}\tau$$

and (k_t) is the solution to the basic differential equation with initial condition K.

PROPOSITION 2 For a given time-path of consumption $x = (x_t)$, $(x_t > 0$, for all $t \geqq 0)$, if a stock of capital K is the associated stock of capital with respect to x, that is, $K = K^0(x)$, then the following equality holds:

$$\int_0^\infty x_t\,\mathrm{e}^{-\hat{\nabla}_t}\,\mathrm{d}t = K$$

where $\hat{\nabla}_t$ is the accumulated average rate of discount:

$$\hat{\nabla}_t = \int_0^t \frac{f(k_\tau)}{k_\tau}\,\mathrm{d}\tau$$

Remark It may be the case that, even if equality in Proposition 2 holds, K is not necessarily the associated stock of capital for the given time-path of consumption $x = (x_t)$.

We next see how the system of efficient prices may be obtained for the process of intertemporal production involving capital accumulation. Let K be a given stock of capital $(K > 0)$, and let us denote by $X(K)$ the set of all time-paths of consumption $x = (x_t)$ which is feasible with respect to initial condition K. We may formally write

$$X(K) = \{x = (x_t) : x_t > 0 (t \geqq 0), K^0(x) \leqq K\}$$

In other words, $X(K)$ is the set of all $x = (x_t)$ such that the solution (k_t) to the basic differential equation with initial condition K exists at all time $t \geqq 0$.

It is straightforward to see that, for every $K > 0$, the set $X(K)$ is non-empty; the time-path $x_t \equiv 0$ is always in $X(K)$. The concavity assumption on $f(k)$ implies that $X(K)$ is a convex set; one has only to note the following inequality:

$$(1-\theta)f(k^0) + \theta f(k^1) \leqq f|(1-\theta)f(k^0) + \theta k^1|$$
$$(0 < \theta < 1)$$

with strict inequality when $k^0 \neq k^1$.

A time-path of consumption $x = (x_t)$ is termed efficient with respect to initial stock of capital K, if $x \in X(K)$ and there exists no $x' = (x'_t)$ such that $x' \in X(K)$ and $x' > x$.

It is evident that, if x is efficient with respect to K, then $K = K^0(x)$; that is, K is the minimum value of the initial condition for which a non-negative solution to the basic differential equation exists for all $t \geqq 0$.

Suppose, to the contrary, $K > K^0(x)$. Then there would exist a K' such that $K' < K$ and the given time-path of consumption $x = (x_t)$ is feasible with respect to the initial stock of capital K'. Hence, there must be a time-path consumption $x' = (x'_t)$ such that $x'_t > x_t$, for all t, and x' is feasible with respect to K; a contradiction.

In contrast, the condition $K = K^0(x)$ may not necessarily imply that x is efficient with respect to K.

Suppose now there exist two time-paths of consumption, $x^0 = (x^0_t)$ and $x^1 = (x^1_t)$, both of which are efficient in $X(K)$, where $K > 0$ is a given stock of capital initially endowed. Let us denote by $x(\theta) = (x_t(\theta))$, $0 \leqq \theta \leqq 1$, a smooth curve connecting x^0 and x^1 on the efficiency frontier; that is, $x(\theta)$, as a function of θ, defined on $|0,1|$, is continuously twice differentiable,

$$x(\theta) \text{ is efficient in } X(K) \quad (0 \leqq \theta \leqq 1)$$

$$x(0) = x^0, \quad x(1) = x^1$$

We denote by $k(\theta) = (k_t(\theta))$ the solution to the basic differential equation with initial condition K and time-path of consumption $x(\theta)$; that is,

$$\dot{k}_t(\theta) = f(k_t(\theta)) - x_t(\theta), \quad k_0(\theta) = K$$

The efficiency assumption on $x(\theta)$ implies that

$$\lim_{t\to\infty} k_t(\theta)\mathrm{e}^{-\nabla_t(\theta)} = 0 \quad (0 \leqq \theta \leqq 1)$$

where $\nabla_t(\theta)$ is the accumulated marginal rate of discount at $k(\theta)$:

$$\nabla_t(\theta) = \int_0^t f'(k_\tau(\theta))\mathrm{d}\tau$$

We now differentiate this differential equation with respect to θ and rearrange it to obtain

$$\dot{k}'_t(\theta) - f'(k_t(\theta))k'_t(\theta) = -x'_t(\theta), \quad k'_0(\theta) = 0$$

By multiplying both sides by $\mathrm{e}^{-\nabla_t(\theta)}$ and integrating from 0 to t, we obtain

$$k'_t(\theta)\mathrm{e}^{-\nabla_t(\theta)} = -\int_0^t x'_t(\theta)\mathrm{e}^{-\nabla_t(\theta)}\,\mathrm{d}\tau$$

We next show that, for $0 < \theta < 1$,

$$\lim_{t\to\infty} k'_t(\theta)\mathrm{e}^{-\nabla_t(\theta)} = 0$$

Suppose, to the contrary, there exists a θ_0 for which the equality above is not satisfied. For the sake of simplicity, we substitute θ for $\theta - \theta_0$ or $\theta_0 - \theta$, so that, for some positive number ε,

$$\lim_{t\to\infty} k'_t(0)\mathrm{e}^{-\nabla_t(0)} > \varepsilon > 0$$

Let us denote by $k^0_t(\theta)$ the solution to the above differential equation with initial condition

$K(\theta) = K - (\theta/2)\varepsilon$.

$$\dot{k}_t^0(\theta) = f(k_t^0(\theta)) - x_t(\theta), \quad k_0^0(\theta) = K - \frac{\theta}{2}\varepsilon$$

Differentiating both sides with respect to θ, multiplying by $\mathrm{e}^{-\nabla_t^0(\theta)}$ and integrating from 0 to ∞ yield

$$\lim_{t\to\infty} k_t^{0\prime}(\theta)\mathrm{e}^{-\nabla_t^0(\theta)} = -\frac{\varepsilon}{2} - \int_0^\infty x_t^{0\prime}(\theta)\mathrm{e}^{-\nabla_t^0(\theta)}\,\mathrm{d}t$$

where $\nabla_t^0(\theta)$ is the accumulated marginal rate of discount at $(k_t^0(\theta))$.

It may be noted that

$$k_t^0(0) = K_t(0), \quad \nabla_t^0(0) = \nabla_t(0)$$

Hence, we have,

$$\lim_{t\to\infty} k_t^{0\prime}(\theta)\mathrm{e}^{-\nabla_t^0(\theta)} = -\varepsilon/2 - \int_0^\infty x_t(\theta)\mathrm{e}^{-\nabla_t^0(\theta)}\,\mathrm{d}t$$

$$> \varepsilon/2 > 0, \quad (0 \leq \theta \leq \bar{\theta})$$

for $\bar{\theta}$ sufficiently small, but positive.

By applying the mean-value theorem, there exist θ_t^* such that

$$k_t^0(\theta) - K_t^0(0) = \theta K_t^{0\prime}(\theta_t^*), \quad 0 < \theta_t^* < \theta$$

which implies that

$$\lim_{t\to\infty} k_t^0(\theta)\mathrm{e}^{-\nabla_t^0(\theta)} > 0, \text{ for sufficiently small } \quad \theta > 0$$

meaning that $x(\theta)$ is feasible with respect to $K(\theta)$; a contradiction.

By taking the limit, as $t \to \infty$, we obtain the following basic equality:

$$\int_0^\infty x_t'(\theta)\mathrm{e}^{-\nabla_t(\theta)}\,\mathrm{d}t = 0$$

We next differentiate the above equation with respect to θ and rearrange to obtain

$$\dot{k}_t''(\theta) - f'(k_t(\theta))k_t''(\theta) = f''(k_t(\theta))k_t'(\theta)^2 - x_t''(\theta)$$

$$k_t''(\theta) = 0$$

By multiplying both sides of the above equation again by $\mathrm{e}^{-\nabla_t}(\theta)$ and integrating it from 0 and t, we obtain

$$k_t''(\theta)\mathrm{e}^{-\nabla_t(\theta)} = \int_0^t f''(k_\tau(\theta))k_\tau'(\theta)^2\mathrm{e}^{-\nabla_t(\theta)}\,\mathrm{d}\tau$$

$$- \int_0^t x_\tau''(\theta)\mathrm{e}^{-\nabla_t(\theta)}\,\mathrm{d}\tau$$

We also can show that

$$\lim_{t\to\infty} k''(\theta)\mathrm{e}^{-\nabla_t(\theta)} \geqq 0$$

which together with the concavity of $f(k)$, implies that

$$\int_0^\infty x_t''(\theta)\mathrm{e}^{-\nabla_t(\theta)}\,\mathrm{d}t \leqq 0$$

with strict inequality when $x^0 \neq x^1$.

We have now established the following.

PROPOSITION 3 Let $x = (x_t)$, $x_t > 0 (t \geqq 0)$, be a given time-path of consumption and $k = (k_t)$ be the solution to the basic differential equation with initial condition $K = K^0(x)$. Then the time-path, $p(x) = (p_t(x))$, defined by

$$p_t(x) = \mathrm{e}^{-\nabla_t(k)}$$

where $\nabla_t(k)$ is the accumulated marginal rate of discount:

$$\nabla_t(k) = \int_0^t f_\tau'(k_\tau)\mathrm{d}\tau$$

represents the system of efficient prices at $x = (x_t)$; that is, $p(x)$ satisfies the following conditions:

$$\int_0^\infty x_t'(\theta)p_t(x(\theta))\mathrm{d}t = 0, \quad \int_0^\infty x_t''(\theta)p_t(x(\theta))\mathrm{d}t \leqq 0$$

where $x(\theta) = (x_t(\theta))$, $0 \leqq \theta \leqq 1$, is a smooth curve connecting two time-paths X^0 and X^1 on the efficiency frontier associated with the initial stock of capital K.

DYNAMIC DUALITY PRINCIPLE: A SIMPLE MODEL OF CAPITAL ACCUMULATION

The concept of efficient prices may be used to formulate the duality principles concerning intertemporal preference and processes of capital accumulation, as discussed in Chapter 7. Before we proceed with the analysis of a general dynamic model involving accumulation or degradation of social overhead capital, we apply the dynamic duality principles to derive criteria for dynamic optimality in the simple model of capital accumulation.

We consider the simple model of capital accumulation as introduced in the previous sections, and analyze the structure of the time-paths of consumption and capital accumulation that are optimum in terms of the given intertemporal preference ordering as described in "Capital accumulation and efficient prices" section.

Let K be the stock of capital endowed in the economy at time 0, and let the production function $f(k)$ satisfy the neoclassical assumptions as specified in "Capital accumulation and efficient prices" section. A time-path of consumption, $x = (x_t)$, $(x_t > 0)$, is feasible with respect to initial stock of capital K if there exists a solution to the following differential equation:

$$\dot{k}_t = f(k_t) - x_t, \quad k_0 = K$$

The time-path of utilities, $u = (u_t)$, associated with time-path of consumption, $x = (x_t)$, is simply given by

$$u_t = u(x_t)$$

where the utility function $u(x)$ satisfies the assumptions described in "Capital accumulation and efficient prices" section.

The time-path of utility functionals, (U_t), where U_t represents the value of utility functional of the time-path of utilities truncated at t, is specified by the following differential equation:

$$U_t = \beta(u_t, U_t) - u_t$$

where $\beta(u,U)$ describes the processes of utility discounting and is assumed to satisfy the conditions referred to in "Capital accumulation and efficient prices" section.

The level of utility functional $U_0 = U(u)$ for the time-path of utilities $u = (u_t)$ itself is to be determined so as to satisfy the transversality condition:

$$\lim_{t \to \infty} U_t \, \mathrm{e}^{-\Delta_t} = 0$$

where the accumulated marginal rate of discount Δ_t is defined by

$$\Delta_t = \int_0^t \beta_U(u_\tau, U_\tau) \mathrm{d}\tau$$

A time-path of capital accumulation, (k_t), or of consumption, (x_t), is defined dynamically optimum when the value of utility functional $U(u)$, $u = (u(x_t))$, is maximized among all feasible time-paths of consumption with the given initial stock of capital K.

The dynamic duality principle implies that a feasible time-path of consumption $x = (x_t)$ is dynamically optimum if, and only if, the two systems of imputed prices, $(p_t(u))$ and $(p_t(x))$ are proportional; that is,

$$u'(x_t)(1 - \beta(u_t, U_t))\mathrm{e}^{-\Delta_t} = \ell f'(k_t)\mathrm{e}^{-\nabla_t}$$

where $u_t = u(x_t)$ and ℓ is a positive constant.

The structure of dynamically optimum time-paths of capital accumulation then is analyzed by logarithmically differentiating both sides of the above equation with respect to time t.

To extend our analysis to the situation where the environment, or social overhead capital, plays a crucial role, in the processes of both consumption and production, it is necessary to closely examine the premises concerning the effect of investment upon the process of capital accumulation.

The basic dynamic equation stipulates that whatever amount of the produced goods that are put aside as investment induces an increase in the stock of capital, exactly in the same amount as the invested goods. With respect to those capital goods that comprise fixed capital in any corporative institution, there exists a diminishing marginal rate of increase in the productive capacity of capital corresponding to investment, referred to as the Penrose effect, as originally introduced by Uzawa (1968) and in detail described in Chapter 9 below.

To incorporate the Penrose effect in our model, let us introduce the real investment A as an explicit variable in our model. The output $f(k_t)$ at each time t then is divided between consumption x_t and investment A_t:

$$f(k_t) = x_t + A_t$$

The accumulation of capital is described by

$$\dot{k}_t = \phi(A_t, k_t)$$

where $\phi(A,k)$ is the Penrose function satisfying the following conditions.

The function $\phi(A,k)$ is defined for all $A \geqq 0$, $k > 0$, continuously twice differentiable, concave with respect to (A,k), and satisfies

$$\phi_A(A,k) > 0, \quad \phi_k(A,k) > 0, \quad \phi_{AA} < 0, \quad \phi_{kk} < 0,$$

$$\phi_{AA}\phi_{kk} - \phi_{Ak}^2 \geqq 0$$

For any given efficient time-path of consumption $x = (x_t)$, the system of imputed prices is given by

$$p_t(x) = \phi_A(A_t, k_t)f'(k_t)\mathrm{e}^{-\nabla_t}$$

where ∇_t is the accumulated marginal rate of discount defined by

$$\nabla_t = \int_0^t |\phi_A(A_\tau, k_\tau)r(k_\tau) + \phi_k(A_\tau, k_\tau)| \mathrm{d}\tau$$

The duality principle is now modified as

$$u'(x_t)(1 - \beta_u(u_t, U_t))\mathrm{e}^{-\Delta_t} = \ell\phi_A(A_t, k_t)f'(k_t)\mathrm{e}^{-\nabla_t}$$

where ∇_t the accumulated marginal rate of discount.

To examine the time-path of capital accumulation k_t and the truncated time-path of utilities U_t for which the duality conditions are satisfied, let us introduce the new time-path of imputed prices $\xi = (\xi_t)$:

$$\xi_t = \ell \, \mathrm{e}^{\Delta_t - \nabla_t} \quad (t \geqq 0)$$

This relation may be written

$$\frac{1 - \beta_u}{\phi_A} = \xi$$

where time suffix t is omitted.

Differentiating the above equation logarithmically with respect to t, we obtain

$$\frac{\dot{\xi}}{\xi} = \beta_U - (r\phi_A + \phi_k)$$

while k_t and U_t have to satisfy the following dynamic equations

$$\dot{k} = \phi(A, k), \quad \dot{U} = \beta(u, U) - u \quad (u_t = u(x_t))$$

with initial conditions $k^0 = K(x)$, $U^0 = U(u)$.

Let us first consider the simple case where both $f(k)$ and $\beta(u,U)$ are homogenous; that is,

$$f'(k) = r > 0 : \text{constant} \quad \beta(u, U) = \delta(z)U, \quad z = \frac{u}{U}$$

where the average rate of discount function $\delta(z)$, $z = u/U$, satisfies the following conditions:

$$\delta(z) > 0, \quad \delta'(z) < 0, \quad \delta''(z) > 0 \quad (z > 0)$$

It is also assumed that the Penrose function $\phi(A, k)$ is homogeneous of order one so that

$$\phi(A, k) = \phi(\alpha)k, \quad \alpha = \frac{A}{k}$$

where $\phi(\alpha)$ satisfies the following conditions:

$$\phi'(\alpha) > 0, \quad \phi''(\alpha) < 0 \quad (\alpha \geqq 0)$$

Furthermore, we consider the case where $u(x) = x$ ($x \geqq 0$). Then, the marginality condition is simply written as

$$\frac{1 - \delta'(z)}{\phi'(\alpha)} = \xi$$

and the feasibility condition may be reduced to

$$r = zw + \alpha$$

where

$$w = \frac{U}{k}, \quad z = \frac{x}{U}$$

The system of differential equations above may be reduced to the following

$$\frac{\dot{\xi}}{\xi} = \hat{\delta}(z) - |\phi(\alpha) + (r - \alpha)\phi'(\alpha)|,$$

$$|\hat{\delta}(z) = \delta(z) - \delta'(z)z|$$

$$\frac{\dot{k}}{k} = \phi(\alpha)$$

$$\frac{\dot{U}}{U} = \delta(z) - z$$

At each time t, the optimum values of z and α are determined by the pair of differential equations above, where the values of w and ξ are given. By taking differentials of right-hand sides of the differential equations and solving with respect to dz and dα, we obtain

$$\begin{pmatrix} \mathrm{d}z \\ \mathrm{d}\alpha \end{pmatrix} = \frac{1}{\delta'' - \xi w \phi''} \begin{pmatrix} z\phi'' & \xi - \phi'' \\ -z\delta'' & w\phi'' \end{pmatrix} \begin{pmatrix} \mathrm{d}w \\ \mathrm{d}\xi \end{pmatrix}$$

$$= \begin{pmatrix} - & - \\ - & + \end{pmatrix} \begin{pmatrix} \mathrm{d}w \\ \mathrm{d}\xi \end{pmatrix}$$

Let us now show that the system of differential equations above has a unique steady state (ξ^0, k^0, U^0). In order to see this, the system of differential equations (64)–(66) may be reduced to the one involving (w,ξ);

$$\frac{\dot{w}}{w} = \delta(z) - z - \phi(\alpha)$$

By taking the differentials of right-hand sides of the above equations, we obtain

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\alpha}\right)_{\dot{w}=0} = -\frac{\phi'(\alpha)}{1 - \delta'(z)} < 0$$

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\alpha}\right)_{\dot{\alpha}=0} = -\frac{(r - \alpha)\phi''(\alpha)}{z\delta''(z)} > 0, \text{ for } \quad \alpha < r$$

Hence, the values of z^0 and α corresponding to the stationary state (w^0, ξ^0) for the system of basic differential equations are uniquely determined as the solution to following pair of equations:

$$\delta(z^0) - z^0 - \phi(\alpha) = 0 \hat{\delta}(z^0) = \phi(\alpha) + (r - \alpha)\phi'(\alpha)$$

The stationary state (w^0, ξ^0) is then given by

$$\xi^0 = \frac{1 - \delta'(z^0)}{\phi'(\alpha)}, \quad w^0 = \frac{r - \alpha}{z^0}$$

The phase diagrams for the system of differential equations are easily analyzed. We have the following relations:

$$\frac{\mathrm{d}\xi}{\mathrm{d}w_{\dot{w}=0}} = -\frac{z(\delta'' - \phi''\xi^2)}{\phi'(\xi - w)} \gtrless 0, \quad \xi \gtrless w$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}w_{\dot{\xi}=0}} = -\frac{(\delta'' - \phi''\xi^2)\phi'}{(\xi - w)z\delta''\phi''w} \lessgtr 0, \quad \xi \lessgtr w$$

The stationarity conditions may be rearranged

$$\frac{U_0}{k_0} = w = \frac{r - \alpha}{\delta(\alpha) - \phi(\alpha)}$$

We obtain

$$\frac{dz}{d\alpha} = -\frac{z}{r - \alpha}$$

$$\frac{dw}{d\alpha} = \frac{1}{(r-\alpha)^2}\left\{-(\delta - \phi) - (r - \alpha)\left(\delta' \frac{dz}{d\alpha} - \phi'\right)\right\}$$

$$= \frac{1}{(r-\alpha)^2}\{ - \delta + \phi + z\delta' + (r - \alpha)\phi\}$$

Therefore,

$$\frac{dw}{d\alpha} = 0 \text{ if, and only if,} \quad \delta - z\delta' = \phi + (r - \alpha)\phi'$$

In contrast, a simple calculation shows that

$$\frac{d^2 w}{d\alpha^2} \sim -\frac{z\delta''}{w} + zw\delta'' < 0$$

where symbol $\sim$ indicates both sides have the same sign.

where $z = x/U$, $\alpha = A/k$, and

$$f(k) = x + \alpha k, \quad z = x/U, \quad (1 - \delta'(z)) = \xi\phi'(\alpha)$$

By taking the differentials of both sides of these equations, we obtain

$$\begin{pmatrix} \delta''/U & \varphi''\xi \\ 1 & k \end{pmatrix}\begin{pmatrix} dx \\ d\alpha \end{pmatrix} = \begin{pmatrix} \delta'' z/U & 0 & -\varphi' \\ 0 & r-\alpha & 0 \end{pmatrix}\begin{pmatrix} dU \\ dk \\ d\xi \end{pmatrix}$$

Hence,

$$\begin{pmatrix} dx \\ d\alpha \end{pmatrix} = 1/\Delta_1 \begin{pmatrix} \delta'' zk/U & -(r-\alpha)\varphi''\xi & -k\varphi' \\ -\delta'' z/U & (r-\alpha)\delta''/U & \varphi' \end{pmatrix}\begin{pmatrix} dU \\ dk \\ d\xi \end{pmatrix}$$

where

$$\Delta_1 = \frac{\delta'' k}{U} - \varphi''\xi > 0$$

$$\begin{pmatrix} d\dot{U} \\ d\dot{k} \\ d\dot{\xi} \end{pmatrix} = \frac{1}{\Delta_1}\begin{pmatrix} -\frac{(1-\delta')\delta'' zk}{U^2} + \Delta_1 \frac{(1-\delta')z}{U} & \frac{(1-\delta')\varphi''\xi(r-\alpha)}{U} & \frac{(1-\delta')k\varphi'}{U} \\ -\frac{\delta'' z\varphi'}{U} & -\frac{\delta''\varphi'(r-\alpha)}{U} & \varphi' 2 \\ -\frac{\delta'' z}{U}\left(\frac{\delta'' zk}{U} - (r-\alpha)\varphi''\right) + \Delta_1 \frac{\delta'' z^2}{U} & \frac{\delta''\varphi''}{U}(r-\alpha)(z\xi - (r-\alpha)) - \Delta_1 r'\varphi' & \varphi'\left(\frac{\delta'' zk}{U} - (r-\alpha)\varphi''\right) \end{pmatrix}$$

$$\times \begin{pmatrix} dU \\ dk \\ d\xi \end{pmatrix}$$

The analysis developed above may be extended to the case where the marginal product of capital is variable. Namely, we consider the case where

$$f(k) > 0, \quad f'(k) > 0, \quad f''(k) < 0 \quad (k \geqq 0)$$

The basic system of differential equations now may be summarized as:

$$\frac{\dot{U}}{U} = \delta(z) - z$$

$$\frac{\dot{k}}{k} = \phi(\alpha)$$

$$\frac{\dot{\xi}}{\xi} = |\delta(z) - z\delta'(z)| - |\phi(\alpha) + (r - \alpha)\phi'(\alpha)|$$

whose determinant D may be shown to be negative.

In contrast, the trace of the above matrix is easily shown to be positive.

Hence, the system of differential equations above is catenary. The stationary state is uniquely determined, and exactly one characteristic root is real and negative.

For any given pair of (U,k), there uniquely exists a value of imputed price ξ such that the solution path to the system the basic differential equations converges to the stationary state, corresponding to the dynamically optimum path.

DYNAMIC OPTIMALITY OF INVESTMENT IN SOCIAL OVERHEAD CAPITAL

The dynamic duality principles discussed in the previous sections may be easily extended to the case where the intertemporal preference ordering is affected by the natural environment, which is conceptualized as the stock of social overhead capital.

Our discussion is carried out within the framework of the simple, dynamic model of capital accumulation as posited in "Intertemporal preferences" section, except for the stock of social overhead capital as another component of the stock of capital in general. We denote by V_t the stock of social overhead capital existing in the society at time t, while the stock of private capital is denoted by k_t, as in "Dynamic duality principle: a simple model of capital accumulation" section. The output $f(k_t)$ at each time t is now divided between consumption x_t, investment in private capital A_t, and investment in social overhead capital B_t:

$$f(k_t) = x_t + A_t + B_t$$

where the production function $f(k)$ is assumed to satisfy all the neoclassical conditions postulated in "Capital accumulation and efficient prices" section above.

The rate of increase in the stock of private capital, $\dot{k}_t$, is determined in terms of the Penrose function $\phi(A_t,k_t)$:

$$k_t = \phi(A_t, k_t)$$

where the Penrose function $\phi(A,k)$ is assumed to satisfy the conditions as specified in this section.

We also assume that the effect of investment in social overhead capital is subject to the Penrose effect, so that the rate of increase in the stock of social overhead capital, $\dot{V}_t$, is determined in terms of the Penrose function $\psi(B,V)$:

$$\dot{V}_t = \psi(B_t, V_t)$$

The Penrose function $\psi(B,V)$ concerning the accumulation of social overhead capital is assumed to satisfy the concavity conditions as in this section.

The function $\psi(B,V)$ is defined for all $B \geqq 0$, $V > 0$, continuously twice differentiable, concave with respect to (B,V), and satisfies

$$\psi_B(B,V) > 0, \quad \psi_V(B,V) > 0, \quad \psi_{BB} < 0,$$

$$\psi_{VV} < 0, \quad \psi_{BB}\psi_{VV} - \psi_{BV}^2 \geqq 0$$

The intertemporal preference ordering is assumed to possess the structure specified in "Capital accumulation and efficient prices" section above. In the present case, however, the utility u_t at each time t is a function of vector of consumption x_t and the stock of environmental capital V_t:

$$u_t = u(x_t, V_t)$$

where $u(x_t,V_t)$ is assumed to be defined for all $(x_t, V_t) \geqq (0,0)$, positive valued, continuously twice-differentiable, and strictly concave:

$$u(x_t, V_t) > 0, \quad u_x(x_t, V_t), \quad u_V(x_t, V_t) > 0,$$

$$u_{xx}, \quad u_{VV}, \quad u_{xx}u_{VV} - u_{xV}^2 > 0$$

$$\text{for all } (x_t, V_t) \geqq (0,0)$$

The utility functional U_t associated with the truncated time-path of utilities ${}^1u = (u_{t+\tau})$ satisfies the basic differential equation in this section. The instantaneous level of utility u_t is assumed to depend upon the stock of social overhead capital V_t as well as the vecter of consumption x_t:

$$u_t = u(x_t, V_t)$$

where $u(x_t,V_t)$ is assumed to be defined for all $(x_t, V_t) \geqq (0,0)$, positive valued, continuously twice-differentiable, and strictly concave:

$$u(x_t, V_t) > 0, \quad u_x(x_t, V_t), \quad u_V(x_t, V_t) > 0,$$

$$u_{xx}, \quad u_{VV} > 0, \quad u_{xx}u_{VV} - u_{xV}^2 \geqq 0,$$

$$\text{for all } (x_t, V_t) \geqq (0,0)$$

Let us denote the imputed prices of private capital and social overhead capital at each time t by ξ_t and η_t, respectively. Then, it is easily shown that the imputed prices are characterized by the following dynamic equations:

$$\frac{\dot{\xi}}{\xi} = -\phi_k - r\phi_A, \quad r = f_k$$

$$\frac{\dot{\eta}}{\eta} = -\psi_V - s\psi_B, \quad s = \frac{u_V}{u_x}$$

To derive formulas for the systems of efficient prices, let us consider two time-paths of consumption and the stocks of private and social overhead capital, (x_t^0, k_t^0, V_t^0) and (x_t^1, k_t^1, V_t^1), such that

$$(k_0^0, V_0^0) = (k_0^1, V_0^1) = (k^0, V^0)$$

and both (x_t^0) and (x_t^1) are efficient.

We denote by $(x_t(\theta), k_t(\theta), V_t(\theta))$ the time-path connecting (x_t^0, k_t^0, V_t^0) and (x_t^I, k_t^I, V_t^I) which is also efficient:

$$\dot{k}_t(\theta) = \phi(A_t(\theta), K_t(\theta)), \quad k_0(\theta) = k^0$$

$$\dot{V}_t(\theta) = \psi(B_t(\theta), V_t(\theta)), \quad V_0(\theta) = V^0$$

where $A_t(\theta)$ and $B_t(\theta)$ are respectively investments in private capital and social overhead capital.

We obtain

$$\dot{k}'_t = \phi_A A'_t + \phi_k k'_t, \quad k'_0 = 0$$

$$\dot{V}'_t = \psi_B B'_t + \psi_V V'_t, \quad V'_0 = 0$$

where symbol $'$ indicates the derivative with respect to θ, while parameter θ is omitted, and ϕ_A, ϕ_k, ψ_B, ψ_V are evaluated at time t.

The feasibility conditions may be written

$$f(k_t(\theta)) = x_t(\theta) + A_t(\theta) + B_t(\theta)$$

which, by differentiating with respect to θ, yields

$$rk'_t = x'_t + A'_t + B'_t, \quad r = f'(k_t)$$

In contrast, we have

$$u'_t = u_x(x'_t + sV'_t), \quad s = u_V/u_x$$

Now let us denote by π_t the imputed price of the output at at time t. Then, we have

$$\xi_t\phi_A = \eta_t\psi_B = \pi_t$$

$$\xi_t\dot{k}'_t + \eta_t\dot{V}'_t = -\pi_t x'_t + \pi_t r k'_t + \xi_t\phi_k k'_t + \eta_t\psi_V$$

$$V'_t\dot{\xi}_t k'_t + \dot{\eta}_t V'_t = -(\xi_t\phi_k k'_t + \eta_t\psi_V V'_t) - \pi_t(rk'_t + sV'_t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\xi_t k'_t + \eta_t V'_t) = -\pi_t(x'_t + sV'_t) = -\xi_t u'_t\frac{\phi_A}{u_x}$$

The transversality conditions imply that

$$\lim_{t\to\infty}(\xi_t k'_t + \eta_t V'_t) = 0,$$

while $k'_0 = V'_0 = 0$. Hence, we obtain the following equality:

$$\int_0^\infty p_t u'_t \,\mathrm{d}t = 0$$

where

$$p_t = \frac{\phi_A}{u_x}\xi_t, \quad \int_0^\infty u''_t p_t \,\mathrm{d}t \leqq 0$$

The system of imputed prices (p_t) may be explicitly written as follows:

$$p_t = \frac{\phi_A}{u_x}\mathrm{e}^{-\nabla_t}$$

where ∇_t is the accumulated marginal rate of discount defined by

$$\nabla_t = \int_0^1 |\phi_k(A_\tau, k_\tau) + r(k_\tau)\phi_A(A_\tau, k_\tau)|\mathrm{d}\tau$$

It may be noted the system of efficient prices is determined independently of the time-path of the stock of social overhead capital. The effect of the changing pattern of the stock of social overhead capital is felt only through the changes in the levels of instantaneous utility $u_t = u(x_t, V_t)$.

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