FIXED POINTS OF CONDENSING MULTIVALUED MAPS IN TOPOLOGICAL VECTOR SPACES

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With the aid of the simplicial approximation property, we show that every admissible multivalued map from a compact convex subset of a complete metric linear space into itself has a fixed point. From this fact we deduce the fixed point property of a closed convex set with respect to pseudocondensing admissible maps.

1. Introduction

The Schauder conjecture that every continuous single-valued map from a compact convex subset of a topological vector space into itself has a fixed point was stated in [12, Problem 54]. In a recent year, Cauty [2] gave a positive answer to this question by a very complicated approximation factorization. Very recently, Dobrowolski [3] established Cauty's proof in a more accessible form by using the fact that a compact convex set in a metric linear space has the simplicial approximation property.

The aim in this paper is to obtain multivalued versions of the Schauder fixed point theorem in complete metric linear spaces. For this we consider three classes of multivalued maps; that is, admissible maps introduced by Górniewicz [4], pseudocondensing maps by Hahn [5], and countably condensing maps by Väth [15], respectively. These pseudocondensing or countably condensing maps are more general than condensing maps.

The main result is that every compact convex set in a complete metric linear space has the fixed point property with respect to admissible maps. The proof is based on the simplicial approximation property and its equivalent version due to Kalton et al. [9], where the latter corresponds to admissibility of the involved set in the sense of Klee [10]; see also [11]. More generally, we apply the main result to prove that every pseudocondensing admissible map from a closed convex subset of a complete metric linear space into itself has a fixed point. Finally, we present a fixed point theorem for countably condensing admissible maps in Fréchet spaces. Here, the fact that we restrict ourselves to countable sets is important in connection with differential and integral operators. The above results include the well-known theorems of Schauder [14], Kakutani [8], Bohnenblust and Karlin [1], and Sadovskii [13].

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For a subset K of a topological vector space E, the closure, the convex hull, and the closed convex hull of K in E are denoted by $\overline{K}$, $\operatorname{co} K$, and $\overline{\operatorname{co}} K$, respectively. By $k(K)$ we denote the collection of all nonempty compact subsets of K.

For topological spaces X and Y, a multivalued map $F : X \multimap Y$ is said to be *upper semicontinuous* on X if, for any open set V in Y, the set $\{x \in X : Fx \subset V\}$ is open in X. F is said to be *compact* if its range $F(X)$ is contained in a compact subset of Y.

Definition 1.1. Given two topological spaces X and Y, an upper semicontinuous map $F : X \to k(Y)$ is said to be *admissible* if there exist a topological space Z and continuous functions $p : Z \to X$ and $q : Z \to Y$ with the following properties:

(1) $\emptyset \neq q(p^{-1}x) \subset Fx$ for each $x \in X$;
(2) p is proper; that is, the inverse image $p^{-1}(A)$ of any compact set $A \subset X$ is compact;
(3) for each $x \in X$, $p^{-1}x$ is an acyclic subspace of Z.

It is well known that an upper semicontinuous map $F : X \to k(Y)$ with acyclic values is admissible and the composition of two admissible maps is also admissible; see [4, Theorem III.2.7].

Throughout this paper we assume that E is a topological vector space that is not necessarily locally convex. $E = (E, \|\cdot\|)$ will be a metric linear space, where $\|\cdot\|$ is an F-norm on E. Hence we have $\|x+y\| \leq \|x\| + \|y\|$ and $\|tx\| \leq \|x\|$ for all $x, y \in E$ and $t \in [-1,1]$. If $E = (E, \|\cdot\|)$ is a complete metric linear space, it is called an *F-space*. A locally convex F-space is called a *Fréchet space*.

2. Admissible maps

With the aid of the simplicial approximation property, we extend Cauty's fixed point theorem to admissible multivalued maps.

We introduce the simplicial approximation property due to Kalton et al. [9] which is a key tool of our main result.

Definition 2.1. A convex subset C of a metric linear space $(E, \|\cdot\|)$ has the *simplicial approximation property* if, for every $\varepsilon > 0$, there exists a finite-dimensional compact convex set $C_\varepsilon \subset C$ such that, if S is any finite-dimensional simplex in C, then there exists a continuous map $h : S \to C_\varepsilon$ with $\|h(x) - x\| < \varepsilon$ for all $x \in S$.

Dobrowolski recently obtained the following result; see [3, Lemma 2.2] and [3, Corollary 2.6].

Lemma 2.2. *Every compact convex set in a metric linear space has the simplicial approximation property.*

The following equivalent formulation of the simplicial approximation property is given in [9, Theorem 9.8].

Lemma 2.3. *If K is an infinite-dimensional compact convex set in an F-space $(E, \|\cdot\|)$, then the following statements are equivalent:*

(1) *K has the simplicial approximation property,*

(2) *if* $\varepsilon > 0$, *there exist a simplex* S *in* K *and a continuous map* $h : K \to S$ *such that* $\|h(x) - x\| < \varepsilon$ *for all* $x \in K$.

Now we can give a multivalued version of Cauty's fixed point theorem [2]. The proof is based on the simplicial approximation property, where we follow the basic line of the proof in [7, Satz 4.2.5].

Theorem 2.4. *Let* K *be a nonempty compact convex set in an* F*-space* $(E, \|\cdot\|)$. *Then any admissible map* $F : K \to k(K)$ *has a fixed point.*

Proof. Suppose that $F : K \to k(K)$ is an admissible map. Since K is a compact convex set and $\overline{\text{co}}F(K) \subset K$, it follows that the set $C := \overline{\text{co}}F(K)$ is compact and convex. By Lemma 2.2, C has the simplicial approximation property. Let $\varepsilon > 0$ be given. Lemma 2.3 implies that there exist a simplex S in C and a continuous map $h_\varepsilon : C \to S$ such that $\|h_\varepsilon(x) - x\| < \varepsilon$ for all $x \in C$.

The composition of F and h_ε, $h_\varepsilon \circ F|_S : S \to k(S)$, is an admissible compact map on S. Notice that every admissible compact multivalued map with compact values defined on an acyclic absolute neighborhood retract has a fixed point; see [4]. Since the simplex S is an acyclic absolute retract, there exists a point x_ε of S such that $x_\varepsilon \in (h_\varepsilon \circ F)x_\varepsilon$. Then there is a point $y_\varepsilon \in Fx_\varepsilon (\subset C)$ such that

$$x_\varepsilon = h_\varepsilon(y_\varepsilon), \qquad ||h_\varepsilon(y_\varepsilon) - y_\varepsilon|| < \varepsilon. \tag{2.1}$$

By the compactness of C we may assume, without loss of generality, that the net (y_ε) converges to some point x in C. Hence it follows that the net (x_ε) also converges to x. Since F is an upper semicontinuous multivalued map with compact values and so F has a closed graph, we conclude that $x \in Fx$. This completes the proof. □

3. Condensing maps

Using a fixed point theorem for admissible maps given in Section 2, we prove that the fixed point property holds for pseudocondensing or countably condensing admissible maps.

In order to generalize the concept of condensing maps in a reasonable way, we need a c-measure of noncompactness introduced by Hahn [5, 6].

Definition 3.1. Let E be a topological vector space, K a nonempty closed convex subset of E, and $\mathcal{M}$ a collection of nonempty subsets of K with the property that, for any $M \in \mathcal{M}$, the sets $\text{co}\, M, \overline{M}, M \cup \{x_0\}$ $(x_0 \in K)$, and every subset of M belong to $\mathcal{M}$. Let c be a real number with $c \geq 1$. A function $\psi : \mathcal{M} \to [0, \infty)$ is said to be a *c-measure of noncompactness* on K provided that the following conditions hold for any $M \in \mathcal{M}$:

(1) $\psi(\overline{M}) = \psi(M)$;
(2) if $x_0 \in K$, then $\psi(M \cup \{x_0\}) = \psi(M)$;
(3) if $N \subset M$, then $\psi(N) \leq \psi(M)$;
(4) $\psi(\text{co}\, M) \leq c\psi(M)$.

The c-measure of noncompactness is said to be *regular* provided that $\psi(M) = 0$ if and only if M is precompact. In particular, if $c = 1$, then ψ is called a *measure of noncompactness* on K.

Definition 3.2. Let K be a closed convex subset of a topological vector space E, Y a nonempty subset of K, and ψ a c-measure of noncompactness on K. An upper semicontinuous map $F : Y \to k(K)$ is said to be *pseudocondensing* on Y provided that, if X is any subset of Y such that $\psi(X) \le c\psi(F(X))$, then $F(X)$ is relatively compact. In particular, if $c = 1$, F is called *condensing*.

In [5] it is shown that the Kuratowski function is a c-measure of noncompactness on a subset of a paranormed space under certain conditions. An example of a pseudocondensing map in the nonlocally convex topological vector space $S(0,1)$ is given in [6].

First we give the following fundamental property of a pseudocondensing map.

Lemma 3.3. *Let K be a closed convex subset of a topological vector space E, Y a nonempty subset of K, and ψ a c-measure of noncompactness on K. If $F : Y \to k(K)$ is a pseudocondensing map, then there exists a closed convex subset C of K with $C \cap Y \neq \emptyset$ such that $F(C \cap Y)$ is a relatively compact subset of C.*

Proof. Choose a point $x_0 \in Y$ and let

$$\Sigma := \{A \subset K : A = \overline{\text{co}}A,\ x_0 \in A,\ F(A \cap Y) \subset A\}. \tag{3.1}$$

Then Σ is nonempty because $K \in \Sigma$. Set $C := \bigcap_{A \in \Sigma} A$ and $C_1 := \overline{\text{co}}(F(C \cap Y) \cup \{x_0\})$. Since $C \in \Sigma$, we have $C_1 \subset C$ and so $F(C_1 \cap Y) \subset F(C \cap Y) \subset C_1$, therefore $C_1 \in \Sigma$. Hence it follows from definition of C that $C = \overline{\text{co}}(F(C \cap Y) \cup \{x_0\})$. Since ψ is a c-measure of noncompactness on K, we have

$$\psi(C \cap Y) \le c\psi\big(F(C \cap Y) \cup \{x_0\}\big) = c\psi\big(F(C \cap Y)\big). \tag{3.2}$$

Since F is pseudocondensing, $F(C \cap Y)$ is a relatively compact subset of C. This completes the proof. □

Now we can prove a fixed point theorem for pseudocondensing admissible maps in F-spaces.

Theorem 3.4. *Let K be a nonempty closed convex set in an F-space E and ψ a regular c-measure of noncompactness on K. Then any pseudocondensing admissible map $F : K \to k(K)$ has a fixed point.*

Proof. Let $F : K \to k(K)$ be a pseudocondensing admissible map. By Lemma 3.3, there exists a nonempty closed convex subset B of K such that $F(B)$ is a relatively compact subset of B. Note that $C := \overline{\text{co}}F(B)$ is compact and $C \subset B$. In fact, since ψ is regular and $c \ge 1$, it follows from $\psi(\text{co}\,F(B)) \le c\psi(F(B))$ that $\psi(\text{co}\,F(B)) = 0$ which implies that $\text{co}\,F(B)$ is precompact. Hence the closed set C is obviously compact in the complete metric space E. The restriction of F to the compact convex set C, $G := F|_C : C \to k(C)$, is an admissible map. Theorem 2.4 implies that G has a fixed point. We conclude that F has a fixed point. This completes the proof. □

Corollary 3.5. *Let K be a nonempty closed convex set in an F-space E. Then any compact admissible map $F : K \to k(K)$ has a fixed point.*

Proof. For any subset X of K, since $F(K)$ is relatively compact, $F(X)$ is also relatively compact. This means that every compact map F is pseudocondensing. Now Theorem 3.4 is applicable. □

Remark 3.6. The more concrete case of a pseudocondensing map $F : K \to k(K)$ with convex values which has a fixed point can be found in [6, Theorem 3], where $K = \{x \in S(0,1) : |x(t)| \leq 1/2 \text{ for all } t \in [0,1]\}$ is a subset of the F-space $S(0,1)$ and ψ is the Kuratowski function on K.

We present another fixed point theorem for condensing admissible maps in Fréchet spaces which includes that of Sadovskii [13].

Theorem 3.7. *Let K be a nonempty closed convex set in a Fréchet space E and ψ a measure of noncompactness on K. Then any condensing admissible map $F : K \to k(K)$ has a fixed point.*

Proof. Let $F : K \to k(K)$ be a condensing admissible map. Applying Lemma 3.3 with $c = 1$, there exists a nonempty closed convex subset B of K such that $F(B)$ is a relatively compact subset of B. Hence $C := \overline{\text{co}}F(B)$ is compact, noting that the closed convex hull of a compact set in a Fréchet space is compact. The restriction $G := F|_C : C \to k(C)$ is an admissible map. Theorem 2.4 implies that G has a fixed point and so does F. This completes the proof. □

Corollary 3.8 (Sadovskii [13]). *If K is a nonempty closed, bounded, and convex subset of a Banach space E, and ψ is the Kuratowski measure of noncompactness on E, then every condensing single-valued map $f : K \to K$ has a fixed point.*

Next we introduce a concept of a countably condensing map due to Väth [15] which is more general than that of a condensing map. The fact that we restrict ourselves to countable sets in the definition is important in connection with differential and integral operators.

Definition 3.9. Let K be a closed convex subset of a topological vector space E, Y a nonempty subset of K, and ψ a measure of noncompactness on K. An upper semicontinuous map $F : Y \to k(K)$ is said to be *countably condensing* on Y provided that if X is any countable subset of Y such that $\psi(X) \leq \psi(F(X))$, then X is relatively compact.

The following result of Väth says that the theory of countably condensing maps reduces to that of compact maps; see [15, Corollary 2.1] or [16, Corollary 3.1].

Lemma 3.10. *Let K be a closed convex subset of a Fréchet space E and Y a nonempty closed subset of K. If $F : Y \to k(K)$ is a countably condensing map, then there exists a closed convex set C in K such that $F(C \cap Y)$ is a subset of C and $\overline{\text{co}}F(C \cap Y)$ is compact.*

Finally, we present the following fixed point theorem for countably condensing admissible maps in Fréchet spaces.

Theorem 3.11. *Let K be a nonempty closed convex set in a Fréchet space E and ψ a measure of noncompactness on K. Then any countably condensing admissible map $F : K \to k(K)$ has a fixed point.*

Proof. Let $F : K \to k(K)$ be a countably condensing admissible map. Then by Lemma 3.10, there exists a closed convex subset B of K such that $F(B)$ is a subset of B and $\overline{\text{co}}F(B)$ is a compact subset of K. The map $G := F|_B : B \to k(B)$ is a compact admissible map. Applying Corollary 3.5, G has a fixed point which is also a fixed point of F. This completes the proof. □

Corollary 3.12. *If K is a nonempty closed convex set in a Fréchet space E, then every countably condensing single-valued map $f : K \to K$ has a fixed point.*

Remark 3.13. In addition, if ψ is a regular measure of noncompactness on a closed convex set K in a Fréchet space E, then an upper semicontinuous map $F : K \to k(K)$ is countably condensing if and only if $F(X)$ is relatively compact for any countable subset X of K such that $\psi(X) \leq \psi(F(X))$. In this situation, Theorem 3.7 is a particular form of Theorem 3.11.

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