A FIXED POINT THEOREM FOR ANALYTIC FUNCTIONS

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We prove that each analytic self-map of the open unit disk which interpolates between certain n-tuples must have a fixed point.

1. Introduction

Let $\mathbb{U}$ denote the open unit disk centered at the origin and $\mathbb{T}$ its boundary. For any pair of distinct complex numbers z and w and any positive constant k, we consider the locus of all points ζ in the complex plane $\mathbb{C}$ having the ratio of the distances to w and z equal to k, that is, we consider the solution set of the equation

$$\frac{|\zeta - w|}{|\zeta - z|} = k. \tag{1.1}$$

We denote that set by $A(z,w,k)$ and (following [1]) call it *the Apollonius circle of constant* k associated to the points z and w. The set $A(z,w,k)$ is a circle for all values of k other than 1 when it is a line.

In this paper, we consider $z,w \in \mathbb{U}$, show that if $z \neq w$, then necessarily $A(z,w,\sqrt{(1-|w|^2)/(1-|z|^2)})$ meets the unit circle twice, consider the arc on the unit circle with those endpoints, situated in the same connected component of $\mathbb{C} \setminus A(z,w,\sqrt{(1-|w|^2)/(1-|z|^2)})$ as z, and denote it by $\Gamma_{z,w}$. We prove that if $Z = (z_1,\dots,z_N)$ and $W = (w_1,\dots,w_N)$ are N-tuples with entries in $\mathbb{U}$ such that $z_j \neq w_j$ for all $j = 1,\dots,N$ and

$$\mathbb{T} = \bigcup_{j=1}^{N} \Gamma_{z_j,w_j}, \tag{1.2}$$

then each analytic self-map of $\mathbb{U}$ interpolating between Z and W must have a fixed point. The next section contains the announced fixed point theorem (Theorem 2.2).

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2. The fixed point theorem

For each $e^{i\theta} \in \mathbb{T}$ and $k > 0$, the set

$$\mathrm{HD}(e^{i\theta},k) := \{z \in \mathbb{U} : |e^{i\theta} - z|^2 < k(1 - |z|^2)\} \tag{2.1}$$

called *the horodisk with constant k tangent at $e^{i\theta}$* is an open disk internally tangent to $\mathbb{T}$ at $e^{i\theta}$ whose boundary $\mathrm{HC}(e^{i\theta},k) := \{z \in \mathbb{U} : |e^{i\theta} - z|^2 = k(1 - |z|^2)\}$ is called *the horocycle with constant k tangent at $e^{i\theta}$*.

The center and radius of $\mathrm{HC}(e^{i\theta},k)$ are given by

$$C = \frac{e^{i\theta}}{1+k}, \qquad R = \frac{k}{1+k}, \tag{2.2}$$

respectively. One should note that $\mathrm{HD}(e^{i\theta},k)$ extends to exhaust $\mathbb{U}$ as $k \to \infty$.

Let φ be a self-map of $\mathbb{U}$. For each positive integer n, $\varphi^{[n]} = \varphi \circ \varphi \circ \cdots \circ \varphi$, n times. The following is a combination of results due to Denjoy, Julia, and Wolff.

Theorem 2.1. *Let φ be an analytic self-map of $\mathbb{U}$. If φ has no fixed point, then there is a remarkable point w on the unit circle such that the sequence $\{\varphi^{[n]}\}$ converges to w uniformly on compact subsets of $\mathbb{U}$ and*

$$\varphi(\mathrm{HD}(w,k)) \subseteq \mathrm{HD}(w,k) \quad k > 0. \tag{2.3}$$

The remarkable point w is called the Denjoy-Wolff point of φ. Relation (2.3) is a consequence of a geometric function-theoretic result known as Julia's lemma. In case φ has a fixed point, but is not the identity or an elliptic disk automorphism, one can use Schwarz's lemma in classical complex analysis to show that $\{\varphi^{[n]}\}$ tends to that fixed point, (which is also regarded as a constant function), uniformly on compact subsets of $\mathbb{U}$. These facts show that if φ is not the identity, then it may have at most a fixed point in $\mathbb{U}$. Good accounts on all the results summarized above can be found in [2, Section 2.3] and [4, Sections 4.4–5.3].

In the sequel, φ will always denote an analytic self-map of $\mathbb{U}$ other than the identity. For each $z \in \mathbb{U}$ such that $\varphi(z) \neq z$, we consider the intersection of the unit circle $\mathbb{T}$ and $A(z,\varphi(z),\sqrt{(1 - |\varphi(z)|^2)/(1 - |z|^2)})$. It necessarily consists of two points.

Indeed, it cannot be a singleton. If one assumes that the aforementioned intersection is the singleton $\{e^{i\theta}\}$, then the relation

$$\frac{|e^{i\theta} - \varphi(z)|^2}{1 - |\varphi(z)|^2} = \frac{|e^{i\theta} - z|^2}{1 - |z|^2} \tag{2.4}$$

must be satisfied, and this means that both z and $\varphi(z)$ are on a horocycle tangent to $\mathbb{T}$ at $e^{i\theta}$, which is contradictory due to the fact of, under our assumptions, $A(z,\varphi(z), \sqrt{(1 - |\varphi(z)|^2)/(1 - |z|^2)})$ is also such a horocycle and hence fails to separate z and $\varphi(z)$ (the points z and $\varphi(z)$ should be in different connected components of $\mathbb{C} \setminus A(z,\varphi(z), \sqrt{(1 - |\varphi(z)|^2)/(1 - |z|^2)})$).

On the other hand, $\mathbb{T} \cap A(z, \varphi(z), \sqrt{(1-|\varphi(z)|^2)/(1-|z|^2)})$ cannot be empty. Indeed, for any $z, w \in \mathbb{U}$, $z \neq w$, $A(z, w, \sqrt{(1-|w|^2)/(1-|z|^2)})$ meets $\mathbb{T}$. To prove that, one can assume without loss of generality that $(1-|w|^2)/(1-|z|^2) > 1$. If, arguing by contradiction, we assume that $A(z, w, \sqrt{(1-|w|^2)/(1-|z|^2)}) \cap \mathbb{T} = \emptyset$, then $\mathbb{T}$ must be exterior to $A(z, w, (1-|w|^2)/(1-|z|^2))$, that is,

$$\left|\frac{e^{i\theta}-w}{e^{i\theta}-z}\right|^2 < \frac{1-|w|^2}{1-|z|^2} \quad \text{or, equivalently,} \quad \frac{|e^{i\theta}-w|^2}{1-|w|^2} < \frac{|e^{i\theta}-z|^2}{1-|z|^2} \quad e^{i\theta} \in \mathbb{T}. \tag{2.5}$$

The last inequality implies that, for each $e^{i\theta} \in \mathbb{T}$, w is interior to the horocycle H tangent to $\mathbb{T}$ at $e^{i\theta}$ that passes through z. This leads to a contradiction since there exist horocycles that are exteriorly tangent to each other at z.

Thus $\mathbb{T} \cap A(z, \varphi(z), \sqrt{(1-|\varphi(z)|^2)/(1-|z|^2)})$ necessarily consists of two points. Let $\Gamma_{z,\varphi(z)}$ denote the open arc of $\mathbb{T}$ with those endpoints, situated in the same connected component of $\mathbb{C} \setminus A(z, \varphi(z), \sqrt{(1-|\varphi(z)|^2)/(1-|z|^2)})$ as z.

By straightforward computations, one can obtain the following formulas for the endpoints $e^{i\theta_1}$ and $e^{i\theta_2}$ of $\Gamma_{z,\varphi(z)}$:

$$e^{i\theta_{1,2}} = \frac{-\mu \pm i\sqrt{|\Lambda|^2 - \mu^2}}{\Lambda}, \tag{2.6}$$

where

$$\Lambda = \bar{z}(1 - |\varphi(z)|^2) - \overline{\varphi(z)}(1-|z|^2), \qquad \mu = |\varphi(z)|^2 - |z|^2. \tag{2.7}$$

It is always true that $\Lambda \neq 0$ and $|\Lambda| > |\mu|$, as the reader can readily check.

We are now ready to state and prove the main result of this mathematical note.

Theorem 2.2. *If there exist $z_1, z_2, \ldots, z_N$ such that $\varphi(z_j) \neq z_j$, $j = 1, \ldots, N$, and*

$$\mathbb{T} = \bigcup_{j=1}^{N} \Gamma_{z_j, \varphi(z_j)}, \tag{2.8}$$

then φ has a fixed point in $\mathbb{U}$. In particular, if $z_1, z_2, \ldots, z_N \in \mathbb{C} \setminus \{0\}$ are zeros of φ and

$$\mathbb{T} = \bigcup_{j=1}^{N} \{e^{i\theta} : |\theta - \arg(z_j)| < \arccos|z_j|\}, \tag{2.9}$$

then φ has a fixed point in $\mathbb{U}$. Conversely, if φ is an analytic self-map of $\mathbb{U}$ other than the identity and φ has a fixed point, then there exist finitely many points $z_1, \ldots, z_k$ in $\mathbb{U}$ such that condition (2.8) is satisfied.

Proof. Observe that if $e^{i\theta} \in \Gamma_{z,\varphi(z)}$, then $e^{i\theta}$ cannot be the Denjoy-Wolff point of φ. Indeed, arguing by contradiction, assume $e^{i\theta}$ is the Denjoy-Wolff point of φ. Note that one can consider a horodisk $\mathrm{HD}(e^{i\theta}, k)$ for which z is interior and $\varphi(z)$ exterior, since $|e^{i\theta} - z|^2/(1-|z|^2) < |e^{i\theta} - \varphi(z)|^2/(1-|\varphi(z)|^2)$. This leads to a contradiction by (2.3).

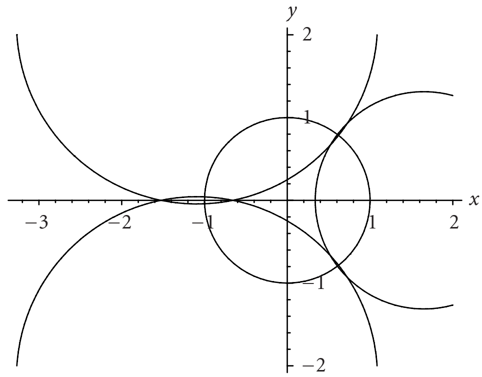

Figure 2.1

Thus if (2.8) holds, then φ does not have a Denjoy-Wolff point, that is, it has a fixed point in $\mathbb{U}$. Finally, observe that if $z \neq 0$ and $\varphi(z) = 0$, a simple computation leads to $\Gamma_{z,\varphi(z)} = \{e^{i\theta} : |\theta - \arg(z)| < \arccos|z|\}$, which takes care of (2.9).

To prove the necessity of condition (2.8) now, assume φ is not the identity and has a fixed point $\omega \in \mathbb{U}$. Let $\rho(z,w) := |z-w|/|1-\overline{w}z|$, $z,w \in \mathbb{U}$, denote the pseudohyperbolic distance on $\mathbb{U}$. For each $z_0 \in \mathbb{U}$ and $r > 0$, let $K(z_0,r) := \{z \in \mathbb{U} : \rho(z,z_0) < r\}$ be the pseudohyperbolic disk of center z_0 and radius r. Pseudohyperbolic disks are also Euclidean disks inside $\mathbb{U}$ (see [3, page 3]), and if $r < 1$, then $K(z_0,r) \neq \mathbb{U}$. By the invariant Schwarz lemma, (see [3, Lemma 1.2]), one has that $\rho(\varphi(z),\omega) \leq \rho(z,\omega)$, $z \in \mathbb{U}$. This means that φ maps closed pseudohyperbolic disks with pseudohyperbolic center ω into themselves. We record this fact for later use and proceed by noting that condition (2.8) is satisfied for some finite set of points in $\mathbb{U}$ if and only if

$$\mathbb{T} = \bigcup_{z \in \mathbb{U} \setminus \{\omega\}} \Gamma_{z,\varphi(z)}, \tag{2.10}$$

which is a direct consequence of the compactness of $\mathbb{T}$. Thus, arguing by contradiction, one should assume that there exists $e^{i\theta} \in \mathbb{T}$ such that, for each $z \neq \omega$, one has that $e^{i\theta} \notin \Gamma_{z,\varphi(z)}$, that is, $|e^{i\theta} - z|^2/(1-|z|^2) > |e^{i\theta} - \varphi(z)|^2/(1-|\varphi(z)|^2)$. One deduces that, for each $z \neq \omega$, $\varphi(z)$ is interior to the horocycle H tangent to $\mathbb{T}$ at $e^{i\theta}$ that passes through z. This generates a contradiction. Indeed, consider some $0 < r < 1$ and the pseudohyperbolic disk $K(\omega,r)$. Let H be the horocycle tangent at $e^{i\theta}$ to $\mathbb{T}$ which is also exteriorly tangent to $\partial K(\omega,r)$. Denote this tangence point by z. Since $\omega \in K(\omega,r)$, $z \neq \omega$. On the other hand, it is impossible that $\varphi(z)$ be simultaneously interior to H and in the closure of $K(\omega,r)$. □

Example 2.3. Any holomorphic self-map of $\mathbb{U}$ interpolating between the triples $(0.34, 0.5i, -0.5i)$ and $(0.335, 0.25+0.125i, 0.25-0.125i)$ has a fixed point in $\mathbb{U}$, because

$$\mathbb{T} = \Gamma_{0.34,0.335} \cup \Gamma_{0.5i,0.25+0.125i} \cup \Gamma_{-0.5i,0.25-0.125i} \tag{2.11}$$

as one can readily check by using relations (2.6) and (2.7) (see also Figure 2.1 which illustrates the equality above). The fact that such holomorphic self-maps exist can be checked by using Pick's interpolation theorem, (see [3, Theorem 2.2]) or (much easier) by noting that $\varphi(z) = (z+1)/4$ is such a map.

References

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