WEAK CONVERGENCE OF AN ITERATIVE SEQUENCE FOR ACCRETIVE OPERATORS IN BANACH SPACES

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Let C be a nonempty closed convex subset of a smooth Banach space E and let A be an accretive operator of C into E. We first introduce the problem of finding a point $u \in C$ such that $\langle Au, J(v-u)\rangle \geq 0$ for all $v \in C$, where J is the duality mapping of E. Next we study a weak convergence theorem for accretive operators in Banach spaces. This theorem extends the result by Gol'shteĭn and Tret'yakov in the Euclidean space to a Banach space. And using our theorem, we consider the problem of finding a fixed point of a strictly pseudocontractive mapping in a Banach space and so on.

1. Introduction

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$, let C be a nonempty closed convex subset of H and let A be a monotone operator of C into H. The *variational inequality* problem is formulated as finding a point $u \in C$ such that

$$(v-u, Au) \geq 0 \tag{1.1}$$

for all $v \in C$. Such a point $u \in C$ is called a solution of the problem. Variational inequalities were initially studied by Stampacchia [13, 17] and ever since have been widely studied. The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C,A)$. In the case when $C = H$, $\mathrm{VI}(H,A) = A^{-1}0$ holds, where $A^{-1}0 = \{u \in H : Au = 0\}$. An element of $A^{-1}0$ is called a zero point of A. An operator A of C into H is said to be *inverse strongly monotone* if there exists a positive real number α such that

$$(x-y, Ax-Ay) \geq \alpha\|Ax-Ay\|^2 \tag{1.2}$$

for all $x, y \in C$; see Browder and Petryshyn [5], Liu and Nashed [18], and Iiduka et al. [11]. For such a case, A is said to be α-inverse strongly monotone. Let T be a nonexpansive mapping of C into itself. It is known that if $A = I - T$, then A is 1/2-inverse strongly

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monotone and $F(T) = \mathrm{VI}(C,A)$, where I is the identity mapping of H and $F(T)$ is the set of fixed points of T; see [11]. In the case of $C = H = \mathbb{R}^N$, for finding a zero point of an inverse strongly monotone operator, Gol'shteĭn and Tret'yakov [8] proved the following theorem.

Theorem 1.1 (see Gol'shteĭn and Tret'yakov [8]). *Let $\mathbb{R}^N$ be the N-dimensional Euclidean space and let A be an α-inverse strongly monotone operator of $\mathbb{R}^N$ into itself with $A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in \mathbb{R}^N$ and*

$$x_{n+1} = x_n - \lambda_n A x_n \tag{1.3}$$

for every $n = 1,2,\ldots$, where $\{\lambda_n\}$ is a sequence in $[0,2\alpha]$. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a,b]$ for some a, b with $0 < a < b < 2\alpha$, then $\{x_n\}$ converges to some element of $A^{-1}0$.

For finding a solution of the variational inequality for an inverse strongly monotone operator, Iiduka et al. [11] proved the following weak convergence theorem.

Theorem 1.2 (see Iiduka et al. [11]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be an α-inverse strongly monotone operator of C into H with $\mathrm{VI}(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = P_C\big(\alpha_n x_n + (1-\alpha_n) P_C(x_n - \lambda_n A x_n)\big) \tag{1.4}$$

for every $n = 1,2,\ldots$, where P_C is the metric projection from H onto C, $\{\alpha_n\}$ is a sequence in $[-1,1]$, and $\{\lambda_n\}$ is a sequence in $[0,2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\alpha_n \in [a,b]$ for some a, b with $-1 < a < b < 1$ and $\lambda_n \in [c,d]$ for some c, d with $0 < c < d < 2(1+a)\alpha$, then $\{x_n\}$ converges weakly to some element of $\mathrm{VI}(C,A)$.

A mapping T of C into itself is said to be *strictly pseudocontractive* [5] if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(I-T)x - (I-T)y\big\|^2 \tag{1.5}$$

for all $x, y \in C$. For such a case, T is said to be k-strictly pseudocontractive. For finding a fixed point of a k-strictly pseudocontractive mapping, Browder and Petryshyn [5] proved the following weak convergence theorem.

Theorem 1.3 (Browder and Petryshyn [5]). *Let K be a nonempty bounded closed convex subset of a real Hilbert space H and let T be a k-strictly pseudocontractive mapping of K into itself. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in K$ and*

$$x_{n+1} = \alpha x_n + (1-\alpha) T x_n \tag{1.6}$$

for every $n = 1,2,\ldots$, where $\alpha \in (k,1)$. Then $\{x_n\}$ converges weakly to some element of $F(T)$.

In this paper, motivated by the above three theorems, we first consider the following generalized variational inequality problem in a Banach space.

Problem 1.4. Let E be a smooth Banach space with norm $\|\cdot\|$, let E^* denote the dual of E, and let $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. Let C be a nonempty closed convex subset of E and let A be an accretive operator of C into E. Find a point $u \in C$ such that

$$\langle Au, J(v-u)\rangle \geq 0, \quad \forall\, v \in C, \tag{1.7}$$

where J is the duality mapping of E into E^*.

This problem is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Kamimura and Takahashi [12]. Second, in order to find a solution of Problem 1.4, we introduce the following iterative scheme for an accretive operator A in a Banach space E: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) Q_C(x_n - \lambda_n A x_n) \tag{1.8}$$

for every $n = 1, 2, \dots$, where Q_C is a sunny nonexpansive retraction from E onto C, $\{\alpha_n\}$ is a sequence in $[0,1]$, and $\{\lambda_n\}$ is a sequence of real numbers. Then we prove a weak convergence (Theorem 3.1) in a Banach space which is generalized simultaneously Gol'shteĭn and Tret'yakov's theorem (Theorem 1.1) and Browder and Petryshyn's theorem (Theorem 1.3).

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual of E. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *uniformly convex* if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x-y\| \geq \varepsilon \text{ implies } \left\|\frac{x+y}{2}\right\| \leq 1-\delta. \tag{2.1}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.2) is attained uniformly for $x, y \in U$. The norm of E is said to be *Fréchet differentiable* if for each $x \in U$, the limit (2.2) is attained uniformly for $y \in U$. And we define a function $\rho : [0,\infty) \to [0,\infty)$ called the *modulus of smoothness* of E as follows:

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E,\ \|x\| = 1,\ \|y\| = \tau\right\}. \tag{2.3}$$

It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. For example, see [1, 23] for more details. We know the following lemma [1, 2].

LEMMA 2.1 [1, 2]. *Let q be a real number with $1 < q \le 2$ and let E be a Banach space. Then E is q-uniformly smooth if and only if there exists a constant $K \ge 1$ such that*

$$\frac{1}{2}\left(\|x+y\|^q + \|x-y\|^q\right) \le \|x\|^q + \|Ky\|^q \tag{2.4}$$

for all $x, y \in E$.

The best constant K in Lemma 2.1 is called the *q-uniformly smoothness constant* of E; see [1]. Let q be a given real number with $q > 1$. The (generalized) *duality mapping J_q* from E into 2^{E^*} is defined by

$$J_q(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^q,\ \|x^*\| = \|x\|^{q-1}\} \tag{2.5}$$

for all $x \in E$. In particular, $J = J_2$ is called the *normalized duality mapping*. It is known that

$$J_q(x) = \|x\|^{q-2}J(x) \tag{2.6}$$

for all $x \in E$. If E is a Hilbert space, then $J = I$. The normalized duality mapping J has the following properties:

(1) if E is smooth, then J is single-valued;
(2) if E is strictly convex, then J is one-to-one and $\langle x-y, x^*-y^*\rangle > 0$ holds for all $(x,x^*),(y,y^*) \in J$ with $x \neq y$;
(3) if E is reflexive, then J is surjective;
(4) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

See [22] for more details. It is also known that

$$q\langle y-x, j_x\rangle \le \|y\|^q - \|x\|^q \tag{2.7}$$

for all $x, y \in E$ and $j_x \in J_q(x)$. Further we know the following result [25]. For the sake of completeness, we give the proof; see also [1, 2].

LEMMA 2.2 [25]. *Let q be a given real number with $1 < q \le 2$ and let E be a q-uniformly smooth Banach space. Then*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q \tag{2.8}$$

for all $x, y \in E$, where J_q is the generalized duality mapping of E and K is the q-uniformly smoothness constant of E.

Proof. Let $x, y \in E$ be given arbitrarily. From (2.7), we have $q\langle y, J_q(x)\rangle \ge \|x\|^q - \|x-y\|^q$. Thus, it follows from Lemma 2.1 that

$$\begin{aligned} q\langle y, J_q(x)\rangle &\ge \|x\|^q - \|x-y\|^q \\ &\ge \|x\|^q - \left(2\|x\|^q + 2\|Ky\|^q - \|x+y\|^q\right) \\ &= -\|x\|^q - 2\|Ky\|^q + \|x+y\|^q. \end{aligned} \tag{2.9}$$

Hence we have $\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q$. □

Let E be a Banach space and let C be a subset of E. Then a mapping T of C into itself is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{2.10}$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A closed convex subset C of a Banach space E is said to have *normal structure* if for each bounded closed convex subset D of C which contains at least two points, there exists an element of D which is not a diametral point of D. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. We know the following theorem [14] related to the existence of fixed points of a nonexpansive mapping.

Theorem 2.3 (Kirk [14]). *Let E be a reflexive Banach space and let D be a nonempty bounded closed convex subset of E which has normal structure. Let T be a nonexpansive mapping of D into itself. Then the set $F(T)$ is nonempty.*

To prove our main result, we also need the following theorem [4].

Theorem 2.4 (see Browder [4]). *Let D be a nonempty bounded closed convex subset of a uniformly convex Banach space E and let T be a nonexpansive mapping of D into itself. If $\{u_j\}$ is a sequence of D such that $u_j \rightharpoonup u_0$ and $\lim_{j\to\infty} \|u_j - Tu_j\| = 0$, then u_0 is a fixed point of T.*

Let D be a subset of C and let Q be a mapping of C into D. Then Q is said to be sunny if

$$Q\big(Qx + t(x - Qx)\big) = Qx \tag{2.11}$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a *retraction* if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D. We know the following two lemmas [15, 20] concerning sunny nonexpansive retractions.

Lemma 2.5 [15]. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Lemma 2.6 (see [20]; see also [6]). *Let C be a nonempty closed convex subset of a smooth Banach space E and let Q_C be a retraction from E onto C. Then the following are equivalent:*

(i) *Q_C is both sunny and nonexpansive;*
(ii) *$\langle x - Q_C x, J(y - Q_C x)\rangle \leq 0$ for all $x \in E$ and $y \in C$.*

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection from E onto C. Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$ and let $x_0 \in C$. Then we have from Lemma 2.6 that $x_0 = Q_C x$ if and only if $\langle x - x_0, J(y - x_0)\rangle \leq 0$ for all $y \in C$, where Q_C is a sunny nonexpansive retraction from E onto C.

Let E be a Banach space and let C be a nonempty closed convex subset of E. An operator A of C into E is said to be *accretive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0 \tag{2.12}$$

for all $x, y \in C$. We can characterize the set of solutions of Problem 1.4 by using sunny nonexpansive retractions.

LEMMA 2.7. *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and let A be an accretive operator of C into E. Then for all $\lambda > 0$,*

$$S(C,A) = F\big(Q_C(I - \lambda A)\big), \tag{2.13}$$

where $S(C,A) = \{u \in C : \langle Au, J(v-u)\rangle \geq 0, \ \forall\ v \in C\}$.

Proof. We have from Lemma 2.6 that $u \in F(Q_C(I-\lambda A))$ if and only if

$$\big\langle (u - \lambda Au) - u, J(y-u)\big\rangle \leq 0 \tag{2.14}$$

for all $y \in C$ and $\lambda > 0$. This inequality is equivalent to the inequality $\langle -\lambda Au, J(y-u)\rangle \leq 0$. Since $\lambda > 0$, we have $u \in S(C,A)$. This completes the proof. □

Now, we define an extension of the inverse strongly monotone operator (1.2) in Banach spaces. Let C be a subset of a smooth Banach space E. For $\alpha > 0$, an operator A of C into E is said to be *α-inverse strongly accretive* if

$$\big\langle Ax - Ay, J(x-y)\big\rangle \geq \alpha \|Ax - Ay\|^2 \tag{2.15}$$

for all $x, y \in C$. Evidently, the definition of the inverse strongly accretive operator is based on that of the inverse strongly monotone operator. It is obvious from (2.15) that

$$\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\| \tag{2.16}$$

for all $x, y \in C$. Let q be a given real number with $q \geq 2$. We also have from (2.6), (2.15), and (2.16) that

$$\begin{aligned}\big\langle Ax - Ay, J_q(x-y)\big\rangle &= \|x-y\|^{q-2}\big\langle Ax - Ay, J(x-y)\big\rangle \\ &\geq \|x-y\|^{q-2}\alpha\|Ax - Ay\|^2 \\ &\geq \big(\alpha\|Ax - Ay\|\big)^{q-2}\alpha\|Ax - Ay\|^2 \\ &= \alpha^{q-1}\|Ax - Ay\|^q\end{aligned} \tag{2.17}$$

for all $x, y \in C$. One should note that no Banach space is q-uniformly smooth for $q > 2$; see [23] for more details. So, in this paper, we study a weak convergence theorem for inverse strongly accretive operators in uniformly convex and 2-uniformly smooth Banach spaces. It is well known that Hilbert spaces and the Lebesgue L^p $(p \geq 2)$ spaces are uniformly convex and 2-uniformly smooth. Let X be a Banach space and let $L^p(X) = L^p(\Omega,\Sigma,\mu;X)$, $1 \le p \le \infty$, be the Lebesgue-Bochner space on an arbitrary measure space (Ω,Σ,μ). Let $1 < q \le 2$ and let $q \le p < \infty$. Then $L^p(X)$ is q-uniformly smooth if and only if X is q-uniformly smooth; see [23]. For convergence theorems in the Lebesgue spaces L^p $(1 < p \le 2)$, see Iiduka and Takahashi [9, 10].

We can know the following property for inverse strongly accretive operators in a 2-uniformly smooth Banach space.

Lemma 2.8. *Let C be a nonempty closed convex subset of a 2-uniformly smooth Banach space E. Let $\alpha > 0$ and let A be an α-inverse strongly accretive operator of C into E. If $0 < \lambda \le \alpha/K^2$, then $I - \lambda A$ is a nonexpansive mapping of C into E, where K is the 2-uniformly smoothness constant of E.*

Proof. We have from Lemma 2.2 that for all $x, y \in C$,

$$\begin{aligned}\left\|(I-\lambda A)x-(I-\lambda A)y\right\|^2 &= \left\|(x-y)-\lambda(Ax-Ay)\right\|^2\\ &\le \|x-y\|^2-2\lambda\langle Ax-Ay,J(x-y)\rangle+2K^2\lambda^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2-2\lambda\alpha\|Ax-Ay\|^2+2K^2\lambda^2\|Ax-Ay\|^2\\ &\le \|x-y\|^2+2\lambda(K^2\lambda-\alpha)\|Ax-Ay\|^2.\end{aligned} \tag{2.18}$$

So, if $0 < \lambda \le \alpha/K^2$, then $I - \lambda A$ is a nonexpansive mapping of C into E. □

Remark 2.9. If $q \ge 2$, we have from (2.17) that for $x, y \in C$,

$$\left\|(I-\lambda A)x-(I-\lambda A)y\right\|^q \le \|x-y\|^q+\lambda\left(2K^q\lambda^{q-1}-q\alpha^{q-1}\right)\|Ax-Ay\|^q. \tag{2.19}$$

Since, for $q > 2$, there exists no Banach space which is q-uniformly smooth, we consider only 2-uniformly smooth Banach spaces. For $1 < q < 2$, the inequalities (2.17) and (2.19) do not hold.

Applying Theorem 2.3, Lemmas 2.7 and 2.8, we have that if D is a nonempty bounded closed convex subset of a uniformly convex and 2-uniformly smooth Banach space E, D is a sunny nonexpansive retract of E and A is an inverse strongly accretive operator of D into E, then the set $S(D,A)$ is nonempty. We know also the following theorem which was proved by Reich [21]; see also Lau and Takahashi [16], Takahashi and Kim [24], and Bruck [7].

Theorem 2.10 (see Reich [21]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space with a Frećhet differentiable norm. Let $\{T_1, T_2, \dots\}$ be a sequence of nonexpansive mappings of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $x \in C$ and $S_n = T_n T_{n-1} \cdots T_1$ for all $n \ge 1$. Then the set*

$$\bigcap_{n=1}^{\infty} \overline{co}\{S_m x : m \ge n\} \cap \bigcap_{n=1}^{\infty} F(T_n) \tag{2.20}$$

consists of at most one point, where $\overline{co}D$ is the closure of the convex hull of D.

3. Weak convergence theorem

In this section, we obtain the following weak convergence theorem for finding a solution of Problem 1.4 for an inverse strongly accretive operator in a uniformly convex and 2-uniformly smooth Banach space.

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$ and let A be an α-inverse strongly accretive operator of C into E with $S(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) Q_C(x_n - \lambda_n A x_n) \tag{3.1}$$

for every $n = 1,2,\ldots$, where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \alpha/K^2]$ for some $a > 0$ and $\alpha_n \in [b,c]$ for some b,c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to some element z of $S(C,A)$, where K is the 2-uniformly smoothness constant of E.

Proof. Put $y_n = Q_C(x_n - \lambda_n A x_n)$ for every $n = 1,2,\ldots$. Let $u \in S(C,A)$. We first prove that $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim_{n\to\infty} \|x_n - y_n\| = 0$. We have from Lemmas 2.7 and 2.8 that

$$\begin{aligned} \|y_n - u\| &= \|Q_C(x_n - \lambda_n A x_n) - Q_C(u - \lambda_n A u)\| \\ &\le \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\| \le \|x_n - u\| \end{aligned} \tag{3.2}$$

for every $n = 1,2,\ldots$. It follows from (3.2) that

$$\begin{aligned} \|x_{n+1} - u\| &= \|\alpha_n (x_n - u) + (1-\alpha_n)(y_n - u)\| \\ &\le \alpha_n \|x_n - u\| + (1-\alpha_n)\|y_n - u\| \\ &\le \alpha_n \|x_n - u\| + (1-\alpha_n)\|x_n - u\| = \|x_n - u\| \end{aligned} \tag{3.3}$$

for every $n = 1,2,\ldots$. Therefore, $\{\|x_n - u\|\}$ is nonincreasing and hence there exists $\lim_{n\to\infty} \|x_n - u\|$. So, $\{x_n\}$ is bounded. We also have from (3.2) and (2.16) that $\{y_n\}$ and $\{Ax_n\}$ are bounded.

Next we will show $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Suppose that $\lim_{n\to\infty} \|x_n - y_n\| \neq 0$. Then there are $\varepsilon > 0$ and a subsequence $\{x_{n_i} - y_{n_i}\}$ of $\{x_n - y_n\}$ such that $\|x_{n_i} - y_{n_i}\| \ge \varepsilon$ for each $i = 1,2,\ldots$. Since E is uniformly convex, the function $\|\cdot\|^2$ is uniformly convex on bounded convex set $B(0, \|x_1 - u\|)$, where $B(0,\|x_1 - u\|) = \{x \in E : \|x\| \le \|x_1 - u\|\}$. So, for ε, there is $\delta > 0$ such that

$$\|x - y\| \ge \varepsilon \text{ implies } \|\lambda x + (1-\lambda) y\|^2 \le \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\delta \tag{3.4}$$

whenever $x, y \in B(0, \|x_1 - u\|)$ and $\lambda \in (0,1)$. Thus, for each $i = 1,2,\ldots$,

$$\begin{aligned} \|x_{n_i+1} - u\|^2 &= \|\alpha_{n_i}(x_{n_i} - u) + (1-\alpha_{n_i})(y_{n_i} - u)\|^2 \\ &\le \alpha_{n_i} \|x_{n_i} - u\|^2 + (1-\alpha_{n_i})\|y_{n_i} - u\|^2 - \alpha_{n_i}(1-\alpha_{n_i})\delta. \end{aligned} \tag{3.5}$$

Therefore, for each $i = 1, 2, \dots$,

$$0 < b(1-c)\delta \le \alpha_{n_i}(1-\alpha_{n_i})\delta \le ||x_{n_i} - u||^2 - ||x_{n_i+1} - u||^2. \tag{3.6}$$

Since the right-hand side of the inequality above converges to 0, we have a contradiction. Hence we conclude that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.7}$$

Since $\{x_n\}$ is bounded, we have that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to z. And since λ_{n_i} is in $[a, \alpha/K^2]$ for some $a > 0$, it holds that $\{\lambda_{n_i}\}$ is bounded. So, there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ which converges to $\lambda_0 \in [a, \alpha/K^2]$. We may assume without loss of generality that $\lambda_{n_i} \to \lambda_0$. We next prove $z \in S(C, A)$. Since Q_C is nonexpansive, it holds from $y_{n_i} = Q_C(x_{n_i} - \lambda_{n_i} A x_{n_i})$ that

$$\begin{aligned} ||Q_C(x_{n_i} - \lambda_0 A x_{n_i}) - x_{n_i}|| &\le ||Q_C(x_{n_i} - \lambda_0 A x_{n_i}) - y_{n_i}|| + ||y_{n_i} - x_{n_i}|| \\ &\le ||(x_{n_i} - \lambda_0 A x_{n_i}) - (x_{n_i} - \lambda_{n_i} A x_{n_i})|| + ||y_{n_i} - x_{n_i}|| \\ &\le M|\lambda_{n_i} - \lambda_0| + ||y_{n_i} - x_{n_i}||, \end{aligned} \tag{3.8}$$

where $M = \sup\{\|Ax_n\| : n = 1, 2, \dots\}$. We obtain from the convergence of $\{\lambda_{n_i}\}$, (3.7), and (3.8) that

$$\lim_{i\to\infty} ||Q_C(I - \lambda_0 A)x_{n_i} - x_{n_i}|| = 0. \tag{3.9}$$

On the other hand, from Lemma 2.8, we have that $Q_C(I - \lambda_0 A)$ is nonexpansive. So, by (3.9), Lemma 2.7, and Theorem 2.4, we obtain $z \in F(Q_C(I - \lambda_0 A)) = S(C, A)$.

Finally, we prove that $\{x_n\}$ converges weakly to some element of $S(C, A)$. We put

$$T_n = \alpha_n I + (1 - \alpha_n) Q_C(I - \lambda_n A) \tag{3.10}$$

for every $n = 1, 2, \dots$. Then we have $x_{n+1} = T_n T_{n-1} \cdots T_1 x$ and $z \in \bigcap_{n=1}^{\infty} \overline{co}\{x_m : m \ge n\}$. We have from Lemma 2.8 that T_n is a nonexpansive mapping of C into itself for every $n = 1, 2, \dots$. And we also have from Lemma 2.7 that $\bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(Q_C(I - \lambda_n A)) = S(C, A)$. Applying Theorem 2.10, we obtain

$$\bigcap_{n=1}^{\infty} \overline{co}\{x_m : m \ge n\} \cap S(C, A) = \{z\}. \tag{3.11}$$

Therefore, the sequence $\{x_n\}$ converges weakly to some element of $S(C, A)$. This completes the proof. □

4. Applications

In this section, we prove some weak convergence theorems in a uniformly convex and 2-uniformly smooth Banach space by using Theorem 3.1. We first study the problem of finding a zero point of an inverse strongly accretive operator. The following theorem is a generalization of Gol'shteĭn and Tret'yakov's theorem (Theorem 1.1).

Theorem 4.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space. Let $\alpha > 0$ and let A be an α-inverse strongly accretive operator of E into itself with $A^{-1}0 \neq \emptyset$, where $A^{-1}0 = \{u \in E : Au = 0\}$. Suppose $x_1 = x \in E$ and $\{x_n\}$ is given by*

$$x_{n+1} = x_n - r_n A x_n \tag{4.1}$$

for every $n = 1,2,\ldots$, where $\{r_n\}$ is a sequence of positive real numbers. If $\{r_n\}$ is chosen so that $r_n \in [s,t]$ for some s,t with $0 < s < t < \alpha/K^2$, then $\{x_n\}$ converges weakly to some element z of $A^{-1}0$, where K is the 2-uniformly smoothness constant of E.

Proof. By assumption, we note that $1 - tK^2/\alpha \in (0,1)$. We define sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ by

$$\alpha_n = 1 - t\frac{K^2}{\alpha}, \qquad \lambda_n = \frac{r_n}{1-\alpha_n} \tag{4.2}$$

for every $n = 1,2,\ldots$, respectively. Then it is easy to check that $\lambda_n \in (0,\alpha/K^2)$ and $S(E, A) = A^{-1}0$. It follows from the definition of $\{x_n\}$ that

$$\begin{aligned} x_{n+1} &= x_n - r_n A x_n = \alpha_n x_n + (1-\alpha_n)\left(x_n - \frac{r_n}{1-\alpha_n} A x_n\right) \\ &= \alpha_n x_n + (1-\alpha_n) I (x_n - \lambda_n A x_n), \end{aligned} \tag{4.3}$$

where I is the identity mapping of E. Obviously, the identity mapping I is a sunny non-expansive retraction from E onto itself. Therefore, by using Theorem 3.1, $\{x_n\}$ converges weakly to some element z of $A^{-1}0$. □

We next study the problem of finding a fixed point of a strictly pseudocontractive mapping. Let $0 \le k < 1$. Let E be a Banach space and let C be a subset of E. Then a mapping T of C into itself is said to be k-strictly pseudocontractive [5, 19] if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \frac{1-k}{2}\|(I-T)x - (I-T)y\|^2 \tag{4.4}$$

for all $x, y \in C$. Then the inequality (4.4) can be written in the form

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \frac{1-k}{2}\|(I-T)x - (I-T)y\|^2. \tag{4.5}$$

If E is a Hilbert space, then the inequality (4.4) (and hence (4.5)) is equivalent to the inequality (1.5). The following theorem is a generalization of Browder and Petryshyn's theorem (Theorem 1.3).

Theorem 4.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset and a sunny nonexpansive retract of E. Let T be a k-strictly pseudocontractive mapping of C into itself with $F(T) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = (1-\beta_n)x_n + \beta_n T x_n \tag{4.6}$$

for every $n = 1,2,\ldots$, where $\{\beta_n\}$ is a sequence in $(0,1)$. If $\{\beta_n\}$ is chosen so that $\beta_n \in [\beta,\gamma]$ for some β, γ with $0 < \beta < \gamma < (1-k)/(2K^2)$, then $\{x_n\}$ converges weakly to some element z of $F(T)$, where K is the 2-uniformly smoothness constant of E.

Proof. By assumption, note that $1 - 2\gamma K^2/(1-k) \in (0,1)$. We define sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ by

$$\alpha_n = 1 - \gamma\frac{2K^2}{1-k}, \qquad \lambda_n = \frac{\beta_n}{1-\alpha_n} \tag{4.7}$$

for every $n = 1,2,\ldots$, respectively. Then we can readily verify that

$$0 < \lambda_n \le \frac{1-k}{2K^2} \le \frac{1}{2} < 1 \tag{4.8}$$

for every $n = 1,2,\ldots$. Put $A = I - T$. We have from (4.5) that A is $(1-k)/2$-inverse strongly accretive. It is easy to show that

$$S(C,A) = S(C,I-T) = F(T) \neq \emptyset. \tag{4.9}$$

Since C is a sunny nonexpansive retract of E and $\lambda_n \in (0,1)$, there exists a sunny nonexpansive retraction Q_C such that $(1-\lambda_n)x_n + \lambda_n Tx_n = Q_C((1-\lambda_n)x_n + \lambda_n Tx_n)$ for every $n = 1,2,\ldots$. It follows from the definition of $\{x_n\}$ that

$$\begin{aligned} x_{n+1} &= (1-\beta_n)x_n + \beta_n Tx_n \\ &= (1-\lambda_n(1-\alpha_n))x_n + \lambda_n(1-\alpha_n)Tx_n \\ &= \alpha_n x_n + (1-\alpha_n)((1-\lambda_n)x_n + \lambda_n Tx_n) \\ &= \alpha_n x_n + (1-\alpha_n)Q_C((1-\lambda_n)x_n + \lambda_n Tx_n) \\ &= \alpha_n x_n + (1-\alpha_n)Q_C(x_n - \lambda_n(I-T)x_n) \\ &= \alpha_n x_n + (1-\alpha_n)Q_C(x_n - \lambda_n Ax_n). \end{aligned} \tag{4.10}$$

Therefore, by using Theorem 3.1, $\{x_n\}$ converges weakly to some element z of $F(T)$. □

Let C be a subset of a smooth Banach space E. Let $\alpha > 0$. An operator A of C into E is said to be α-*strongly accretive* if

$$\langle Ax - Ay, J(x-y)\rangle \ge \alpha\|x-y\|^2 \tag{4.11}$$

for all $x, y \in C$. Let $\beta > 0$. An operator A of C into E is said to be β-*Lipschitz continuous* if

$$\|Ax - Ay\| \le \beta\|x-y\| \tag{4.12}$$

for all $x, y \in C$. Let C be a nonempty closed convex subset of a Hilbert space H. One method of finding a point $u \in \mathrm{VI}(C,A)$ is the *projection algorithm* which starts with any $x_1 = x \in C$ and updates iteratively x_{n+1} according to the formula

$$x_{n+1} = P_C(x_n - \lambda Ax_n) \tag{4.13}$$

for every $n = 1,2,\ldots$, where P_C is the metric projection from H onto C, A is a monotone (accretive) operator of C into H, and λ is a positive real number. It is well known that if A is an α-strongly accretive and β-Lipschitz continuous operator of C into H and $\lambda \in (0, 2\alpha/\beta^2)$, then the operator $P_C(I - \lambda A)$ is a contraction of C into itself. Hence, the Banach contraction principle guarantees that the sequence generated by (4.13) converges strongly to the unique solution of $\mathrm{VI}(C,A)$; see [3]. Motivated by this result, we prove the following weak convergence theorem for strongly accretive and Lipschitz continuous operators.

Theorem 4.3. *Let E be a uniformly convex and 2-uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, let $\alpha > 0$, let $\beta > 0$, and let A be an α-strongly accretive and β-Lipschitz continuous operator of C into E with $S(C,A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) Q_C(x_n - \lambda_n A x_n) \tag{4.14}$$

for every $n = 1,2,\ldots$, where $\{\lambda_n\}$ is a sequence of positive real numbers and $\{\alpha_n\}$ is a sequence in $[0,1]$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \alpha/(K^2\beta^2)]$ for some $a > 0$ and $\alpha_n \in [b,c]$ for some b,c with $0 < b < c < 1$, then $\{x_n\}$ converges weakly to a unique element z of $S(C,A)$, where K is the 2-uniformly smoothness constant of E.

Proof. Since A is an α-strongly accretive and β-Lipschitz continuous operator of C into E, we have

$$\langle Ax - Ay, J(x - y)\rangle \geq \alpha \|x - y\|^2 \geq \frac{\alpha}{\beta^2}\|Ax - Ay\|^2 \tag{4.15}$$

for all $x, y \in C$. Therefore, A is α/β^2-inverse strongly accretive. Since A is strongly accretive and $S(C,A) \neq \emptyset$, the set $S(C,A)$ consists of one point z. Using Theorem 3.1, $\{x_n\}$ converges weakly to a unique element z of $S(C,A)$. □

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