WEAK AND STRONG CONVERGENCE OF FINITE FAMILY WITH ERRORS OF NONEXPANSIVE NONSELF-MAPPINGS

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We are concerned with the study of a multistep iterative scheme with errors involving a finite family of nonexpansive nonself-mappings. We approximate the common fixed points of a finite family of nonexpansive nonself-mappings by weak and strong convergence of the scheme in a uniformly convex Banach space. Our results extend and improve some recent results, Shahzad (2005) and many others.

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1. Introduction

Let K be a subset of a real normed linear space E and let T be a self-mapping on K. T is said to be nonexpansive provided $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$.

Fixed-point iteration process for nonexpansive mappings in Banach spaces including Mann and Ishikawa iteration processes has been studied extensively by many authors to solve the nonlinear operator equations in Hilbert spaces and Banach spaces; see [3, 7, 10, 11, 15, 16]. Tan and Xu [15] introduced and studied a modified Ishikawa process to approximate fixed points of nonexpansive mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space E. Five years later, Xu [18] introduced iterative schemes known as Mann iterative scheme with errors and Ishikawa iterative scheme with errors. Takahashi and Tamura [14] introduced and studied a generalization of Ishikawa iterative schemes for a pair of nonexpansive mappings in Banach spaces. Recently, Khan and Fukhar-ud-din [6] extended their scheme to the modified Ishikawa iterative schemes with errors for two mappings and gave weak and strong convergence theorems. On the other hand, iterative techniques for approximating fixed points of nonexpansive nonself-mappings have been studied by various authors; see [4, 8, 13, 19]. Shahzad [12] introduced and studied an iteration scheme for

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approximating a fixed point of nonexpansive nonself-mappings (when such a fixed point exists) and gave some strong and weak convergence theorems for such mappings.

Inspired and motivated by these facts, we introduce and study a multistep iterative scheme with errors for a finite family of nonexpansive nonself-mappings. Our schemes can be viewed as an extension for two-step iterative schemes of Shahzad [12]. The scheme is defined as follows.

Let K be a nonempty closed convex subset of a uniformly convex Banach space E, which is also a nonexpansive retract of E. And let $T_1, T_2, \ldots, T_N : K \to E$ be nonexpansive mappings, the following iteration scheme is studied:

$$\begin{aligned}
x_n^1 &= P\left(\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1\right),\\
x_n^2 &= P\left(\alpha_n^2 T_2 x_n^1 + \beta_n^2 x_n + \gamma_n^2 u_n^2\right),\\
&\vdots \qquad\qquad \vdots\\
x_{n+1} = x_n^N &= P\left(\alpha_n^N T_N x_n^N - 1 + \beta_n^N x_n + \gamma_n^N u_n^N\right)
\end{aligned} \tag{1.1}$$

with $x_1 \in K$, $n \geqslant 1$, where P is a nonexpansive retraction with respect to K and $\{\alpha_n^1\}$, $\{\alpha_n^2\}, \ldots, \{\alpha_n^N\}$, $\{\beta_n^1\}, \{\beta_n^2\}, \ldots, \{\beta_n^N\}$, $\{\gamma_n^1\}, \{\gamma_n^2\}, \ldots, \{\gamma_n^N\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \ldots, N$, and $\{u_n^1\}, \{u_n^2\}, \ldots, \{u_n^N\}$ are bounded sequences in K.

For $N = 2$, $T_1 = T_2 \equiv T$, $\beta_n = \alpha_n^1$, $\alpha_n = \alpha_n^2$, and $\gamma_n^1 = \gamma_n^2 \equiv 0$, then (1.1) reduces to the scheme for a mapping defined by Shahzad [12]:

$$\begin{aligned}
x_1 &= x \in K,\\
x_{n+1} &= P\left((1-\alpha_n)x_n + \alpha_n TP\left[(1-\beta_n)x_n + \beta_n T x_n\right]\right),
\end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$.

For $N = 2$, $T_1, T_2 : K \to K$, $T_1 = T$, $T_2 = S$, and $y_n = x_n^1$, then (1.1) reduces to the scheme with errors for two mappings defined by

$$\begin{aligned}
x_1 &= x \in K,\\
y_n &= \alpha_n^1 T x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1,\\
x_{n+1} = x_n^2 &= \alpha_n^2 S y_n + \beta_n^2 x_n + \gamma_n^2 u_n^2,
\end{aligned} \tag{1.3}$$

where $\{\alpha_n^1\}$, $\{\alpha_n^2\}$, $\{\beta_n^1\}$, $\{\beta_n^2\}$, $\{\gamma_n^1\}$, $\{\gamma_n^2\}$ are sequences in $[0,1]$ with $\alpha_n^1 + \beta_n^1 + \gamma_n^1 = 1 = \alpha_n^2 + \beta_n^2 + \gamma_n^2$ and $\{u_n^1\}$, $\{u_n^2\}$ are bounded sequences in K.

It is our purpose in this paper to establish several weak and strong convergence theorems of the multistep iterative scheme with errors for a finite family of nonexpansive nonself-mappings. More precisely, we prove weak convergence of these implicit iteration processes in a uniformly convex Banach space which has the Kadec-Klee property. The results presented in this paper extend and improve the corresponding ones announced by Shahzad [12], and many others.

2. Preliminaries

In this section, we recall the well-known concepts and results.

Let E be a real Banach space. A subset K of E is said to be a *retract* of E if there exists a continuous map $P : E \to K$ such that $Px = x$ for all $x \in K$. A map $P : E \to E$ is said to be a *retraction* if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P. A mapping $T : K \to E$ is called *demiclosed* with respect to $y \in E$ if for each sequence $\{x_n\}$ in K and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in K$ and $Tx = y$. A Banach space E is said to satisfy *Opial's condition* [9] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y|| \tag{2.1}$$

for all $y \in E$ with $x \neq y$. A Banach space E is said to have the *Kadec-Klee property* if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$. A family $\{T_i : i = 1,2,\dots,N\}$ of N nonself-mappings of K (i.e., $T_i : K \to E$) with $F = \bigcap_{i=1}^N F(T_i) \neq \varnothing$ is said to satisfy condition (B) on K if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that for all $x \in K$,

$$\max_{1\leqslant i\leqslant N} \{||x - T_i x||\} \geqslant f(d(x,F)). \tag{2.2}$$

The family $\{T_i : i = 1,2,\dots,N\}$ is said to satisfy condition (A^N) if (2.2) is replaced by $1/N \sum_{i=1}^N \|x - T_i x\| \geqslant f(d(x,F))$ for all $x \in K$. Note that condition (B) reduces to condition (A^N) when $\|x - T_1 x\| = \|x - T_2 x\| = \cdots = \|x - T_N x\|$.

A mapping $T : K \to E$ is called *semicompact* if any sequence $\{x_n\}$ in K satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Lemma 2.1 (Tan and Xu [15]). *Let $\{s_n\}$, $\{t_n\}$ be two nonnegative sequences satisfying*

$$s_{n+1} \leqslant s_n + t_n, \quad \forall n \geqslant 1. \tag{2.3}$$

If $\sum_{n=1}^\infty t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists. Moreover, if there exists a subsequence $\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j} \to 0$ as $j \to \infty$, then $s_n \to 0$ as $n \to \infty$.

Lemma 2.2 (Xu [17]). *Let $p > 1$ and $R > 0$ be two fixed numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that $\|\lambda x + (1-\lambda)y\|^p \leqslant \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|)$ for all $x, y \in B_R(0) = \{x \in E : \|x\| \leqslant R\}$, and $\lambda \in [0,1]$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.*

Lemma 2.3 (Kaczor [5]). *Let E be a real reflexive Banach space such that its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $x^*, y^* \in \omega_w(x_n)$; here $\omega_w(x_n)$*

denote the set of all weak subsequential limits of $\{x_n\}$. *Suppose* $\lim_{n\to\infty}\|tx_n+(1-t)x^*-y^*\|$ *exists for all* $t\in[0,1]$. *Then* $x^*=y^*$.

Lemma 2.4 (Browder [1]). *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E, and* $T:K\to E$ *a nonexpansive mapping. Then* $I-T$ *is demiclosed at zero.*

3. Main results

In this section, we prove weak and strong convergence theorems of the iterative scheme given in (1.1) for a finite family of nonexpansive mappings in a Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let* $T_1,T_2,\dots,T_N:K\to E$ *be nonexpansive mappings. Let* $\{x_n\}$ *be the sequence defined by (1.1) with* $\sum_{n=1}^{\infty}\gamma_n^i<\infty$ *for each* $i=1,2,\dots,N$. *If* $\bigcap_{i=1}^N F(T_i)\neq\varnothing$, *then* $\lim_{n\to\infty}\|x_n-x^*\|$ *exists for all* $x^*\in\bigcap_{i=1}^N F(T_i)$.

Proof. For each $n\geqslant 1$, we note that

$$\begin{aligned}||x_n^1-x^*|| &\leqslant \alpha_n^1||T_1x_n-x^*||+\beta_n^1||x_n-x^*||+\gamma_n^1||u_n^1-x^*||\\ &\leqslant \alpha_n^1||x_n-x^*||+\beta_n^1||x_n-x^*||+\gamma_n^1||u_n^1-x^*||\\ &\leqslant ||x_n-x^*||+d_n^0,\end{aligned}\tag{3.1}$$

where $d_n^0=\gamma_n^1\|u_n^1-x^*\|$. Since $\sum_{n=1}^{\infty}\gamma_n^1<\infty$, $\sum_{n=1}^{\infty}d_n^0<\infty$. Next, we note that

$$\begin{aligned}||x_n^2-x^*|| &\leqslant \alpha_n^2||T_2x_n^1-x^*||+\beta_n^2||x_n-x^*||+\gamma_n^2||u_n^2-x^*||\\ &\leqslant \alpha_n^2||x_n^1-x^*||+\beta_n^2||x_n-x^*||+\gamma_n^2||u_n^2-x^*||\\ &= (\alpha_n^2+\beta_n^2)||x_n-x^*||+\alpha_n^2d_n^0+\gamma_n^2||u_n^2-x^*||\\ &\leqslant ||x_n-x^*||+d_n^1,\end{aligned}\tag{3.2}$$

where $d_n^1=\alpha_n^2d_n^0+\gamma_n^2\|u_n^2-x^*\|$. Since $\sum_{n=1}^{\infty}d_n^0<\infty$ and $\sum_{n=1}^{\infty}\gamma_n^2<\infty$, $\sum_{n=1}^{\infty}d_n^1<\infty$. Similarly, we have

$$\begin{aligned}||x_n^3-x^*|| &\leqslant \alpha_n^3||x_n^2-x^*||+\beta_n^3||x_n-x^*||+\gamma_n^3||u_n^3-x^*||\\ &\leqslant \alpha_n^3\left[||x_n-x^*||+d_n^1\right]+\beta_n^3||x_n-x^*||+\gamma_n^3||u_n^3-x^*||\\ &\leqslant ||x_n-x^*||+\alpha_n^3d_n^1+\gamma_n^3||u_n^3-x^*||=||x_n-x^*||+d_n^2,\end{aligned}\tag{3.3}$$

where $d_n^2=\alpha_n^3d_n^1+\gamma_n^3\|u_n^3-x^*\|$, so $\sum_{n=1}^{\infty}d_n^2<\infty$.

By continuing the above method, there exists a nonnegative real sequence $\{d_n^{i-1}\}$ such that $\sum_{n=1}^{\infty} d_n^{i-1} < \infty$ and

$$\|x_n^i - x^*\| \leqslant \|x_n - x^*\| + d_n^{i-1}, \quad \forall n \geqslant 1,\ \forall i = 1,2,\ldots,N. \tag{3.4}$$

Thus $\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \leqslant \|x_n - x^*\| + d_n^{N-1}$ for all $n \in N$. Hence, by Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. This completes the proof. □

Lemma 3.2. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2, \ldots, T_N : K \to E$ be nonexpansive mappings. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1,2,\ldots,N$, for some $\varepsilon \in (0,1)$. If $\bigcap_{i=1}^{N} F(T_i) \neq \varnothing$, then $\lim_{n\to\infty}\|x_n - T_i x_n\| = 0$ for all $i = 1,2,\ldots,N$.*

Proof. Let $x^* \in \bigcap_{i=1}^{N} F(T_i)$. Then, by Lemma 3.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Let $\lim_{n\to\infty}\|x_n - x^*\| = r$. If $r = 0$, then by the continuity of each T_i the conclusion follows. Suppose that $r > 0$. Firstly, we are now to show that $\lim_{n\to\infty}\|T_N x_n - x_n\| = 0$. Since $\{x_n\}$ and $\{u_n^i\}$ are bounded for all $i = 1,2,\ldots,N$, there exists $R > 0$ such that $x_n - x^* + \gamma_n^i(u_n^i - x_n)$, $T_i x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n) \in B_R(0)$ for all $n \geqslant 1$ and for all $i = 1,2,\ldots,N$. Using Lemma 2.2, we have

$$\begin{aligned}
\|x_n^N - x^*\|^2 &\leqslant \|\alpha_n^N T_N x_n^{N-1} + \beta_n^N x_n + \gamma_n^N u_n^N - x^*\|^2 \\
&= \|\alpha_n^N (T_N x_n^{N-1} - x^* + \gamma_n^N(u_n^N - x_n)) + (1-\alpha_n^N)(x_n - x^* + \gamma_n^N(u_n^N - x_n))\|^2 \\
&\leqslant \alpha_n^N \|T_N x_n^{N-1} - x^* + \gamma_n^N(u_n^N - x_n)\|^2 + (1-\alpha_n^N)\|x_n - x^* + \gamma_n^N(u_n^N - x_n)\|^2 \\
&\quad - W_2(\alpha_n^N) g(\|T_N x_n^{N-1} - x_n\|) \\
&\leqslant \alpha_n^N (\|x_n^{N-1} - x^*\| + \gamma_n^N\|u_n^N - x_n\|)^2 + (1-\alpha_n^N)(\|x_n - x^*\| + \gamma_n^N\|u_n^N - x_n\|)^2 \\
&\quad - W_2(\alpha_n^N) g(\|T_N x_n^{N-1} - x_n\|) \\
&\leqslant \alpha_n^N (\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N\|u_n^N - x_n\|)^2 \\
&\quad + (1-\alpha_n^N)(\|x_n - x^*\| + d_n^{N-2} + \gamma_n^N\|u_n^N - x_n\|)^2 \\
&\quad - W_2(\alpha_n^N) g(\|T_N x_n^{N-1} - x_n\|) \\
&= (\|x_n - x^*\| + \lambda_n^{N-2})^2 - W_2(\alpha_n^N) g(\|T_N x_n^{N-1} - x_n\|),
\end{aligned} \tag{3.5}$$

where $\lambda_n^{N-2} := d_n^{N-2} + \gamma_n^N\|u_n^N - x^*\|$. Observe that $\varepsilon^3 \leqslant W_2(\alpha_n^N)$ now (3.5) implies that $\varepsilon^3 g(\|T_N x_n^{N-1} - x_n\|) \leqslant \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{N-2}$, where $\rho_n^{N-2} := 2\lambda_n^{N-2}\|x_n - x^*\|^2 + (\lambda_n^{N-2})^2$. Since $\sum_{n=1}^{\infty} d_n^{N-2} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^{N-2} < \infty$, we get $\sum_{n=1}^{\infty} \rho_n^{N-2} < \infty$. This implies that $\lim_{n\to\infty} g(\|T_N x_n^{N-1} - x_n\|) = 0$. Since g is strictly increasing and continuous at 0, it follows that $\lim_{n\to\infty}\|T_N x_n^{N-1} - x_n\| = 0$. Note that

$$\begin{aligned}||x_n - x^*|| &\leqslant ||x_n - T_N x_n^{N-1}|| + ||T_N x_n^{N-1} - x^*|| \\ &\leqslant ||x_n - T_N x_n^{N-1}|| + ||x_n^{N-1} - x^*||,\end{aligned} \tag{3.6}$$

for all $n \geqslant 1$. Thus $r = \lim_{n\to\infty}\|x_n - x^*\| \leqslant \liminf_{n\to\infty}\|x_n^{N-1} - x^*\| \leqslant \limsup_{n\to\infty}\|x_n^{N-1} - x^*\| \leqslant r$ and therefore $\lim_{n\to\infty}\|x_n^{N-1} - x^*\| = r$. Using the same argument in the proof above, we have

$$\begin{aligned}||x_n^{N-1} - x^*||^2 &\leqslant \alpha_n^{N-1}\left(||x_n^{N-2} - x^*|| + \gamma_n^{N-1}||u_n^{N-1} - x^*||\right)^2 \\ &\quad + \left(1 - \alpha_n^{N-1}\right)\left(||x_n - x^*|| + \gamma_n^{N-1}||u_n^{N-1} - x^*||\right)^2 \\ &\quad - W_2\left(\alpha_n^{N-1}\right)g\left(||T_{N-1}x_n^{N-2} - x_n||\right) \\ &\leqslant \alpha_n^{N-1}\left(||x_n - x^*|| + d_n^{N-3} + \gamma_n^{N-1}||u_n^{N-1} - x^*||\right)^2 \\ &\quad + \left(1 - \alpha_n^{N-1}\right)\left(||x_n - x^*|| + d_n^{N-3} + \gamma_n^{N-1}||u_n^{N-1} - x^*||\right)^2 \\ &\quad - W_2\left(\alpha_n^{N-1}\right)g\left(||T_{N-1}x_n^{N-2} - x_n||\right) \\ &\leqslant ||x_n - x^*||^2 + \rho_n^{N-3} - W_2\left(\alpha_n^{N-1}\right)g\left(||T_{N-1}x_n^{N-2} - x_n||\right).\end{aligned} \tag{3.7}$$

This implies that $\varepsilon^3 g(\|T_{N-1}x_n^{N-2} - x_n\|) \leqslant \|x_n - x^*\|^2 - \|x_n^{N-1} - x^*\|^2 + \rho_n^{N-3}$ and therefore $\lim_{n\to\infty}\|T_{N-1}x_n^{N-2} - x_n\| = 0$. Thus, we have

$$\begin{aligned}||x_n - T_N x_n|| &\leqslant ||x_n - T_N x_n^{N-1}|| + ||T_N x_n^{N-1} - T_N x_n|| \\ &\leqslant ||x_n - T_N x_n^{N-1}|| + ||x_n^{N-1} - x_n|| \\ &= ||x_n - T_N x_n^{N-1}|| + ||P(\alpha_n^{N-1}T_{N-1}x_n^{N-2} + \beta_n^{N-1}x_n + \gamma_n^{N-1}u_n^{N-1}) - Px_n|| \\ &\leqslant ||x_n - T_N x_n^{N-1}|| + \alpha_n^{N-1}||x_n - T_{N-1}x_n^{N-2}|| + \gamma_n^{N-1}||u_n^{N-1} - x_n||.\end{aligned} \tag{3.8}$$

Since $\lim_{n\to\infty}\|x_n - T_N x_n^{N-1}\| = 0$, $\lim_{n\to\infty}\|x_n - T_{N-1}x_n^{N-2}\| = 0$, and $\sum_{n=1}^{\infty}\gamma_n^{N-1} < \infty$, it follows that $\lim_{n\to\infty}\|x_n - T_N x_n\| = 0$. Similarly, by using the same argument as in the proof above, we have $\lim_{n\to\infty}\|x_n - T_{N-2}x_n^{N-3}\| = \lim_{n\to\infty}\|x_n - T_{N-3}x_n^{N-4}\| =,\dots,= \lim_{n\to\infty}\|x_n - T_2 x_n^1\| = 0$. This implies that $\lim_{n\to\infty}\|x_n - T_{N-1}x_n\| = \lim_{n\to\infty}\|x_n - T_{N-2}x_n\| =,\dots,= \lim_{n\to\infty}\|x_n - T_3 x_n\| = 0$. It remains to show that

$$\lim_{n\to\infty}||x_n - T_1 x_n|| = 0, \qquad \lim_{n\to\infty}||x_n - T_2 x_n|| = 0. \tag{3.9}$$

Note that

$$\begin{aligned}||x_n^1 - x^*||^2 &\leqslant \alpha_n^1 (||x_n - x^*|| + \gamma_n^1 ||u_n^1 - x^*||)^2 \\ &\quad + (1-\alpha_n^1)(||x_n - x^*|| + \gamma_n^1 ||u_n^1 - x^*||)^2 - W_2(\alpha_n^1) g(||T_1 x_n - x_n||) \\ &= (||x_n - x^*|| + \gamma_n^1 ||u_n^1 - x^*||)^2 - W_2(\alpha_n^1) g(||T_1 x_n - x_n||).\end{aligned} \tag{3.10}$$

Thus, we have $\varepsilon^3 g(\|T_1 x_n - x_n\|) \leqslant (\|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\|)^2 - \|x_n^1 - x^*\|^2$ and therefore $\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0$. Since $\|x_n - T_2 x_n\| \leqslant \|x_n - T_2 x_n^1\| + \alpha_n^1 \|T_1 x_n - x_n\| + \gamma_n^1 \|u_n^1 - x_n\|$, it implies that $\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0$. Therefore $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, N$. □

Theorem 3.3. *Let E be a uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2, \ldots, T_N : K \to E$ be nonexpansive mappings which are satisfying condition (B). Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ for some $\varepsilon \in (0,1)$. If $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. By Lemma 3.2, $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, N$. Now by condition (B), $f(d(x_n, F)) \leqslant M_n := \max_{1\leqslant i\leqslant N} \{\|T_i x_n - x_n\|\}$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$.

Now we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{y_j\} \in F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. By the following method of the proof of Tan and Xu [15], we get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let $y_j \to y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \to y$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$, $x_n \to y \in F$. □

For $N = 2$, $T_1 = T_2 \equiv T$, $\beta_n = \alpha_n^1$, $\alpha_n = \alpha_n^2$, and $\gamma_n^1 = \gamma_n^2 \equiv 0$ in Theorem 3.3, we obtain the following results.

Corollary 3.4 (see [12, Theorem 3.6]). *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T : K \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.2). Suppose T satisfies condition (A^1). Then $\{x_n\}$ converges strongly to some fixed point of T.*

When $N = 2$, $S = T_1$, $T = T_2 : C \to C$, and $y_n = x_n^1$ in Theorem 3.3, we obtain strong convergence theorem as follows.

Corollary 3.5. *Let E be a uniformly convex Banach space and let C be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let S, T be nonexpansive mappings of C into itself satisfying condition (A^2), and let $\{x_n\}$ be sequence defined by (1.3) with $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$ and $0 < \delta \leqslant \alpha_n^1, \alpha_n^2 \leqslant 1 - \delta < 1$ for all $n \in \mathbb{N}$. If $F := F(S) \cap F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.*

Theorem 3.6. *Let E be a uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2, \dots, T_N : K \to E$ be nonexpansive mappings. Suppose that one of the mappings in $\{T_i : i = 1,2,\dots,N\}$ is semicompact. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1,2,\dots,N$ for some $\varepsilon \in (0,1)$. If $F := \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$, then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. Suppose that T_{i_0} is *semicompact* for some $i_0 = 1,2,\dots,N$. By Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - T_{i_0}x_n\| = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in K$ as $j \to \infty$. Now Lemma 3.2 guarantees that $\lim_{j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0$ for all $l = 1,2,\dots,N$ and so $\|x^* - T_l x^*\| = 0$ for all $l = 1,2,\dots,N$. This implies that $x^* \in F$. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and then $\lim_{n\to\infty} \|x_n - x^*\| = \lim_{j\to\infty} \|x_{n_j} - x^*\| = 0$. This completes the proof. □

Theorem 3.7. *Let E be a uniformly convex Banach space satisfying the Opial's condition and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2, \dots, T_N : K \to E$ be nonexpansive mappings and let $\{x_n\}$ be a sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1,2,\dots,N$ for some $\varepsilon \in (0,1)$. If $F := \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$, then $\{x_n\}$ converges weakly to a common fixed point in F.*

Proof. Let $x^* \in F$. Then as proved in Lemma 3.1, $\lim_{x\to\infty} \|x_n - x^*\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in F. To prove this, let $x_{n_i} \rightharpoonup z_1$ and $x_{n_j} \rightharpoonup z_2$ for some subsequences $\{x_{n_i}\}$, $\{x_{n_j}\}$ of $\{x_n\}$. By Lemma 3.2,

$$\lim_{i\to\infty} ||x_{n_i} - T_k x_{n_i}|| = 0 = \lim_{j\to\infty} ||x_{n_j} - T_k x_{n_j}|| \tag{3.11}$$

for all $k = 1,2,\dots,N$ and by Lemma 2.4 insures that $I - T_k$ are demiclosed at zero for all $k = 1,2,\dots,N$. Therefore we obtain $T_k z_1 = z_1$ and $T_k z_2 = z_2$ for all $k = 1,2,\dots,N$. Then $z_1, z_2 \in F$. Next, we prove the uniqueness. Suppose that $z_1 \neq z_2$, then by the Opial's condition $\lim_{n\to\infty} \|x_n - z_1\| = \lim_{i\to\infty} \|x_{n_i} - z_1\| < \lim_{i\to\infty} \|x_{n_i} - z_2\| = \lim_{j\to\infty} \|x_{n_j} - z_2\| < \lim_{j\to\infty} \|x_{n_j} - z_1\| = \lim_{n\to\infty} \|x_n - z_1\|$. This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in F. □

Lemma 3.8. *Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2, \dots, T_N : K \to E$ be nonexpansive mappings. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.1) with for each $i = 1,2,\dots,N$, $\sum_{n=1}^{\infty} \gamma_n^i < \infty$. If $F := \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$, then for all $u, v \in F$, the limit*

$$\lim_{n\to\infty} ||tx_n + (1-t)u - v|| \tag{3.12}$$

exists for all $t \in [0,1]$.

Proof. By Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$. This implies that $\{x_n\}$ is bounded. Observe that there exists $R > 0$ such that $\{x_n\} \subset C := B_R(0) \cap K$, and hence C is a nonempty closed convex bounded subset of E. Let $a_n(t) := \|tx_n + (1-t)u - v\|$. Then $\lim_{n\to\infty} a_n(0) = \|u - v\|$, and from Lemma 3.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - v\|$ exists. Without loss of generality, we may assume that $\lim_{n\to\infty} \|x_n - u\| = r > 0$ and

$t \in (0,1)$. For any $n \geqslant 1$ and for all $i = 1,2,\dots,N$, we define $A_n^i : C \to C$ by

$$\begin{aligned} A_n^1 &:= P(\alpha_n^1 T_1 + \beta_n^1 I + \gamma_n^1 u_n^1), \\ A_n^2 &:= P(\alpha_n^2 T_2 A_n^1 + \beta_n^2 I + \gamma_n^2 u_n^2), \\ &\vdots \qquad \vdots \qquad \vdots \\ A_n^N &:= P(\alpha_n^N T_N A_n^{N-1} + \beta_n^N I + \gamma_n^N u_n^N). \end{aligned} \tag{3.13}$$

Thus, for all $x, y \in K$, we have $\|A_n^i x - A_n^i y\| \leqslant \alpha_n^i \|A_n^{i-1} x - A_n^{i-1} y\| + \beta_n^i \|x - y\|$ for all $i = 2,\dots,N$, and $\|A_n^1 x - A_n^1 y\| \leqslant \alpha_n^1 \|x - y\| + \beta_n^1 \|x - y\|$. This implies that

$$||A_n^N x - A_n^N y|| \leqslant \|x - y\|. \tag{3.14}$$

Set $S_{n,m} := A_{n+m-1}^N A_{n+m-2}^N \cdots A_n^N$, $m \geqslant 1$, and $b_{n,m} := \|S_{n,m}(tx_n + (1-t)u) - (tS_{n,m}x_n + (1-t)S_{n,m}u)\|$. It easy to see that $A_n^N x_n = x_{n+1}$, $S_{n,m}x_n = x_{n+m}$, and $\|S_{n,m}x - S_{n,m}y\| \leqslant \|x - y\|$.

We show first that, for any $x^* \in F$, $\|S_{n,m}x^* - x^*\| \to 0$ uniformly for all $m \geqslant 1$ as $n \to \infty$. Indeed, for any $x^* \in F$, we have

$$||A_n^i x^* - x^*|| \leqslant \alpha_n^i ||A_n^{i-1} x^* - x^*|| + \gamma_n^i ||u_n^i - x^*|| \tag{3.15}$$

for all $i = 2,\dots,N$, and $\|A_n^1 x^* - x^*\| \leqslant \gamma_n^1 \|u_n^1 - x^*\|$. Therefore

$$\begin{aligned} ||A_n^N x^* - x^*|| &\leqslant \sigma_n^2 \gamma_n^1 ||u_n^1 - x^*|| + \sigma_n^3 \gamma_n^2 ||u_n^2 - x^*|| + \cdots + \sigma_n^N \gamma_n^{N-1} ||u_n^{N-1} - x^*|| \\ &\quad + \gamma_n^N ||u_n^N - x^*|| \leqslant M \sum_{i=1}^N \gamma_n^i, \end{aligned} \tag{3.16}$$

for all $n \geqslant 1$, where $M = \max\{\sup_{n\geqslant 1}\{\|u_n^1 - x^*\|\},\dots,\sup_{n\geqslant 1}\{\|u_n^N - x^*\|\}\}$ and $\sigma_n^k = \prod_{i=k}^N \alpha_n^i$. Hence

$$\begin{aligned} ||S_{n,m}x^* - x^*|| &\leqslant ||A_{n+m-1}^N A_{n+m-2}^N \cdots A_n^N x^* - A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+1}^N x^*|| \\ &\quad + ||A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+1}^N x^* - A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+2}^N x^*|| \\ &\quad \vdots \qquad \vdots \\ &\quad + ||A_{n+m-1}^N x^* - x^*|| \\ &\leqslant ||A_n^N x^* - x^*|| + ||A_{n+1}^N x^* - x^*|| + \cdots + ||A_{n+m-1}^N x^* - x^*|| \\ &\leqslant M \sum_{i=1}^N (\gamma_n^i + \gamma_{n+1}^i + \cdots + \gamma_{n+m-1}^i) \\ &\leqslant M \sum_{i=1}^N \sum_{k=n}^\infty \gamma_k^i := \delta_n^{x^*}. \end{aligned} \tag{3.17}$$

Since $\sum_{n=1}^{\infty}\gamma_n^i<\infty$, for all $i=1,2,\dots,N$, we have $\delta_n^{x^*}\to 0$ as $n\to\infty$ and hence $\|S_{n,m}x^*-x^*\|\to 0$ as $n\to\infty$. Observe that

$$\begin{aligned}
a_{n+m}(t) &= \|tS_{n,m}x_n+(1-t)u-v\| \\
&\leqslant \|tS_{n,m}x_n+(1-t)u-S_{n,m}(tx_n+(1-t)u)\| \\
&\quad +\|S_{n,m}(tx_n+(1-t)u)-v\| \\
&= \|tS_{n,m}x_n+(1-t)S_{n,m}u-S_{n,m}(tx_n+(1-t)u)+(1-t)(u-S_{n,m}u)\| \\
&\quad +\|S_{n,m}(tx_n+(1-t)u)-v\| \\
&\leqslant b_{n,m}+\|S_{n,m}(tx_n+(1-t)u)-v\|+(1-t)\|u-S_{n,m}u\| \\
&\leqslant b_{n,m}+\|S_{n,m}(tx_n+(1-t)u)-S_{n,m}v\|+\|S_{n,m}v-v\| \\
&\quad +(1-t)\|u-S_{n,m}u\| \\
&\leqslant b_{n,m}+a_n(t)+\|S_{n,m}v-v\|+(1-t)\|u-S_{n,m}u\| \\
&\leqslant b_{n,m}+a_n(t)+\delta_n^v+(1-t)\delta_n^u.
\end{aligned} \tag{3.18}$$

By using [2, Theorem 2.3], we have

$$\begin{aligned}
b_{n,m} &\leqslant \varphi^{-1}(\|x_n-u\|-\|S_{n,m}x_n-S_{n,m}u\|) \\
&= \varphi^{-1}(\|x_n-u\|-\|x_{n+m}-u+u-S_{n,m}u\|) \\
&\leqslant \varphi^{-1}(\|x_n-u\|-(\|x_{n+m}-u\|-\|S_{n,m}u-u\|)),
\end{aligned} \tag{3.19}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0 as $n\to\infty$ for all $m\geqslant 1$. Thus, fixing n and letting $m\to\infty$ in (3.19), we have

$$\limsup_{m\to\infty} a_{n+m}(t)\leqslant \varphi^{-1}\Big(\|x_n-u\|-\Big(\lim_{m\to\infty}\|x_m-u\|-\delta_n^u\Big)\Big)+a_n(t)+\delta_n^v+(1-t)\delta_n^u \tag{3.20}$$

and again letting $n\to\infty$,

$$\limsup_{n\to\infty} a_n(t)\leqslant \varphi^{-1}(0)+\liminf_{n\to\infty} a_n(t)+0+0=\liminf_{n\to\infty} a_n(t). \tag{3.21}$$

This completes the proof. □

Theorem 3.9. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kaded-Klee property and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1,T_2,\dots,T_N : K\to E$ be nonexpansive mappings with $F:=\bigcap_{i=1}^N F(T_i)\neq\varnothing$. From arbitrary $x_1\in K$, define the sequence $\{x_n\}$ by the recursion (1.1) with for each $i=1,2,\dots,N$, $\sum_{n=1}^{\infty}\gamma_n^i<\infty$ and $\alpha_n^i\in[\varepsilon,1-\varepsilon]$ for some $\varepsilon\in(0,1)$. Then $\{x_n\}$ converges weakly to some fixed point of T.*

Proof. Lemma 3.1 guarantees that $\{x_n\}$ is bounded. Since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to some $x^*\in K$. By Lemma 3.2, we have

$\lim_{j\to\infty}\|x_{n_j} - T_i x_{n_j}\| = 0$ for all $i = 1,2,\dots,N$. Now Lemma 2.4 guarantees that $I - T_i$ is demiclosed at zero for all $i = 1,2,\dots,N$. This implies that $T_i x^* = x^*$ for all $i = 1,2,\dots,N$, hence this means that $x^* \in F$. It remains to show that $\{x_n\}$ converges weakly to x^*. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converging weakly to some y^*. Then $y^* \in K$ and so $x^*, y^* \in \omega_w(x_n) \cap F$. By Lemma 3.8, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)x^* - y^*\| \tag{3.22}$$

exists for all $t \in [0,1]$. By Lemma 2.3 we have $x^* = y^*$. As a result, $\omega_w(x_n) \cap F$ is a singleton, and so $\{x_n\}$ converges weakly to some fixed point of T. □

Corollary 3.10 (see [12, Theorem 3.5]). *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T : K \to E$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. From arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion (1.2). Then $\{x_n\}$ converges weakly to some fixed point of T.*

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