ON ALMOST COINCIDENCE POINTS IN GENERALIZED CONVEX SPACES

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We prove an almost coincidence point theorem in generalized convex spaces. As an application, we derive a result on the existence of a maximal element and an almost coincidence point theorem in hyperconvex spaces. The results of this paper generalize some known results in the literature.

1. Introduction and preliminaries

The notion of a generalized convex space we work with in this paper was introduced by Park and Kim in [10]. In generalized convex spaces, many results on fixed points, coincidence points, equilibrium problems, variational inequalities, continuous selections, saddle points, and others have been obtained, see, for example, [6, 8, 10–13].

In this paper, we obtain an almost coincidence point theorem in generalized convex spaces. Some applications to the existence of a maximal element of an almost fixed point theorem in hyperconvex spaces are given.

A multimap or map $F : X \multimap Y$ is a function from a set X into the power set of a set Y. For $A \subset X$, let $F(A) = \bigcup\{Fx : x \in A\}$. For any $B \subset Y$, the lower inverse and upper inverse of B under F are defined by

$$\begin{aligned} F^-(B) &= \{x \in X : Fx \cap B \neq \varnothing\}, \\ F^+(B) &= \{x \in X : Fx \subset B\}, \end{aligned} \tag{1.1}$$

respectively. The lower inverse of $F : X \multimap Y$ is the map $F^- : Y \multimap X$ defined by $x \in F^- y$ if and only if $y \in Fx$.

A map $F : X \multimap Y$ is upper (lower) semicontinuous on X if and only if for every open $V \subset Y$, the set $F^+(V)$ $(F^-(V))$ is open. A map $F : X \multimap Y$ is continuous if and only if it is upper and lower semicontinuous.

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For a nonempty subset D of X, let $\langle D\rangle$ denote the set of all nonempty finite subsets of D. Let Δ_n denote the standard n-simplex with vertices $e_1,e_2,\ldots,e_{n+1}$, where e_i is the ith unit vector in $\mathbb{R}^{n+1}$.

A generalized convex space or G-convex space $(X,D;\Gamma)$ consists of a topological space X, a nonempty set D, and a function $\Gamma:\langle D\rangle \multimap X$ with nonempty values such that for each $A\in\langle D\rangle$ with $|A|=n+1$, there exists a continuous function $\varphi_A:\Delta_n\to\Gamma(A)$, such that $\varphi_A(\Delta_J)\subset\Gamma(J)$, where Δ_J denote the faces of Δ_n corresponding to $J\in\langle A\rangle$.

Particular forms of G-convex space are convex subsets of a topological vector space, Lassonde's convex space, a metric space with Michael's convex structure, S-contractible space, H-space, Komiya's convex space, Bielawski's simplicial convexity, Joó's pseudoconvex space, see, for example, [11–13].

For each $A\in\langle D\rangle$, we may write $\Gamma(A)=\Gamma_A$. Note that Γ_A does not need to contain A. For $(X,D;\Gamma)$, a subset C of X is said to be G-convex if for each $A\in\langle D\rangle$, $A\subset C$ implies $\Gamma_A\subset C$. If $D=X$, then $(X,D;\Gamma)$ will be denoted by (X,Γ). The G-convex hull of K, denoted by $G-\operatorname{co}(K)$, is the set

$$\bigcap\{B\subset X : B \text{ is a } G\text{-convex subset of } X \text{ containing } K\}. \tag{1.2}$$

Let C be a G-convex subset of X, a map $F:C\multimap X$ is called G-quasiconvex if

$$F(d)\cap S\neq\varnothing \quad \text{for each } d\in D \Longrightarrow F(u)\cap S\neq\varnothing \quad \text{for each } u\in\Gamma_D, \tag{1.3}$$

for each $D\in\langle C\rangle$, and for each G-convex subset S of X. If X is a topological vector space and $\Gamma_A=\operatorname{co}A$, we obtain the class of quasiconvex maps, see, for example, [7, page 18].

Let C be a subset of X, a map $F:C\multimap X$ is called G-KKM map if $\Gamma_A\subset F(A)$ for each $A\in\langle C\rangle$.

The following version of G-KKM-type theorem, see, for example, [13, page 49], will be used to prove the main result of this paper.

Theorem 1.1. *Let (X,Γ) be a G-convex space, K a nonempty subset of X, and $H:K\multimap X$ a map with closed (open) values and G-KKM map. Then $\bigcap_{x\in D}H(x)\neq\varnothing$ for each $D\in\langle K\rangle$.*

2. Almost-like coincidence point theorem

Theorem 2.1. *Let (X,Γ) be a G-convex space, K a nonempty subset of X, U a nonempty closed (open) G-convex subset of X, and $\mu:K\times K\multimap X$ a map such that*

(1) *for each fixed $y\in K$, the map $x\mapsto\mu(x,y)$ is upper (lower) semicontinuous map,*

(2) *for each fixed $x\in K$, the map $y\mapsto\mu(x,y)$ is G-quasiconvex map,*

(3) *there exists a set $D\in\langle K\rangle$ such that $\Gamma_D\subseteq K$ and $\mu(x,D)\cap U\neq\varnothing$ for each $x\in K$.*

Then there exists $x_U\in K$ such that

$$\mu(x_U,x_U)\cap U\neq\varnothing. \tag{2.1}$$

Proof. Let for every $y\in K$, $H:K\multimap K$ be defined by

$$H(y)=\{x\in K:\mu(x,y)\cap U=\varnothing\}. \tag{2.2}$$

From assumption (1), we obtain that $H(y)$ is closed (open) set for each $y \in K$. We can prove that H is not a G-KKM map. Namely,

$$\bigcap_{y\in D} H(y) = \{x \in K : \mu(x,D) \cap U = \varnothing\}, \tag{2.3}$$

and from assumption (3), we obtain that

$$\bigcap_{y\in D} H(y) = \varnothing. \tag{2.4}$$

So, by Theorem 1.1, $H : K \multimap K$ is not a G-KKM map. This implies that there exists $A \in \langle D\rangle$ such that

$$\Gamma_A \not\subseteq H(A), \tag{2.5}$$

and hence there is an $x_U \in \Gamma_A$ such that $x_U \notin H(A)$. This implies that

$$\mu(x_U,a) \cap U \neq \varnothing \quad \text{for each } a \in A. \tag{2.6}$$

From assumption (2), we obtain

$$\mu(x_U,x_U) \cap U \neq \varnothing. \tag{2.7}$$

□

From Theorem 2.1, we have the following almost coincidence point theorem for topological vector space.

Theorem 2.2. *Let X be a topological vector space, K a nonempty subset of X, U a nonempty open (closed) convex neighborhood of 0 in X, and $F_1 : K \multimap X$, $F_2 : K \multimap X$ ($F_2 : K \to X$) are maps such that*

(1) *the map F_1 is lower (upper) semicontinuous map with convex values,*
(2) *the map F_2 is quasiconvex,*
(3) *there exists a set $D \in \langle K\rangle$ such that $\operatorname{co} D \subseteq K$ and $F_1(x) \cap (F_2(D)+U) \neq \varnothing$ for each $x \in K$.*

Then there exists $x_U \in K$ such that

$$F_1(x_U) \cap (F_2(x_U)+U) \neq \varnothing. \tag{2.8}$$

Proof. Taking $\mu(x,y) = F_1(x) - F_2(y)$ and $\Gamma_A = \operatorname{co} A$ in Theorem 2.1, we get the proof. □

As an application of Theorem 2.2, we obtain the following result of existence of almost fixed point of Park [9, Theorem 2.1].

Corollary 2.3. *Let X be a topological vector space, K a nonempty subset of X, U a nonempty open (closed) convex neighborhood of 0 in X, and $F : K \multimap X$ a lower (upper) semicontinuous map with convex values such that there exists a set $D \in \langle K\rangle$ such that $\operatorname{co} D \subseteq K$ and $F(x) \cap (D+U) \neq \varnothing$ for each $x \in K$. Then there exists $x_U \in K$ such that*

$$F(x_U) \cap (x_U+U) \neq \varnothing. \tag{2.9}$$

Remark 2.4. The assumption

$$F(x) \cap (D+U) \neq \emptyset, \quad \text{for each } x \in K, \tag{2.10}$$

in Corollary 2.3 can be replaced by the following condition:

$$F(X) \subseteq D+U. \tag{2.11}$$

In this case, we obtain the result of Kim and Park [4, Theorem 1.2].

3. Almost coincidence point theorem in metrizable G-convex spaces

Let (X,Γ) be a metrizable G-convex space with metric d. For any nonnegative real number r and any subset A of X, we define

$$B(A,r) = \bigcup \{B(a,r) : a \in A\}, \tag{3.1}$$

where $B(a,r) = \{x \in X : d(a,x) < r\}$.

Similarly, we define

$$B[A,r] = \bigcup \{B[a,r] : a \in A\}, \tag{3.2}$$

where $B[a,r] = \{x \in X : d(a,x) \leq r\}$.

In this case, we obtain the following result.

Theorem 3.1. *Let (X,Γ) be a metrizable G-convex space, K a nonempty subset of X, $F_1 : K \multimap X$ a map with G-convex values, and $F_2 : K \multimap X$ a map such that*

(1) *the map F_1 is lower semicontinuous,*

(2) *there exists a $\lambda \geq 1$ such that $G - \mathrm{co}(B(F_2^-(A),r)) \subseteq F_2^-(B(A,\lambda r))$, for all G-convex subsets A of X and nonnegative real number r,*

(3) *there exists a set $D \in \langle K \rangle$ such that $\Gamma_D \subseteq K$ and $F_1(x) \cap B(F_2(D),\varepsilon) \neq \emptyset$ for each $x \in K$, where $\varepsilon > 0$.*

Then there exists $x_\varepsilon \in K$ such that

$$F_1(x_\varepsilon) \cap B(F_2(x_\varepsilon),\lambda\varepsilon) \neq \emptyset. \tag{3.3}$$

Proof. Let for every $y \in K$, $H : K \multimap K$ be defined by

$$H(y) = \{x \in K : F_1(x) \cap B(F_2(y),\varepsilon) = \emptyset\}. \tag{3.4}$$

From assumption (1), we obtain that $H(y)$ is open for each $y \in K$, further, from assumption (3), we obtain that

$$\bigcap_{y \in D} H(y) = \emptyset. \tag{3.5}$$

So, by Theorem 1.1, $H : K \multimap K$ is not a G-KKM map. This implies that there exists $A \in \langle D \rangle$ such that

$$\Gamma_A \nsubseteq H(A), \tag{3.6}$$

and hence there is an $x_\varepsilon \in \Gamma_A$ such that

$$F_1(x_\varepsilon) \cap B(F_2(a),\varepsilon) \neq \varnothing \quad \text{for each } a \in A. \tag{3.7}$$

Hence, we obtain

$$F_2(a) \cap B(F_1(x_\varepsilon),\varepsilon) \neq \varnothing \quad \text{for each } a \in A. \tag{3.8}$$

So, from assumption (2), we have

$$F_2(x_\varepsilon) \cap B(F_1(x_\varepsilon),\lambda\varepsilon) \neq \varnothing, \tag{3.9}$$

that is,

$$F_1(x_U) \cap B(F_2(x_U),\lambda\varepsilon) \neq \varnothing. \tag{3.10}$$

□

Note that if in Theorem 3.1 a map $F_2(x) = \{x\}$, $x \in K$, and open balls are replaced by closed balls, we obtain following result.

Theorem 3.2. *Let (X,Γ) be a metrizable G-convex space, K a nonempty subset of X, $F : K \multimap X$ an upper semicontinuous map with G-convex values, and there exists a $\lambda \geq 1$ such that $G - \operatorname{co} B[A,r] \subseteq B[A,\lambda r]$, for all G-convex subsets A of X and nonnegative real number r. If there exists a set $D \in \langle K \rangle$ such that $\Gamma_D \subseteq K$ and $F(x) \cap B[D,\varepsilon] \neq \varnothing$ for each $x \in K$, where $\varepsilon > 0$, then there exists $x_\varepsilon \in K$ such that*

$$F(x_\varepsilon) \cap B[x_\varepsilon,\lambda\varepsilon] \neq \varnothing. \tag{3.11}$$

Corollary 3.3. *Let X be a metrizable G-convex space, K a nonempty subset of X, $f : K \to X$ a continuous map, and there exists a $\lambda \geq 1$ such that $G - \operatorname{co} B[A,r] \subseteq B[A,\lambda r]$, for all G-convex subsets A of X and nonnegative real number r. If there exists a set $D \in \langle K \rangle$ such that $\operatorname{co} D \subseteq K$ and $f(K) \subseteq B[D,\varepsilon] \neq \varnothing$, where $\varepsilon > 0$, then there exists $x_\varepsilon \in K$ such that*

$$f(x_\varepsilon) \in B[x_\varepsilon,\lambda\varepsilon]. \tag{3.12}$$

Corollary 3.4. *Let X be a metrizable G-convex space, K a nonempty G-convex compact subset of X, $f : K \to K$ a continuous map, and there exists a $\lambda \geq 1$ such that $G - \operatorname{co} B[A,r] \subseteq B[A,\lambda r]$, for all G-convex subsets A of X and nonnegative real number r. Then there exists $x \in K$ such that $f(x) = x$.*

Remark 3.5. (1) Note that if X is locally G-convex space, see, for example, [13, page 190], set K is a compact set and $F : K \multimap K$ is map with closed values, from Theorem 3.2 we obtain a famous Fan-Glicksberg-type fixed point theorem.

(2) If X is a normed space, then Corollary 3.3 reduces to the result of Kim and Park [4, Theorem 2.1].

(3) Note that from Corollary 3.4, we obtain famous Schauder fixed point theorem.

Example 3.6. Let X be a hyperconvex metric space, see, for example, [2, 3]. For a nonempty bounded subset A of X, put

$$\operatorname{co} A = \bigcap\{B : B \text{ is closed ball in } X \text{ containing } A\}. \tag{3.13}$$

Let $\mathcal{A}(X) = \{A \subset X : A = \operatorname{co} A\}$. The elements of $\mathcal{A}(X)$ are called admissible subsets of X. It is known that any hyperconvex metric space (X,d) is a G-convex space (X,Γ), with $\Gamma_A = \operatorname{co} A$ for each $A \in \langle X\rangle$.

The $B(A,r)$ of an admissible subset A of a hyperconvex metric space is also an admissible set, see [2, Lemma 4.10]. Let $F_2 : K \multimap X$ be a G-quasiconvex map, that is, $F_2^-(A)$ is an admissible set for each admissible subset A of X. Then the map F_2 satisfies the condition (2) in Theorem 3.1 for each real number λ such that $\lambda \geq 1$.

From Theorem 3.1, we have the following almost coincidence point theorem and almost fixed point theorem in hyperconvex metric spaces.

Theorem 3.7. *Let X be a hyperconvex metric space, K a nonempty subset of X, $F_1 : K \multimap X$ a map with admissible values, and $F_2 : K \multimap X$ a map such that*

(1) *the map F_1 is lower semicontinuous,*

(2) *the map F_2 is quasiconvex,*

(3) *there exists a set $D \in \langle K\rangle$ such that $\operatorname{co} D \subseteq K$ and $F_1(x) \cap B(F_2(D),\varepsilon) \neq \emptyset$ for each $x \in K$, where $\varepsilon > 0$.*

Then there exists $x_\varepsilon \in K$ such that

$$F_1(x_\varepsilon) \cap B(F_2(x_\varepsilon),\varepsilon) \neq \emptyset. \tag{3.14}$$

Note that if K is a bounded set and $\alpha(\cdot)$ is a measure of noncompactness, then for each $\varepsilon > 0$, there exists a finite set $D \subseteq K$ such that $K \subseteq B[D,\alpha(K)+\varepsilon)]$. In this case, lower semicontinuous map can be replaced by upper semicontinuous map.

Theorem 3.8. *Let X be a hyperconvex metric space, K a nonempty bounded admissible subset of X, $F : K \multimap B[K,\mu]$ an upper semicontinuous map with admissible values, where $\mu > 0$. Then for each $\varepsilon > 0$, there exists $x_\varepsilon \in K$ such that*

$$x_\varepsilon \in B[F(x_\varepsilon),\alpha(K)+\varepsilon+\mu]. \tag{3.15}$$

If in Theorem 3.8 set K is a compact set and map F with closed values, then as an immediate consequence, we obtain the result of existence of fixed point of Kirk and Shin [5, Corollary 3.5].

Finally, we obtain the result of existence of maximal elements for hyperconvex metric spaces.

Let $F : K \to 2^X$, where 2^X denotes the set of all subsets of X. An element $x \in K$ is a maximal element of K if $F(x) = \emptyset$, see, for example, [1, page 33]. The F-maximal set of F is defined as $M_F = \{x \in K : F(x) = \emptyset\}$.

Corollary 3.9. *Let X be a hyperconvex metric space, K a nonempty subset of X, $F_1 : K \to 2^X$ a map with admissible values, and $F_2 : K \to 2^X$ a map such that*

(1) *the map F_1 is lower semicontinuous,*
(2) *the map F_2 is quasiconvex,*
(3) *there exists a set $D \in \langle K \rangle$ such that $\operatorname{co} D \subseteq K$ and $F_1(x) \cap B(F_2(D), \varepsilon) \neq \emptyset$ for each $x \in K$, where $\varepsilon > 0$.*

If $x \notin F_1^-(B(F_2(x), \varepsilon))$ for each $x \in K$, then $M_{F_1} \cup M_{F_2}$ is a nonempty set.

Corollary 3.10. *Let X be a hyperconvex metric space, K a nonempty bounded admissible subset of X, $F : K \to 2^X$ an upper semicontinuous map with admissible values, and let $\varepsilon > 0$ such that $x \in F^-(B[K, \varepsilon]) \setminus F^-(B[x, \alpha(K) + \varepsilon])$ for each $x \in K$. Then F has a maximal element.*

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References

[1] K. C. Border, *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, Cambridge, 1985.
[2] R. Espínola and M. A. Khamsi, *Introduction to Hyperconvex Spaces*, Kluwer Academic, Dordrecht, 2001.
[3] M. A. Khamsi, *KKM and Ky Fan theorems in hyperconvex metric spaces*, Journal of Mathematical Analysis and Applications **204** (1996), no. 1, 298–306.
[4] I.-S. Kim and S. Park, *Almost fixed point theorems of the Fort type*, Indian Journal of Pure and Applied Mathematics **34** (2003), no. 5, 765–771.
[5] W. A. Kirk and S. S. Shin, *Fixed point theorems in hyperconvex spaces*, Houston Journal of Mathematics **23** (1997), no. 1, 175–188.
[6] L.-J. Lin, *Applications of a fixed point theorem in G-convex space*, Nonlinear Analysis **46** (2001), no. 5, 601–608.
[7] K. Nikodem, *K-Convex and K-Concave Set-Valued Functions*, Politechnika, Lodzka, 1989.
[8] S. Park, *Continuous selection theorems in generalized convex spaces*, Numerical Functional Analysis and Optimization **20** (1999), no. 5-6, 567–583.
[9] ———, *Remarks on fixed point theorems for new classes of multimaps*, Journal of the Academy of Natural Sciences, Republic of Korea **43** (2004), 1–20.
[10] S. Park and H. Kim, *Admissible classes of multifunction on generalized convex spaces*, Proceedings of College Nature Science, Seoul National University **18** (1993), 1–21.
[11] ———, *Coincidence theorems for admissible multifunctions on generalized convex spaces*, Journal of Mathematical Analysis and Applications **197** (1996), no. 1, 173–187.
[12] ———, *Foundations of the KKM theory on generalized convex spaces*, Journal of Mathematical Analysis and Applications **209** (1997), no. 2, 551–571.
[13] G. X.-Z. Yuan, *KKM Theory and Applications in Nonlinear Analysis*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 218, Marcel Dekker, New York, 1999.

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