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*Research Article*

# Fixed Points of Weakly Compatible Maps Satisfying a General Contractive Condition of Integral Type

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We prove a fixed point theorem for weakly compatible maps satisfying a general contractive condition of integral type.



## 1. Introduction

Branciari [1] obtained a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle for an integral-type inequality. The authors in [2–6] proved some fixed point theorems involving more general contractive conditions. Also in [7], Suzuki shows that Meir-Keeler contractions of integral type are still Meir-Keeler contractions. In this paper, we establish a fixed point theorem for weakly compatible maps satisfying a general contractive inequality of integral type. This result substantially extends the theorems of [1, 4, 6].

Sessa [8] generalized the concept of commuting mappings by calling self-mappings $A$ and $S$ of metric space $(X,d)$ a weakly commuting pair if and only if $d(ASx,SAx) \le d(Ax, Sx)$ for all $x \in X$. He and others proved some common fixed point theorems of weakly commuting mappings [8–11]. Then, Jungck [12] introduced the concept of compatibility and he and others proved some common fixed point theorems using this concept [12–16].

Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible. Examples in [8, 12] show that neither converse is true.

Recently, Jungck and Rhoades [14] defined the concept of weak compatibility.

*Definition 1.1* (see [14, 17]). Two maps $A,S : X \to X$ are said to be weakly compatible if they commute at their coincidence points.

Again, it is obvious that compatible mappings are weakly compatible. Examples in [14, 17] show that neither converse is true. Many fixed point results have been obtained for weakly compatible mappings (see [14, 17–21]).

Lemma 1.2 (see [22]). *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a right continuous function such that $\psi(t) < t$ for every $t > 0$, then $\lim_{n\to\infty} \psi^n(t) = 0$, where $\psi^n$ denotes the $n$-times repeated composition of $\psi$ with itself.*

## 2. Main result

Now we give our main theorem.

Theorem 2.1. *Let $A$, $B$, $S$, and $T$ be self-maps defined on a metric space $(X,d)$ satisfying the following conditions:*

(i) $S(X) \subseteq B(X)$, $T(X) \subseteq A(X)$,

(ii) *for all $x,y \in X$, there exists a right continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(0) = 0$, and $\psi(s) < s$ for $s > 0$ such that*

$$\int_0^{d(Sx,Ty)} \varphi(t)dt \le \psi\left(\int_0^{M(x,y)} \varphi(t)dt\right), \tag{2.1}$$

*where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesque integrable mapping which is summable, nonnegative and such that*

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \quad \text{for each } \varepsilon > 0, \tag{2.2}$$

$$M(x,y) = \max\left\{d(Ax,By), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)+d(Ty,Ax)}{2}\right\}. \tag{2.3}$$

*If one of $A(X)$, $B(X)$, $S(X)$, or $T(X)$ is a complete subspace of $X$, then*

(1) *$A$ and $S$ have a coincidence point, or*

(2) *$B$ and $T$ have a coincidence point.*

*Further, if $S$ and $A$ as well as $T$ and $B$ are weakly compatible, then*

(3) *$A$, $B$, $S$, and $T$ have a unique common fixed point.*

*Proof.* Let $x_0 \in X$ be an arbitrary point of $X$. From (i) we can construct a sequence $\{y_n\}$ in $X$ as follows:

$$y_{2n+1} = Sx_{2n} = Bx_{2n+1}, \qquad y_{2n+2} = Tx_{2n+1} = Ax_{2n+2} \tag{2.4}$$

for all $n = 0,1,\ldots$. Define $d_n = d(y_n, y_{n+1})$. Suppose that $d_{2n} = 0$ for some $n$. Then $y_{2n} = y_{2n+1}$; that is, $Tx_{2n-1} = Ax_{2n} = Sx_{2n} = Bx_{2n+1}$, and $A$ and $S$ have a coincidence point. □

Similarly, if $d_{2n+1} = 0$, then $B$ and $T$ have a coincidence point. Assume that $d_n \neq 0$ for each $n$.

Then, by (ii),

$$\int_0^{d(Sx_{2n},Tx_{2n+1})} \varphi(t)dt \le \psi\left(\int_0^{M(x_{2n},x_{2n+1})} \varphi(t)dt\right), \tag{2.5}$$

where

$$M(x_{2n},x_{2n+1}) = \max\left\{d(Ax_{2n},Bx_{2n+1}),d(Sx_{2n},Ax_{2n}),d(Tx_{2n+1},Bx_{2n+1}),\right.$$
$$\left.\frac{d(Sx_{2n},Bx_{2n+1})+d(Tx_{2n+1},Ax_{2n})}{2}\right\} \tag{2.6}$$
$$= \max\{d_{2n},d_{2n+1}\}.$$

Thus from (2.5), we have

$$\int_0^{d_{2n+1}} \varphi(t)dt \le \psi\left(\int_0^{\max\{d_{2n},d_{2n+1}\}} \varphi(t)dt\right). \tag{2.7}$$

Now, if $d_{2n+1} \ge d_{2n}$ for some $n$, then, from (2.7), we have

$$\int_0^{d_{2n+1}} \varphi(t)dt \le \psi\left(\int_0^{d_{2n+1}} \varphi(t)dt\right) < \int_0^{d_{2n+1}} \varphi(t)dt, \tag{2.8}$$

which is a contradiction. Thus $d_{2n} > d_{2n+1}$ for all $n$, and so, from (2.7), we have

$$\int_0^{d_{2n+1}} \varphi(t)dt \le \psi\left(\int_0^{d_{2n}} \varphi(t)dt\right). \tag{2.9}$$

Similarly,

$$\int_0^{d_{2n}} \varphi(t)dt \le \psi\left(\int_0^{d_{2n-1}} \varphi(t)dt\right). \tag{2.10}$$

In general, we have for all $n = 1,2,\dots$,

$$\int_0^{d_n} \varphi(t)dt \le \psi\left(\int_0^{d_{n-1}} \varphi(t)dt\right). \tag{2.11}$$

From (2.11), we have

$$\begin{aligned}\int_0^{d_n} \varphi(t)dt &\le \psi\left(\int_0^{d_{n-1}} \varphi(t)dt\right)\\ &\le \psi^2\left(\int_0^{d_{n-2}} \varphi(t)dt\right)\\ &\ \ \vdots\\ &\le \psi^n\left(\int_0^{d_0} \varphi(t)dt\right),\end{aligned} \tag{2.12}$$

and, taking the limit as $n \to \infty$ and using Lemma 1.2, we have

$$\lim_{n\to\infty}\int_0^{d_n} \varphi(t)dt \le \lim_{n\to\infty}\psi^n\left(\int_0^{d_0} \varphi(t)dt\right) = 0, \tag{2.13}$$

which, from (2.2), implies that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.14}$$

We now show that $\{y_n\}$ is a Cauchy sequence. For this it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ such that for each even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$d(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon. \tag{2.15}$$

For every even integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (2.15) such that

$$d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon. \tag{2.16}$$

Now

$$0 < \delta := \int_0^{\varepsilon} \varphi(t)dt \leq \int_0^{d(y_{2n(k)}, y_{2m(k)})} \varphi(t)dt \leq \int_0^{d(y_{2n(k)}, y_{2m(k)-2})+d_{2m(k)-2}+d_{2m(k)-1}} \varphi(t)dt. \tag{2.17}$$

Then by (2.14), (2.15), and (2.16), it follows that

$$\lim_{k\to\infty} \int_0^{d(y_{2n(k)}, y_{2m(k)})} \varphi(t)dt = \delta. \tag{2.18}$$

Also, by the triangular inequality,

$$\begin{aligned} |d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| &\leq d_{2m(k)-1}, \\ |d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| &\leq d_{2m(k)-1} + d_{2n(k)}, \end{aligned} \tag{2.19}$$

and so

$$\begin{aligned} \int_0^{|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})|} \varphi(t)dt &\leq \int_0^{d_{2m(k)-1}} \varphi(t)dt, \\ \int_0^{|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})|} \varphi(t)dt &\leq \int_0^{d_{2m(k)-1}+d_{2n(k)}} \varphi(t)dt. \end{aligned} \tag{2.20}$$

Using (2.18), we get

$$\int_0^{d(y_{2n(k)}, y_{2m(k)-1})} \varphi(t)dt \longrightarrow \delta, \tag{2.21}$$

$$\int_0^{d(y_{2n(k)+1}, y_{2m(k)-1})} \varphi(t)dt \longrightarrow \delta, \tag{2.22}$$

as $k \to \infty$. Thus

$$d(y_{2n(k)}, y_{2m(k)}) \leq d_{2n(k)} + d(y_{2n(k)+1}, y_{2m(k)}) \leq d_{2n(k)} + d(Sx_{2n(k)}, Tx_{2m(k)-1}), \tag{2.23}$$

and so

$$\int_0^{d(y_{2n(k)},y_{2m(k)})} \varphi(t)dt \le \int_0^{d_{2n(k)}+d(Sx_{2n(k)},Tx_{2m(k)-1})} \varphi(t)dt. \tag{2.24}$$

Letting $k \to \infty$ on both sides of the last inequality, we have

$$\delta \le \lim_{k\to\infty} \int_0^{d(Sx_{2n(k)},Tx_{2m(k)-1})} \varphi(t)dt \le \lim_{k\to\infty} \psi\left(\int_0^{M(x_{2n(k)},x_{2m(k)-1})} \varphi(t)dt\right), \tag{2.25}$$

where

$$\begin{aligned} M(x_{2n(k)},x_{2m(k)-1}) = \max\Big\{ & d(y_{2n(k)},y_{2m(k)-1}), d_{2n(k)}, d_{2m(k)-1}, \\ & \frac{d(y_{2n(k)+1},y_{2m(k)-1}) + d(y_{2n(k)},y_{2m(k)})}{2}\Big\}. \end{aligned} \tag{2.26}$$

Combining (2.14), (2.15), (2.16), (2.18), (2.21), and (2.22) yields the following contradiction from (2.25):

$$\delta \le \psi(\delta) < \delta. \tag{2.27}$$

Thus $\{y_{2n}\}$ is a Cauchy sequence and so $\{y_n\}$ is a Cauchy sequence.

Now, suppose that $A(X)$ is complete. Note that the sequence $\{y_{2n}\}$ is contained in $A(X)$ and has a limit in $A(X)$. Call it $u$. Let $v \in A^{-1}u$. Then $Av = u$. We will use the fact that the sequence $\{y_{2n-1}\}$ also converges to $u$. To prove that $Sv = u$, let $r = d(Sv,u) > 0$. Then taking $x = v$ and $y = x_{2n-1}$ in (ii),

$$\int_0^{d(Sv,y_{2n})} \varphi(t)dt = \int_0^{d(Sv,Tx_{2n-1})} \varphi(t)dt \le \psi\left(\int_0^{M(v,x_{2n-1})} \varphi(t)dt\right), \tag{2.28}$$

where

$$\begin{aligned} M(v,x_{2n-1}) = \max\Big\{ & d(u,y_{2n-1}), d(Sv,u), d(y_{2n},y_{2n-1}), \\ & \frac{d(Sv,y_{2n-1}) + d(y_{2n},u)}{2}\Big\}. \end{aligned} \tag{2.29}$$

Since $\lim_n d(Sv,y_{2n}) = r$, $\lim_n d(u,y_{2n-1}) = \lim_n d(y_{2n},y_{2n-1}) = 0$, and $\lim_n[d(Sv,y_{2n-1}) + d(y_{2n},u)] = r$, we may conclude that

$$\int_0^r \varphi(t)dt \le \psi\left(\int_0^r \varphi(t)dt\right) < \int_0^r \varphi(t)dt, \tag{2.30}$$

which is a contradiction. Hence from (2.2), $Sv = u$. This proves (1).

Since $S(X) \subseteq B(X)$, $Sv = u$ implies that $u \in B(X)$. Let $w \in B^{-1}u$. Then $Bw = u$. By using the argument of the previous section, it can be easily verified that $Tw = u$. This proves (2).

The same result holds if we assume that $B(X)$ is complete instead of $A(X)$.

Now if $T(X)$ is complete, then by (i), $u \in T(X) \subseteq A(X)$. Similarly if $S(X)$ is complete, then $u \in S(X) \subseteq B(X)$. Thus (1) and (2) are completely established.

To prove (3), note that $S$, $A$ and $T$, $B$ are weakly compatible and

$$u = Sv = Av = Tw = Bw, \tag{2.31}$$

then

$$\begin{aligned} Au &= ASv = SAv = Su, \\ Bu &= BTw = TBw = Tu. \end{aligned} \tag{2.32}$$

If $Tu \neq u$ then, from (ii), (2.31) and (2.32),

$$\begin{aligned} \int_0^{d(u,Tu)} \varphi(t)dt &= \int_0^{d(Sv,Tu)} \varphi(t)dt \leq \psi\left(\int_0^{M(v,u)} \varphi(t)dt\right) \\ &= \psi\left(\int_0^{d(u,Tu)} \varphi(t)dt\right) < \int_0^{d(u,Tu)} \varphi(t)dt, \end{aligned} \tag{2.33}$$

which is a contradiction. So $Tu = u$. Similarly $Su = u$. Then, evidently from (2.32), $u$ is a common fixed point of $A$, $B$, $S$, and $T$.

The uniqueness of the common fixed point follows easily from condition (ii).

*Remark 2.2.* Theorem 2.1 is a generalization of the main theorem of [1], Theorem 2 of [4], and Theorem 2 of [6].

If $\varphi(t) \equiv 1$, then Theorem 2.1 of this paper reduces to Theorem 2.1 of [17].

If $\varphi(t) \equiv 1$ and $\psi = ht$, $0 \leq h < 1$, then Theorem 2.1 of this paper reduces to Corollary 3.1 of [20].

The following example shows that our main theorem is generalization of Corollary 3.1 of [20].

*Example 2.3.* Let $X = \{1/n : n \in N\} \cup \{0\}$ with Euclidean metric and $S$, $T$, $A$, $B$ are self maps of $X$ defined by

$$S\left(\frac{1}{n}\right) = \begin{cases} \dfrac{1}{n+1} & \text{if } n \text{ is odd}, \\ \dfrac{1}{n+2} & \text{if } n \text{ is even}, \\ 0 & \text{if } n = \infty, \end{cases} \qquad T\left(\frac{1}{n}\right) = \begin{cases} \dfrac{1}{n+1} & \text{if } n \text{ is even}, \\ \dfrac{1}{n+2} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n = \infty, \end{cases}$$
$$A\left(\frac{1}{n}\right) = B\left(\frac{1}{n}\right) = \frac{1}{n} \quad \forall n \in N \cup \{\infty\}. \tag{2.34}$$

Clearly $S(X) \subseteq B(X)$, $T(X) \subseteq A(X)$, $A(X)$ is a complete subspace of $X$ and $A, S$ and $B, T$ are weakly compatible.

Now suppose that the contractive condition of Corollary 3.1 of [20] is satisfying, that is, there exists $h \in [0,1)$ such that

$$d(Sx,Ty) \leq hM(x,y) \tag{2.35}$$

for all $x, y \in X$. Therefore, for $x \neq y$, we have

$$\frac{d(Sx,Ty)}{M(x,y)} \leq h < 1, \tag{2.36}$$

but since $\sup_{x\neq y}(d(Sx,Ty)/M(x,y)) = 1$, one has a contradiction. Thus the condition (2.35) is not satisfied.

Now we define $\varphi(t) = \max\{0, t^{1/t-2}[1-\log t]\}$ for $t > 0$, $\varphi(0) = 0$. Then for any $\tau \in (0,e)$,

$$\int_0^{\tau} \varphi(t)dt = \tau^{1/\tau}. \tag{2.37}$$

Thus we must show that there exists a right continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(s) < s$ for $s > 0$, $\psi(0) = 0$ such that

$$\left(d(Sx,Ty)\right)^{1/d(Sx,Ty)} \leq \psi\left((M(x,y))^{1/M(x,y)}\right) \tag{2.38}$$

for all $x, y \in X$. Now we claim that (2.38) is satisfying with $\psi(s) = s/2$, that is,

$$\left(d(Sx,Ty)\right)^{1/d(Sx,Ty)} \leq \frac{1}{2}\left((M(x,y))^{1/M(x,y)}\right) \tag{2.39}$$

for all $x, y \in X$. Since the function $\tau \to \tau^{1/\tau}$ is nondecreasing, we show sufficiently that

$$\left(d(Sx,Ty)\right)^{1/d(Sx,Ty)} \leq \frac{1}{2}\left((d(x,y))^{1/d(x,y)}\right) \tag{2.40}$$

instead of (2.39). Now using Example 4 of [6], we have (2.40), thus the condition (2.38) is satisfied.


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