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Research Article

Iteration Scheme with Perturbed Mapping for Common Fixed Points of a Finite Family of Nonexpansive Mappings

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We propose an iteration scheme with perturbed mapping for approximation of common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$. We show that the proposed iteration scheme converges to the common fixed point $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ which solves some variational inequality.

1. Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, respectively. A mapping T with domain $D(T)$ and range $R(T)$ in H is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in D(T). \tag{1.1}$$

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of H. Denote the common fixed points set of $\{T_i\}_{i=1}^N$ by $\bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Let $F : H \to H$ be a mapping such that for some constants $k, \eta > 0$, F is k-Lipschitzian and η-strongly monotone. Let $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$, $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$ and take a fixed number $\mu \in (0, 2\eta/k^2)$. The iterative schemes concerning nonlinear operators have been studied extensively by many authors, you may refer to [1–12]. Especially, in [13], Zeng and Yao introduced the following implicit iteration process with perturbed mapping F.

For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\left[T_n x_n - \lambda_n \mu F(T_n x_n)\right], \quad n \ge 1, \tag{1.2}$$

where $T_n := T_{n \bmod N}$.

Using this iteration process, they proved the following weak and strong convergence theorems for nonexpansive mappings in Hilbert spaces.

Theorem 1.1 (see [13]). *Let H be a real Hilbert space and let $F : H \to H$ be a mapping such that for some constants $k,\eta > 0$, F is k-Lipschitzain vcommentand η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0,2\eta/k^2)$ and $x_0 \in H$. Let $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$ and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ satisfying the conditions $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \le \alpha_n \le \beta$, $n \ge 1$, for some $\alpha,\beta \in (0,1)$. Then the sequence $\{x_n\}_{n=1}^\infty$ defined by (1.2) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.*

Theorem 1.2 (see [13]). *Let H be a real Hilbert space and let $F : H \to H$ be a mapping such that for some constants $k,\eta > 0$, F is k-Lipschitzain and η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0,2\eta/k^2)$ and $x_0 \in H$. Let $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$ and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ satisfying the conditions $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \le \alpha_n \le \beta$, $n \ge 1$, for some $\alpha,\beta \in (0,1)$. Then the sequence $\{x_n\}_{n=1}^\infty$ defined by (1.2) converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$ if and only if*

$$\liminf_{n\to\infty} d\left(x_n, \bigcap_{i=1}^N \mathrm{Fix}(T_i)\right) = 0. \tag{1.3}$$

Very recently, Wang [14] considered an explicit iterative scheme with perturbed mapping F and obtained the following result.

Theorem 1.3. *Let H be a Hilbert space, let $T : H \to H$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and let $F : H \to H$ be an η-strongly monotone and k-Lipschitzian mapping. For any given $x_0 \in H$, $\{x_n\}$ is defined by*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T^{\lambda_{n+1}} x_n, \quad n \ge 0, \tag{1.4}$$

where $T^{\lambda_{n+1}} x_n = T x_n - \lambda_{n+1}\mu F(T x_n)$, $\{\alpha_n\}$ and $\{\lambda_n\} \subset [0,1)$ satisfy the following conditions:

(1) *$\alpha \le \alpha_n \le \beta$ for some $\alpha,\beta \in (0,1)$;*
(2) *$\sum_{n=1}^\infty \lambda_n < \infty$;*
(3) *$0 < \mu < 2\eta/k^2$.*

Then

(1) *$\{x_n\}$ converges weakly to a fixed point of T,*
(2) *$\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0. \tag{1.5}$$

This naturally brings us the following questions.

Questions 1.4. Let $T_i : H \to H$ $(i = 1,2,\dots,N)$ be a finite family of nonexpansive mappings and F is k-Lipschitzain and η-strongly monotone.

(i) Could we construct an explicit iterative algorithm to approximate the common fixed points of the mappings $\{T_i\}_{i=1}^N$?
(ii) Could we remove the assumption (2) imposed on the sequence $\{x_n\}$?

Motivated and inspired by the above research work of Zeng and Yao [13] and Wang [14], in this paper, we will propose a new explicit iteration scheme with perturbed mapping for approximation of common fixed points of a finite family of nonexpansive self-mappings of H. We will establish strong convergence theorem for this explicit iteration scheme. To be more specific, let $\alpha_{n1},\alpha_{n2},\dots,\alpha_{nN} \in (0,1]$, $n \in N$. Given the mappings $T_1,T_2,\dots,T_N$, following [15], one can define, for each n, mappings $U_{n1},U_{n2},\dots,U_{nN}$ by

$$\begin{aligned} U_{n1} &= \alpha_{n1}T_1 + (1-\alpha_{n1})I, \\ U_{n2} &= \alpha_{n2}T_2U_{n1} + (1-\alpha_{n2})I, \\ &\vdots \\ U_{n,N-1} &= \alpha_{n,N-1}T_{N-1}U_{n,N-2} + (1-\alpha_{n,N-1})I, \\ W_n := U_{nN} &= \alpha_{nN}T_NU_{n,N-1} + (1-\alpha_{nN})I. \end{aligned} \tag{1.6}$$

Such a mapping W_n is called the W-mapping generated by $T_1,\dots,T_N$ and $\alpha_{n1},\dots,\alpha_{nN}$.

First we introduce the following explicit iteration scheme with perturbed mapping F.

For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated iteratively by

$$x_{n+1} = \beta x_n + (1-\beta)\big[W_nx_n - \lambda_n\mu F(W_nx_n)\big], \quad n \geq 0, \tag{1.7}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$, β is a constant in $(0,1)$, F is k-Lipschitzian and η-strongly monotone, and W_n is the W-mapping defined by (1.6).

We have the following crucial conclusion concerning W_n.

Proposition 1.5 (see [15]). *Let C be a nonempty closed convex subset of a Banach space E. Let $T_1,T_2,\dots,T_N$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N \operatorname{Fix}(T_i)$ is nonempty, and let $\alpha_{n1},\alpha_{n2},\dots,\alpha_{nN}$ be real numbers such that $0 < \alpha_{ni} \leq b < 1$ for any $i \in N$. For any $n \in N$, let W_n be the W-mapping of C into itself generated by $T_N,T_{N-1},\dots,T_1$ and $\alpha_{nN},\alpha_{n,N-1},\dots,\alpha_{n1}$. Then W_n is nonexpansive. Further, if E is strictly convex, then $\operatorname{Fix}(W_n) = \bigcap_{i=1}^N \operatorname{Fix}(T_i)$.*

Now we recall some basic notations. Let $T : H \to H$ be nonexpansive mapping and $F : H \to H$ be a mapping such that for some constants $k,\eta > 0$, F is k-Lipschitzian and η-strongly monotone; that is, F satisfies the following conditions:

$$\begin{aligned} \|Fx - Fy\| &\leq k\|x-y\|, \quad \forall x,y \in H, \\ \langle Fx - Fy, x-y\rangle &\geq \eta\|x-y\|^2, \quad \forall x,y \in H, \end{aligned} \tag{1.8}$$

respectively. We may assume, without loss of generality, that $\eta \in (0,1)$ and $k \in [1,\infty)$. Under these conditions, it is well known that the variational inequality problem—find $x^* \in \bigcap_{i=1}^N \operatorname{Fix}(T_i)$ such that

$$VI\left(F, \bigcap_{i=1}^{N} \operatorname{Fix}(T_i)\right) : \langle F(x^*), x - x^*\rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i), \tag{1.9}$$

has a unique solution $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. [Note: the unique existence of the solution $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ is guaranteed automatically because F is k-Lipschitzian and η-strongly monotone over $\bigcap_{i=1}^N \mathrm{Fix}(T_i)$.]

For any given numbers $\lambda \in [0,1)$ and $\mu \in (0, 2\eta/k^2)$, we define the mapping $T^\lambda : H \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in H. \tag{1.10}$$

Concerning the corresponding result of $T^\lambda x$, you can find it in [16].

Lemma 1.6 (see [16]). *If $0 \le \lambda < 1$ and $0 < \mu < 2\eta/k^2$, then there holds for $T^\lambda : H \to H$,*

$$||T^\lambda x - T^\lambda y|| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in H, \tag{1.11}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)} \in (0,1)$.

Next, let us state four preliminary results which will be needed in the sequel. Lemma 1.7 is very interesting and important, you may find it in [17], the original prove can be found in [18]. Lemmas 1.8 and 1.9 well-known demiclosedness principle and subdifferential inequality, respectively. Lemma 1.10 is basic and important result, please consult it in [19].

Lemma 1.7 (see [17]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1. \tag{1.12}$$

Suppose

$$x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n, \tag{1.13}$$

for all integers $n \ge 0$ and

$$\limsup_{n\to\infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \le 0. \tag{1.14}$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.8 (see [20]). *Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here, I is the identity operator of H.*

Lemma 1.9 (see [21]). *$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$ for all $x, y \in H$.*

Lemma 1.10 (see [19]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n) a_n + \delta_n, \tag{1.15}$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

2. Main result

Now we state and prove our main result.

Theorem 2.1. *Let H be a real Hilbert space and let $F : H \to H$ be a k-Lipschitzian and η-strongly monotone mapping. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of H such that $\bigcap_{i=1}^N \operatorname{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/k^2)$. Suppose the sequences $\{\alpha_{n,i}\}_{i=1}^N$ satisfy $\lim_{n\to\infty}(\alpha_{n,i} - \alpha_{n-1,i}) = 0$, for all $i = 1,2,\ldots,N$. If $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1)$ satisfy the following conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$,

then the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1.7) converges strongly to a common fixed point $x^* \in \bigcap_{i=1}^N \operatorname{Fix}(T_i)$ which solves the variational inequality (1.9).

Proof. Let x^* be an arbitrary element of $\bigcap_{i=1}^N \operatorname{Fix}(T_i)$. Observe that

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|\beta x_n + (1-\beta) W_n^{\lambda_n} x_n - x^*\| \\ &\leq \beta \|x_n - x^*\| + (1-\beta)\|W_n^{\lambda_n} x_n - x^*\|, \end{aligned} \tag{2.1}$$

where $W_n^{\lambda_n} x := W_n x - \lambda_n \mu F(W_n x)$. Note that

$$W_n^{\lambda_n} x^* = x^* - \lambda_n \mu F(x^*). \tag{2.2}$$

Utilizing Lemma 1.6, we have

$$\begin{aligned} \|W_n^{\lambda_n} x_n - x^*\| &= \|W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^* + W_n^{\lambda_n} x^* - x^*\| \\ &\leq \|W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*\| + \|W_n^{\lambda_n} x^* - x^*\| \\ &\leq (1 - \lambda_n \tau)\|x_n - x^*\| + \lambda_n \mu \|F(x^*)\|. \end{aligned} \tag{2.3}$$

From (2.1) and (2.3), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq [\beta + (1-\beta)(1-\lambda_n\tau)]\|x_n - x^*\| + (1-\beta)\lambda_n\mu\|F(x^*)\| \\ &= [1 - (1-\beta)\lambda_n\tau]\|x_n - x^*\| + (1-\beta)\lambda_n\mu\|F(x^*)\| \\ &\leq \max\left\{\|x_0 - x^*\|, \left(\frac{\mu}{\tau}\right)\|F(x^*)\|\right\}. \end{aligned} \tag{2.4}$$

Hence, $\{x_n\}$ is bounded. We also can obtain that $\{W_n x_n\}$, $\{T_i U_{n,j} x_n\}(i = 1,\ldots,N;\ j = 1,\ldots,N)$, and $\{F(W_n x_n)\}$ are all bounded.

We will use M to denote the possible different constants appearing in the following reasoning.

We note that

$$\begin{aligned}
&\|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| \\
&\quad = \|W_{n+1}x_{n+1} - W_nx_n - \lambda_{n+1}\mu F(W_{n+1}x_{n+1}) + \lambda_n\mu F(W_nx_n)\| \\
&\quad \le \|W_{n+1}x_{n+1} - W_nx_n\| + \lambda_{n+1}\mu\|F(W_{n+1}x_{n+1})\| + \lambda_n\mu\|F(W_nx_n)\| \\
&\quad \le \|W_{n+1}x_{n+1} - W_{n+1}x_n\| + \|W_{n+1}x_n - W_nx_n\| + (\lambda_{n+1}+\lambda_n)M \\
&\quad \le \|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_nx_n\| + (\lambda_{n+1}+\lambda_n)M.
\end{aligned} \tag{2.5}$$

From (1.6), since T_N and $U_{n,N}$ are nonexpansive,

$$\begin{aligned}
&\|W_{n+1}x_n - W_nx_n\| \\
&\quad = \|\alpha_{n+1,N}T_NU_{n+1,N-1}x_n + (1-\alpha_{n+1,N})x_n - \alpha_{n,N}T_NU_{n,N-1}x_n - (1-\alpha_{n,N})x_n\| \\
&\quad \le \|\alpha_{n+1,N}T_NU_{n+1,N-1}x_n - \alpha_{n,N}T_NU_{n,N-1}x_n\| + |\alpha_{n+1,N} - \alpha_{n,N}|\|x_n\| \\
&\quad \le \|\alpha_{n+1,N}(T_NU_{n+1,N-1}x_n - T_NU_{n,N-1}x_n)\| + |\alpha_{n+1,N} - \alpha_{n,N}|\|T_NU_{n,N-1}x_n\| \\
&\qquad + |\alpha_{n+1,N} - \alpha_{n,N}|\|x_n\| \\
&\quad \le \alpha_{n+1,N}\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| + 2M|\alpha_{n+1,N} - \alpha_{n,N}|.
\end{aligned} \tag{2.6}$$

Again, from (1.6), we have

$$\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad = \|\alpha_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n + (1-\alpha_{n+1,N-1})x_n \\
&\qquad - \alpha_{n,N-1}T_{N-1}U_{n,N-2}x_n - (1-\alpha_{n,N-1})x_n\| \\
&\quad \le \|\alpha_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n - \alpha_{n,N-1}T_{N-1}U_{n,N-2}x_n\| \\
&\qquad + |\alpha_{n+1,N-1} - \alpha_{n,N-1}|\|x_n\| \\
&\quad \le |\alpha_{n+1,N-1} - \alpha_{n,N-1}|\|x_n\| + |\alpha_{n+1,N-1} - \alpha_{n,N-1}|M \\
&\qquad + \alpha_{n+1,N-1}\|T_{N-1}U_{n+1,N-2}x_n - T_{N-1}U_{n,N-2}x_n\| \\
&\quad \le 2M|\alpha_{n+1,N-1} - \alpha_{n,N-1}| + \alpha_{n+1,N-1}\|U_{n+1,N-2}x_n - U_{n,N-2}x_n\| \\
&\quad \le 2M|\alpha_{n+1,N-1} - \alpha_{n,N-1}| + \|U_{n+1,N-2}x_n - U_{n,N-2}x_n\|.
\end{aligned} \tag{2.7}$$

Therefore, we have

$$\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad \le 2M\,|\,\alpha_{n+1,N-1}-\alpha_{n,N-1}\,| + 2M\,|\,\alpha_{n+1,N-2}-\alpha_{n,N-2}\,| \\
&\qquad + \|U_{n+1,N-3}x_n - U_{n,N-3}x_n\| \\
&\quad \le 2M\sum_{i=2}^{N-1}|\,\alpha_{n+1,i}-\alpha_{n,i}\,| + \|U_{n+1,1}x_n - U_{n,1}x_n\| \\
&\quad = \|\alpha_{n+1,1}T_1x_n + (1-\alpha_{n+1,1})x_n - \alpha_{n,1}T_1x_n - (1-\alpha_{n,1})x_n\| \\
&\qquad + 2M\sum_{i=2}^{N-1}|\,\alpha_{n+1,i}-\alpha_{n,i}\,|,
\end{aligned} \tag{2.8}$$

then

$$\begin{aligned}
&\|U_{n+1,N-1}x_n - U_{n,N-1}x_n\| \\
&\quad \le |\,\alpha_{n+1,1}-\alpha_{n,1}\,|\,\|x_n\| + \|\alpha_{n+1,1}T_1x_n - \alpha_{n,1}T_1x_n\| \\
&\qquad + 2M\sum_{i=2}^{N-1}|\,\alpha_{n+1,i}-\alpha_{n,i}\,| \le 2M\sum_{i=1}^{N-1}|\,\alpha_{n+1,i}-\alpha_{n,i}\,|.
\end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.6), we have

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| &\le 2M\,|\,\alpha_{n+1,N}-\alpha_{n,N}\,| + 2\alpha_{n+1,N}M\sum_{i=1}^{N-1}|\,\alpha_{n+1,i}-\alpha_{n,i}\,| \\
&\le 2M\sum_{i=1}^{N}|\,\alpha_{n+1,i}-\alpha_{n,i}\,|.
\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.5), we have

$$\|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| \le \|x_{n+1}-x_n\| + 2M\sum_{i=1}^{N}|\,\alpha_{n+1,i}-\alpha_{n,i}\,| + (\lambda_{n+1}+\lambda_n)M, \tag{2.11}$$

which implies that

$$\limsup_{n\to\infty}\left(\|W_{n+1}^{\lambda_{n+1}}x_{n+1} - W_n^{\lambda_n}x_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{2.12}$$

We note that $x_{n+1} = \beta x_n + (1-\beta)W_n^{\lambda_n}x_n$ and $0<\beta<1$, then from Lemma 1.7 and (2.12), we have $\lim_{n\to\infty}\|W_n^{\lambda_n}x_n - x_n\| = 0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta)\|W_n^{\lambda_n}x_n - x_n\| = 0. \tag{2.13}$$

On the other hand,

$$\begin{aligned}
\|x_n - W_nx_n\| &\le \|x_{n+1}-x_n\| + \|x_{n+1}-W_nx_n\| \\
&\le \|x_{n+1}-x_n\| + \beta\|x_n - W_nx_n\| + (1-\beta)\lambda_n\mu\|F(W_nx_n)\|,
\end{aligned} \tag{2.14}$$

that is,

$$\|x_n - W_n x_n\| \le \frac{1}{1-\beta}\|x_{n+1} - x_n\| + \lambda_n \mu \|F(W_n x_n)\|, \tag{2.15}$$

this together with (i) and (2.13) imply

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0. \tag{2.16}$$

We next show that

$$\limsup_{n\to\infty} \langle -F(x^*), x_n - x^* \rangle \le 0. \tag{2.17}$$

To prove this, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle -F(x^*), x_n - x^* \rangle = \lim_{i\to\infty} \langle -F(x^*), x_{n_i} - x^* \rangle. \tag{2.18}$$

Without loss of generality, we may further assume that $x_{n_i} \to z$ weakly for some $z \in H$.

By Lemma 1.8 and (2.16), we have

$$z \in \operatorname{Fix}(W_n), \tag{2.19}$$

this together with Proposition 1.5 imply that

$$z \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \tag{2.20}$$

Since x^* solves the variational inequality (1.9), then we obtain

$$\limsup_{n\to\infty} \langle -F(x^*), x_n - x^* \rangle = \langle -F(x^*), z - x^* \rangle \le 0. \tag{2.21}$$

Finally, we show that $x_n \to x^*$. Indeed, from Lemma 1.9, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|\beta(x_n - x^*) + (1-\beta)(W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*) + (1-\beta)(W_n^{\lambda_n} x^* - x^*)\|^2 \\
&\quad \le \|\beta(x_n - x^*) + (1-\beta)(W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*)\|^2 + 2(1-\beta)\langle W_n^{\lambda_n} x^* - x^*, x_{n+1} - x^* \rangle \\
&\quad \le [\beta\|x_n - x^*\| + (1-\beta)\|W_n^{\lambda_n} x_n - W_n^{\lambda_n} x^*\|]^2 + 2(1-\beta)\lambda_n \mu \langle -F(x^*), x_{n+1} - x^* \rangle \\
&\quad \le [\beta\|x_n - x^*\| + (1-\beta)(1-\lambda_n \tau)\|x_n - x^*\|]^2 + 2(1-\beta)\lambda_n \mu \langle -F(x^*), x_{n+1} - x^* \rangle \\
&\quad \le [1 - (1-\beta)\tau\lambda_n]\|x_n - x^*\|^2 + (1-\beta)\tau\lambda_n \left\{ 2\frac{\mu}{\tau} \langle -F(x^*), x_{n+1} - x^* \rangle \right\}.
\end{aligned} \tag{2.22}$$

Now applying Lemma 1.10 and (2.21) to (2.22) concludes that $x_n \to x^*$ $(n \to \infty)$. This completes the proof. □

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