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Research Article

A Fixed Point Approach to the Stability of a Volterra Integral Equation

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We will apply the fixed point method for proving the Hyers-Ulam-Rassias stability of a Volterra integral equation of the second kind.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality $\| f(x+y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have the Hyers-Ulam stability.

Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$||f(x+y) - f(x) - f(y)|| \le \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

and proved the Hyers theorem. That is, Rassias proved the Hyers-Ulam-Rassias stability of the Cauchy additive functional equation. Since then, the stability of several functional equations has been extensively investigated [4–10].

The terminologies Hyers-Ulam-Rassias stability and Hyers-Ulam stability can also be applied to the case of other functional equations, differential equations, and of various integral equations.

For a given continuous function f and a fixed real number c, the integral equation

$$y(x) = \int_c^x f(\tau, y(\tau))\, d\tau \tag{1.2}$$

is called a Volterra integral equation of the second kind. If for each function $y(x)$ satisfying

$$\left| y(x) - \int_c^x f(\tau, y(\tau))\, d\tau \right| \le \psi(x), \tag{1.3}$$

where $\psi(x) \ge 0$ for all x, there exists a solution $y_0(x)$ of the Volterra integral equation (1.2) and a constant $C > 0$ with

$$| y(x) - y_0(x) | \le C\psi(x) \tag{1.4}$$

for all x, where C is independent of $y(x)$ and $y_0(x)$, then we say that the integral equation (1.2) has the Hyers-Ulam-Rassias stability. If $\psi(x)$ is a constant function in the above inequalities, we say that the integral equation (1.2) has the Hyers-Ulam stability.

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies the following:

(M_1) $d(x, y) = 0$ if and only if $x = y$;

(M_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(M_3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We remark that the only one difference of the generalized metric from the usual metric is that the range of the former is permitted to include the infinity.

We now introduce one of the fundamental results of fixed point theory. For the proof, we refer to [11]. This theorem will play an important role in proving our main theorems.

Theorem 1.1. *Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then the followings are true:*

(a) *the sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ;*

(b) *x^* is the unique fixed point of Λ in*

$$X^* = \{ y \in X \mid d(\Lambda^k x, y) < \infty \}; \tag{1.5}$$

(c) *If $y \in X^*$, then*

$$d(y, x^*) \le \frac{1}{1-L} d(\Lambda y, y). \tag{1.6}$$

In this paper, we will adopt the idea of Cădariu and Radu [12] and prove the Hyers-Ulam-Rassias stability and the Hyers-Ulam stability of the Volterra integral equation (1.2).

2. Hyers-Ulam-Rassias stability

Recently, Cădariu and Radu [12] applied the fixed point method to the investigation of the Cauchy additive functional equation. Using such a clever idea, they could present another proof for the Hyers-Ulam stability of that equation [13–15].

In this section, by using the idea of Cădariu and Radu, we will prove the Hyers-Ulam-Rassias stability of the Volterra integral equation (1.2).

Theorem 2.1. *Let K and L be positive constants with $0 < KL < 1$ and let $I = [a,b]$ be given for fixed real numbers a,b with $a < b$. Assume that $f : I \times \mathbb{C} \to \mathbb{C}$ is a continuous function which satisfies a Lipschitz condition*

$$|f(x,y) - f(x,z)| \leq L|y - z| \tag{2.1}$$

for any $x \in I$ and all $y,z \in \mathbb{C}$. If a continuous function $y : I \to \mathbb{C}$ satisfies

$$\left| y(x) - \int_c^x f(\tau, y(\tau))\,d\tau \right| \leq \varphi(x) \tag{2.2}$$

for all $x \in I$ and for some $c \in I$, where $\varphi : I \to (0,\infty)$ is a continuous function with

$$\left| \int_c^x \varphi(\tau)\,d\tau \right| \leq K\varphi(x) \tag{2.3}$$

for each $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{C}$ such that

$$y_0(x) = \int_c^x f(\tau, y_0(\tau))\,d\tau, \tag{2.4}$$

$$|y(x) - y_0(x)| \leq \frac{1}{1-KL}\varphi(x) \tag{2.5}$$

for all $x \in I$.

Proof. First, we define a set

$$X = \{h : I \longrightarrow \mathbb{C} \mid h \text{ is continuous}\} \tag{2.6}$$

and introduce a generalized metric on X as follows:

$$d(g,h) = \inf\{C \in [0,\infty] \mid |g(x) - h(x)| \leq C\varphi(x)\ \forall x \in I\}. \tag{2.7}$$

(Here, we give a proof for the triangle inequality. Assume that $d(g,h) > d(g,k) + d(k,h)$ would hold for some $g,h,k \in X$. Then, there should exist an $x_0 \in I$ with

$$|g(x_0) - h(x_0)| > \{d(g,k) + d(k,h)\}\varphi(x_0) = d(g,k)\varphi(x_0) + d(k,h)\varphi(x_0). \tag{2.8}$$

In view of (2.7), this inequality would yield

$$|g(x_0) - h(x_0)| > |g(x_0) - k(x_0)| + |k(x_0) - h(x_0)|, \tag{2.9}$$

a contradiction.)

Our task is to show that (X,d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (X,d). Then, for any $\varepsilon > 0$ there exists an integer $N_\varepsilon > 0$ such that $d(h_m,h_n) \leq \varepsilon$ for all $m,n \geq N_\varepsilon$. In view of (2.7), we have

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N}\ \forall m,n \geq N_\varepsilon\ \forall x \in I : |h_m(x) - h_n(x)| \leq \varepsilon\varphi(x). \tag{2.10}$$

If x is fixed, (2.10) implies that $\{h_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $\{h_n(x)\}$ converges for each $x \in I$. Thus, we can define a function $h : I \to \mathbb{C}$ by

$$h(x) = \lim_{n\to\infty} h_n(x). \tag{2.11}$$

Since φ is continuous on the compact interval I, φ is bounded. Thus, (2.10) implies that $\{h_n\}$ converges uniformly to h in the usual topology of $\mathbb{C}$. Hence, h is continuous, that is, $h \in X$. (It has not been proved yet that $\{h_n\}$ converges to h in (X,d).)

If we let m increase to infinity, it follows from (2.10) that

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N}\ \forall n \geq N_\varepsilon\ \forall x \in I : |h(x) - h_n(x)| \leq \varepsilon\varphi(x). \tag{2.12}$$

By considering (2.7), we get

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \quad \forall n \geq N_\varepsilon : d(h,h_n) \leq \varepsilon. \tag{2.13}$$

This means that the Cauchy sequence $\{h_n\}$ converges to h in (X,d). Hence, (X,d) is complete.

We now define an operator $\Lambda : X \to X$ by

$$(\Lambda h)(x) = \int_c^x f(\tau,h(\tau))\,d\tau \tag{2.14}$$

for all $h \in X$ and $x \in I$. Then, according to the fundamental theorem of Calculus, Λh is continuously differentiable on I, since f is a continuous function. Hence, we conclude that $\Lambda h \in X$.

We assert that Λ is strictly contractive on X. Given any $g,h \in X$, let $C_{gh} \in [0,\infty]$ be an arbitrary constant with $d(g,h) \leq C_{gh}$, that is,

$$|g(x) - h(x)| \leq C_{gh}\varphi(x) \tag{2.15}$$

for any $x \in I$. Then, it follows from (2.1), (2.3), (2.14), and (2.15) that

$$\begin{aligned} |(\Lambda g)(x) - (\Lambda h)(x)| &= \left| \int_c^x \{f(\tau, g(\tau)) - f(\tau, h(\tau))\} d\tau \right| \\ &\le \left| \int_c^x |f(\tau, g(\tau)) - f(\tau, h(\tau))| d\tau \right| \\ &\le L \left| \int_c^x |g(\tau) - h(\tau)| d\tau \right| \\ &\le LC_{gh} \left| \int_c^x \varphi(\tau) d\tau \right| \le KLC_{gh}\varphi(x) \end{aligned} \tag{2.16}$$

for all $x \in I$, that is, $d(\Lambda g, \Lambda h) \le KLC_{gh}$. Hence, we may conclude that $d(\Lambda g, \Lambda h) \le KLd(g,h)$ for any $g, h \in X$ and we note that $0 < KL < 1$.

Let $h_0 \in X$ be given. By (2.6) and (2.14), there exists a constant $0 < C < \infty$ such that

$$|(\Lambda h_0)(x) - h_0(x)| = \left| \int_c^x f(\tau, h_0(\tau)) d\tau - h_0(x) \right| \le C\varphi(x) \tag{2.17}$$

for every $x \in I$, since f, h_0 are bounded on I and $\min_{x \in I} \varphi(x) > 0$. Thus, (2.7) implies that

$$d(\Lambda h_0, h_0) < \infty. \tag{2.18}$$

Therefore, it follows from Theorem 1.1(a) that there exists a continuous function $y_0 : I \to \mathbb{C}$ such that $\Lambda^n h_0 \to y_0$ in (X,d) and $\Lambda y_0 = y_0$, or equivalently, y_0 satisfies (2.4) for every $x \in I$.

We show that $\{g \in X \mid d(h_0, g) < \infty\} = X$, where h_0 was chosen with the property (2.18). Given any $g \in X$, since g, h_0 are bounded on I and $\min_{x \in I} \varphi(x) > 0$, there exists a constant $0 < C_g < \infty$ such that

$$|h_0(x) - g(x)| \le C_g \varphi(x) \tag{2.19}$$

for any $x \in I$. Hence, we have $d(h_0, g) < \infty$ for all $g \in X$, that is, $\{g \in X \mid d(h_0, g) < \infty\} = X$. Now, Theorem 1.1(b) implies that y_0 is the unique continuous function with the property (2.4).

Finally, Theorem 1.1(c) implies that

$$d(y, y_0) \le \frac{1}{1 - KL} d(\Lambda y, y) \le \frac{1}{1 - KL}, \tag{2.20}$$

since inequality (2.2) means that $d(y, \Lambda y) \le 1$. In view of (2.7), we can conclude that the inequality (2.5) holds for all $x \in I$. □

In the previous theorem, we have investigated the Hyers-Ulam-Rassias stability of the Volterra integral equation (1.2) defined on compact domains. We will now prove the last theorem for the case of unbounded domains. More precisely, Theorem 2.1 is also true if I is replaced by an unbounded interval $(-\infty, a]$, $\mathbb{R}$, or $[a, \infty)$, as we see in the following theorem.

Theorem 2.2. *Let K and L be positive constants with $0 < KL < 1$ and let I denote either $(-\infty, a]$ or $\mathbb{R}$ or $[a, \infty)$ for a given real number a. Assume that $f : I \times \mathbb{C} \to \mathbb{C}$ is a continuous function which satisfies a Lipschitz condition (2.1) for all $x \in I$ and all $y, z \in \mathbb{C}$. If a continuous function $y : I \to \mathbb{C}$ satisfies inequality (2.2) for all $x \in I$ and for some $c \in I$, where $\varphi : I \to (0, \infty)$ is a continuous function satisfying (2.3) for any $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{C}$ which satisfies (2.4) and (2.5) for all $x \in I$.*

Proof. We will prove our theorem for the case $I = \mathbb{R}$. We can similarly prove our theorem for $I = (-\infty, a]$ or $I = [a, \infty)$.

For any $n \in \mathbb{N}$, we define $I_n = [c - n, c + n]$. According to Theorem 2.1, there exists a unique continuous function $y_{0,n} : I_n \to \mathbb{C}$ such that

$$y_{0,n}(x) = \int_c^x f(\tau, y_{0,n}(\tau))\, d\tau, \tag{2.21}$$

$$|\, y(x) - y_{0,n}(x) \,| \le \frac{1}{1 - KL} \varphi(x) \tag{2.22}$$

for all $x \in I_n$. The uniqueness of $y_{0,n}$ implies that if $x \in I_n$, then

$$y_{0,n}(x) = y_{0,n+1}(x) = y_{0,n+2}(x) = \cdots. \tag{2.23}$$

For any $x \in \mathbb{R}$, let us define $n(x) \in \mathbb{N}$ as

$$n(x) = \min\{n \in \mathbb{N} \mid x \in I_n\}. \tag{2.24}$$

Moreover, we define a function $y_0 : \mathbb{R} \to \mathbb{C}$ by

$$y_0(x) = y_{0,n(x)}(x), \tag{2.25}$$

and we assert that y_0 is continuous. For an arbitrary $x_1 \in \mathbb{R}$, we choose the integer $n_1 = n(x_1)$. Then, x_1 belongs to the interior of I_{n_1+1} and there exists an $\varepsilon > 0$ such that $y_0(x) = y_{0,n_1+1}(x)$ for all x with $x_1 - \varepsilon < x < x_1 + \varepsilon$. Since y_{0,n_1+1} is continuous at x_1, so is y_0. That is, y_0 is continuous at x_1 for any $x_1 \in \mathbb{R}$.

We will now show that y_0 satisfies (2.4) and (2.5) for all $x \in \mathbb{R}$. For an arbitrary $x \in \mathbb{R}$, we choose the integer $n(x)$. Then, it holds that $x \in I_{n(x)}$ and it follows from (2.21) that

$$y_0(x) = y_{0,n(x)}(x) = \int_c^x f(\tau, y_{0,n(x)}(\tau))\, d\tau = \int_c^x f(\tau, y_0(\tau))\, d\tau, \tag{2.26}$$

where the last equality holds true because $n(\tau) \le n(x)$ for any $\tau \in I_{n(x)}$ and it follows from (2.23) that

$$y_0(\tau) = y_{0,n(\tau)}(\tau) = y_{0,n(x)}(\tau). \tag{2.27}$$

Since $y_0(x) = y_{0,n(x)}(x)$ and $x \in I_{n(x)}$ for all $x \in \mathbb{R}$, (2.22) implies that

$$|\, y(x) - y_0(x) \,| = |\, y(x) - y_{0,n(x)}(x) \,| \le \frac{1}{1 - KL} \varphi(x). \tag{2.28}$$

Finally, we assert that y_0 is unique. Assume that $y_1 : \mathbb{R} \to \mathbb{C}$ is another continuous function which satisfies (2.4) and (2.5), with y_1 in place of y_0, for all $x \in \mathbb{R}$. Suppose x is an arbitrary real number. Since the restrictions $y_0|_{I_{n(x)}} (= y_{0,n(x)})$ and $y_1|_{I_{n(x)}}$ both satisfy (2.4) and (2.5) for all $x \in I_{n(x)}$, the uniqueness of $y_{0,n(x)} = y_0|_{I_{n(x)}}$ implies that

$$y_0(x) = y_0\,|_{I_{n(x)}}(x) = y_1\,|_{I_{n(x)}}(x) = y_1(x) \tag{2.29}$$

as required. □

Example 2.3. We introduce some examples for I and φ which satisfy the condition (2.3). Let α and ρ be constants with $\rho > 0$ and $\alpha > L$.

(a) If $I = [0,\infty)$, then the continuous function $\varphi(x) = \rho e^{\alpha x}$ satisfies the condition (2.3) with $c = 0$, for all $x \in I$.

(b) If $I = (-\infty,0]$, then the continuous function $\varphi(x) = \rho e^{-\alpha x}$ satisfies the condition (2.3) with $c = 0$, for any $x \in I$.

(c) If we let $I = \mathbb{R}$ and define

$$\varphi(x) = \begin{cases} \rho e^{\alpha x} & (\text{for } x \geq 0), \\ \rho e^{-\alpha x} & (\text{for } x < 0) \end{cases} \tag{2.30}$$

for all $x \in \mathbb{R}$, then the continuous function φ satisfies the condition (2.3) with $c = 0$, for all $x \in \mathbb{R}$.

3. Hyers-Ulam stability

In the following theorem, we prove the Hyers-Ulam stability of the Volterra integral equation (1.2) defined on any compact interval.

Theorem 3.1. *Given $a \in \mathbb{R}$ and $r > 0$, let $I(a;r)$ denote a closed interval $\{x \in \mathbb{R} \mid a - r \leq x \leq a + r\}$ and let $f : I(a;r) \times \mathbb{C} \to \mathbb{C}$ be a continuous function which satisfies a Lipschitz condition (2.1) for all $x \in I(a;r)$ and $y,z \in \mathbb{C}$, where L is a constant with $0 < Lr < 1$. If a continuous function $y : I(a;r) \to \mathbb{C}$ satisfies*

$$\left| y(x) - b - \int_a^x f(\tau, y(\tau))\,d\tau \right| \leq \theta \tag{3.1}$$

for all $x \in I(a;r)$ and for some $\theta \geq 0$, where b is a complex number, then there exists a unique continuous function $y_0 : I(a;r) \to \mathbb{C}$ such that

$$y_0(x) = b + \int_a^x f(\tau, y_0(\tau))\,d\tau, \tag{3.2}$$

$$|\,y(x) - y_0(x)\,| \leq \frac{\theta}{1 - Lr} \tag{3.3}$$

for all $x \in I(a;r)$.

Proof. Let us define a set

$$X = \{h : I(a;r) \to \mathbb{C} \mid h \text{ is continuous }\} \tag{3.4}$$

and introduce a generalized metric on X as follows:

$$d(g,h) = \inf\{C \in [0,\infty] \mid |g(x) - h(x)| \le C \ \forall x \in I(a;r)\}. \tag{3.5}$$

Then, analogously to the proof of Theorem 2.1, we can show that (X,d) is complete.

If we define an operator $\Lambda : X \to X$ by

$$(\Lambda h)(x) = b + \int_a^x f(\tau, h(\tau))\, d\tau \tag{3.6}$$

for all $x \in I(a;r)$, then the fundamental theorem of Calculus implies that $\Lambda h \in X$ for every $h \in X$ because Λh is continuously differentiable on $I(a;r)$.

We assert that Λ is strictly contractive on X. Given $g,h \in X$, let $C_{gh} \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le C_{gh}$, that is,

$$|g(x) - h(x)| \le C_{gh} \tag{3.7}$$

for any $x \in I(a;r)$. It then follows from (2.1) that

$$\begin{aligned} |(\Lambda g)(x) - (\Lambda h)(x)| &\le \left| \int_a^x |f(\tau, g(\tau)) - f(\tau, h(\tau))|\, d\tau \right| \le \left| \int_a^x L|g(\tau) - h(\tau)|\, d\tau \right| \\ &\le LC_{gh}|x - a| \le LC_{gh}r \end{aligned} \tag{3.8}$$

for all $x \in I(a;r)$, that is, $d(\Lambda g, \Lambda h) \le LrC_{gh}$. Hence, we conclude that $d(\Lambda g, \Lambda h) \le Lrd(g,h)$ for any $g,h \in X$ and we note that $0 < Lr < 1$.

Similarly as in the proof of Theorem 2.1, we can choose an $h_0 \in X$ with $d(\Lambda h_0, h_0) < \infty$. Hence, it follows from Theorem 1.1(a) that there exists a continuous function $y_0 : I(a;r) \to \mathbb{C}$ such that $\Lambda^n h_0 \to y_0$ in (X,d) as $n \to \infty$, and such that y_0 satisfies the Volterra integral equation (3.2) for any $x \in I(a;r)$.

By applying a similar argument of the proof of Theorem 2.1 to this case, we can show that $\{g \in X \mid d(h_0, g) < \infty\} = X$. Therefore, Theorem 1.1(b) implies that y_0 is a unique continuous function with the property (3.2). Furthermore, Theorem 1.1(c) implies that

$$|y(x) - y_0(x)| \le \frac{\theta}{1 - Lr} \tag{3.9}$$

for all $x \in I(a;r)$. □

Unfortunately, we could not prove the Hyers-Ulam stability of the integral equation defined on an infinite interval. So, it is an open problem whether the Volterra integral equation (1.2) has the Hyers-Ulam stability for the case of infinite intervals.

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