Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2007, Article ID 64363, 12 pages
doi:10.1155/2007/64363

Research Article

Convergence Theorem for Equilibrium Problems and Fixed Point Problems of Infinite Family of Nonexpansive Mappings

Yonghong Yao, Yeong-Cheng Liou, and Jen-Chih Yao

Received 17 March 2007; Accepted 20 August 2007

Recommended by Billy E. Rhoades

We introduce an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of infinite nonexpansive mappings in a Hilbert space. We prove a strong-convergence theorem under mild assumptions on parameters.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $h : C \times C \to R$ be an equilibrium bifunction, that is, $h(u,u) = 0$ for every $u \in C$. Then one can define the equilibrium problem that is to find an element $u \in C$ such that

$$\mathrm{EP}(h) : h(u,v) \geq 0 \quad \forall v \in C. \tag{1.1}$$

Denote the set of solutions of $\mathrm{EP}(h)$ by $\mathrm{SEP}(h)$. This problem contains fixed point problems, optimization problems, variational inequality problems, and Nash equilibrium problems as special cases, see [1]. Some methods have been proposed to solve the equilibrium problem, please consult [2–4].

Recently, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{SEP}(h) \neq \emptyset$ and proved a strong convergence theorem. Motivated by the idea of Combettes and Hirstoaga, very recently, Takahashi and Takahashi [4] introduced a new iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Their results extend and improve the corresponding results announced by Combettes and Hirstoaga [2], Moudafi [5], Wittmann [6], and Tada and Takahashi [7].

In this paper, motivated and inspired by Combettes and Hirstoaga [2] and Takahashi and Takahashi [4], we introduce an iterative scheme for finding a common element of the set of solutions of EP(h) and the set of fixed points of infinite nonexpansive mappings in a Hilbert space. We obtain a strong convergence theorem which improves and extends the corresponding results of [2, 4].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Then for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \le \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \Longleftrightarrow \langle x - x^*, x^* - y\rangle \ge 0 \quad \forall y \in C. \tag{2.1}$$

Recall that a mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. Denote the set of fixed points of T by $F(T)$. It is well known that if C is a bounded closed convex and $T : C \to C$ is nonexpansive, then $F(T) \ne \emptyset$; see, for instance, [8]. We call a mapping $f : H \to H$ contractive if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \le \alpha\|x - y\|$ for all $x, y \in H$.

For an equilibrium bifunction $h : C \times C \to R$, we call h satisfying condition (A) if h satisfies the following three conditions:

(i) h is monotone, that is, $h(x,y) + h(y,x) \le 0$ for all $x, y \in C$;

(ii) for each $x, y, z \in C$, $\lim_{t\downarrow 0} h(tz + (1-t)x, y) \le h(x,y)$;

(iii) for each $x \in C$, $y \mapsto h(x,y)$ is convex and lower semicontinuous.

If an equilibrium bifunction $h : C \times C \to R$ satisfies condition (A), then we have the following two important results. You can find the first lemma in [1] and the second one in [2].

Lemma 2.1. *Let C be a nonempty closed convex subset of H and let h be an equilibrium bifunction of $C \times C$ into R, satisfying condition (A). Let $r > 0$ and $x \in H$. Then there exists $y \in C$ such that*

$$h(y,z) + \frac{1}{r}\langle z - y, y - x\rangle \ge 0 \quad \forall z \in C. \tag{2.2}$$

Lemma 2.2. *Assume that h satisfies the same assumptions as Lemma 2.1. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \left\{ y \in C : h(y,z) + \frac{1}{r}\langle z - y, y - x\rangle \ge 0,\ \forall z \in C \right\} \tag{2.3}$$

for all $y \in H$. Then the following holds:

(1) S_r *is single-valued and* S_r *is firmly nonexpansive, that is, for any* $x, y \in H$,

$$\|S_r x - S_r y\|^2 \le \langle S_r x - S_r y, x - y\rangle; \tag{2.4}$$

(2) $F(S_r) = \mathrm{SEP}(h)$ *and* $\mathrm{SEP}(h)$ *is closed and convex.*
We also need the following lemmas for proving our main results.

Lemma 2.3 (see [9]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.4 (see [10]). *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1} \le (1-\gamma_n)a_n + \delta_n$, *where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that* $\sum_{n=1}^{\infty}\gamma_n = \infty$ *and* $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

3. Iterative scheme and strong convergence theorems

In this section, we first introduce our iterative scheme. Consequently, we will establish strong convergence theorems for this iteration scheme. To be more specific, let $T_1, T_2, \ldots$ be infinite mappings of C into C and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i \in N$. For any $n \in N$, define a mapping W_n of C into C as follows:

$$\begin{aligned}
U_{n,n+1} &= I, \\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\
&\ \ \vdots \\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\
&\ \ \vdots \\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{3.1}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and λ_n, $\lambda_{n-1}, \ldots, \lambda_1$; see [11].

Now we introduce the following iteration scheme: Let f be a contraction of H into itself with coefficient $\alpha \in (0,1)$ and given $x_0 \in H$ arbitrarily. Suppose the sequences $\{x_n\}_{n=1}^{\infty}$

and $\{y_n\}_{n=1}^{\infty}$ are generated iteratively by

$$\begin{gathered} h(y_n,x)+\frac{1}{r_n}\langle x-y_n,y_n-x_n\rangle\geq 0,\quad \forall x\in C,\\ x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n W_n y_n, \end{gathered} \tag{3.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$, $\{r_n\}$ is a real sequence in $(0,\infty)$, h is an equilibrium bifunction, and W_n is the W-mapping defined by (3.1).

We have the following crucial conclusions concerning W_n. You can find them in [12, 13]. Now we only need the following similar version in Hilbert spaces.

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T_1, $T_2,\ldots$ be nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty}F(T_i)$ is nonempty, and let $\lambda_1,\lambda_2,\ldots$ be real numbers such that $0<\lambda_i\leq b<1$ for any $i\in N$. Then for every $x\in C$ and $k\in N$, the limit $\lim_{n\to\infty}U_{n,k}x$ exists.*

Remark 3.2. From Lemma 3.1, we have that if C is bounded, then for all $\varepsilon>0$, there exists a common positive integer number N_0 such that for $n>N_0$, $\|U_{n,k}x-U_k(x)\|<\varepsilon$ for all $x\in C$. Indeed, by the similar argument to Lemma 3.2 in [13], let $w\in\bigcap_{n=1}^{\infty}F(T_n)$. Since C is bounded, there exists a constant $M>0$ such that $\|x-w\|\leq M$ for all $x\in C$. Fix $k\in N$. Then for all $x\in C$ and any $n\in N$ with $n\geq k$, we have $\|U_{n+1,k}x-U_{n,k}x\|\leq 2(\prod_{i=k}^{n+1}\lambda_i)\|x-w\|\leq 2M(\prod_{i=k}^{n+1}\lambda_i)$.

Let $\varepsilon>0$. Then there exists $n_0\in N$ with $n_0\geq k$ such that for all $x\in C$, $b^{n_0-k+2}<\varepsilon(1-b)/2M$. So for all $x\in C$ and every m, n with $m>n>n_0$, we have

$$\begin{aligned} &||U_{m,k}x-U_{n,k}x||\\ &\quad\leq\sum_{j=n}^{m-1}||U_{j+1,k}x-U_{j,k}x||\leq\sum_{j=n}^{m-1}\left\{2\left(\prod_{i=k}^{j+1}\lambda_i\right)\|x-w\|\right\}\\ &\quad\leq 2M\sum_{j=n}^{m-1}b^{j-k+2}\leq\frac{2Mb^{n-k+2}}{1-b}<\varepsilon. \end{aligned} \tag{3.3}$$

Remark 3.3. Using Lemma 3.1, one can define a mapping W of C into C as $Wx=\lim_{n\to\infty}W_nx=\lim_{n\to\infty}U_{n,1}x$ for every $x\in C$. Such a W is called the W-mapping generated by $T_1,T_2,\ldots$ and $\lambda_1,\lambda_2,\ldots$. We observe that if $\{x_n\}$ is a bounded sequence in C, then we have

$$\lim_{n\to\infty}||Wx_n-W_nx_n||=0. \tag{3.4}$$

Indeed, from Remark 3.1, we have: for any $\varepsilon>0$, there is n_0 such that $\|Wx-W_nx\|\leq\varepsilon$ for all $x\in\{x_n\}$ and for all $n\geq n_0$. In particular, $\|Wx_n-W_nx_n\|\leq\varepsilon$ for all $n\geq n_0$. Consequently, $\lim_{n\to\infty}\|Wx_n-W_nx_n\|=0$, as claimed.

Throughout this paper, we will assume that $0<\lambda_i\leq b<1$ for every $i\in N$.

Lemma 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T_1, $T_2,\dots$ be nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty}F(T_i)$ is nonempty, and let $\lambda_1,\lambda_2,\dots$ be real numbers such that $0<\lambda_i\leq b<1$ for any $i\in N$. Then $F(W)=\bigcap_{i=1}^{\infty}F(T_i)$.*

Now we state and prove our main results.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $h: C\times C\to R$ be an equilibrium bifunction satisfying condition (A) and let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{SEP}(h)\neq\emptyset$. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$ and $\{r_n\}\subset(0,\infty)$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(ii) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(iii) $\liminf_{n\to\infty}r_n>0$ *and* $\lim_{n\to\infty}(r_{n+1}-r_n)=0$.

Let f be a contraction of H into itself and given $x_0\in H$ arbitrarily. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated iteratively by (3.2) converge strongly to $x^*\in\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)$, where $x^*=P_{\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)}f(x^*)$.

Proof. Let $Q=P_{\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)}$. Note that f is a contraction mapping with coefficient $\alpha\in(0,1)$. Then $\|Qf(x)-Qf(y)\|\leq\|f(x)-f(y)\|\leq\alpha\|x-y\|$ for all $x,y\in H$. Therefore, Qf is a contraction of H into itself, which implies that there exists a unique element $x^*\in H$ such that $x^*=Qf(x^*)$. At the same time, we note that $x^*\in C$.

Let $p\in\bigcap_{i=1}^{\infty}F(T_i)\cap\mathrm{SEP}(h)$. From the definition of S_r, we note that $y_n=S_{r_n}x_n$. It follows that $\|y_n-p\|=\|S_{r_n}x_n-S_{r_n}p\|\leq\|x_n-p\|$. Next, we prove that $\{x_n\}$ and $\{y_n\}$ are bounded. From (3.1) and (3.2), we obtain

$$\begin{aligned}||x_{n+1}-p||&\leq\alpha_n||f(x_n)-p||+\beta_n||x_n-p||+\gamma_n||W_ny_n-p||\\&\leq\alpha_n(||f(x_n)-f(p)||+||f(p)-p||)+\beta_n||x_n-p||+\gamma_n||y_n-p||\\&\leq\alpha_n(\alpha||x_n-p||+||f(p)-p||)+(1-\alpha_n)||x_n-p||\\&\leq\max\left\{||x_0-p||,\frac{1}{1-\alpha}||f(p)-p||\right\}.\end{aligned}\tag{3.5}$$

Therefore, $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{W_nx_n\}$, and $\{f(x_n)\}$ are all bounded. We shall use M to denote the possible different constants appearing in the following reasoning.

Setting $x_{n+1}=\beta_nx_n+(1-\beta_n)z_n$ for all $n\geq 0$, we have that

$$\begin{aligned}z_{n+1}-z_n&=\frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}-\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}\\&=\frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-f(x_n))+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n)\\&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(W_{n+1}y_{n+1}-W_ny_n)+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)W_ny_n.\end{aligned}\tag{3.6}$$

So we have

$$\begin{aligned}||z_{n+1}-z_n|| \le{}& \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}||x_{n+1}-x_n|| \\ &+ \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(||f(x_n)||+||W_ny_n||) \\ &+ \frac{\gamma_{n+1}}{1-\beta_{n+1}}||W_{n+1}y_{n+1}-W_ny_n||.\end{aligned} \tag{3.7}$$

Since T_i and $U_{n,i}$ are nonexpansive from (3.1), we have

$$\begin{aligned}||W_{n+1}y_n-W_ny_n|| &= ||\lambda_1T_1U_{n+1,2}y_n-\lambda_1T_1U_{n,2}y_n|| \\ &\le \lambda_1||U_{n+1,2}y_n-U_{n,2}y_n|| \\ &\le \lambda_1\lambda_2||U_{n+1,3}y_n-U_{n,3}y_n|| \\ &\le \cdots \\ &\le \lambda_1\lambda_2\cdots\lambda_n||U_{n+1,n+1}y_n-U_{n,n+1}y_n|| \\ &\le M\prod_{i=1}^{n}\lambda_i,\end{aligned} \tag{3.8}$$

and hence

$$\begin{aligned}||W_{n+1}y_{n+1}-W_ny_n|| &\le ||W_{n+1}y_{n+1}-W_{n+1}y_n||+||W_{n+1}y_n-W_ny_n|| \\ &\le ||y_{n+1}-y_n||+M\prod_{i=1}^{n}\lambda_i.\end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.7), we have

$$\begin{aligned}||z_{n+1}-z_n|| \le{}& \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}||x_{n+1}-x_n||+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(||f(x_n)||+||W_ny_n||) \\ &+\frac{\gamma_{n+1}}{1-\beta_{n+1}}||y_{n+1}-y_n||+\frac{M\gamma_{n+1}}{1-\beta_{n+1}}\prod_{i=1}^{n}\lambda_i.\end{aligned} \tag{3.10}$$

On the other hand, from $y_n = S_{r_n}x_n$ and $y_{n+1} = S_{r_{n+1}}x_{n+1}$, we have

$$h(y_n,x)+\frac{1}{r_n}\langle x-y_n,y_n-x_n\rangle \ge 0 \quad \forall x\in C, \tag{3.11}$$

$$h(y_{n+1},x)+\frac{1}{r_{n+1}}\langle x-y_{n+1},y_{n+1}-x_{n+1}\rangle \ge 0 \quad \forall x\in C. \tag{3.12}$$

Putting $x = y_{n+1}$ in (3.11) and $x = y_n$ in (3.12), we have

$$h(y_n, y_{n+1}) + \frac{1}{r_n}\langle y_{n+1} - y_n, y_n - x_n\rangle \geq 0, \tag{3.13}$$

$$h(y_{n+1}, y_n) + \frac{1}{r_{n+1}}\langle y_n - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.14}$$

From the monotonicity of h, we have

$$h(y_n, y_{n+1}) + h(y_{n+1}, y_n) \leq 0. \tag{3.15}$$

So from (3.13), we can conclude that

$$\left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0, \tag{3.16}$$

and hence

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - x_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1})\right\rangle \geq 0. \tag{3.17}$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, we may assume that there exists a real number τ such that $r_n > \tau > 0$ for all $n \in N$. Then we have

$$\begin{aligned} ||y_{n+1} - y_n||^2 &\leq \left\langle y_{n+1} - y_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(y_{n+1} - x_{n+1})\right\rangle \\ &\leq ||y_{n+1} - y_n||\left\{||x_{n+1} - x_n|| + \left|1 - \frac{r_n}{r_{n+1}}\right| ||y_{n+1} - x_{n+1}||\right\}, \end{aligned} \tag{3.18}$$

and hence

$$||y_{n+1} - y_n|| \leq ||x_{n+1} - x_n|| + \frac{M}{\tau}|r_{n+1} - r_n|. \tag{3.19}$$

Substituting (3.19) into (3.10), we have

$$\begin{aligned} ||z_{n+1} - z_n|| \leq{}& \frac{\alpha\alpha_{n+1}}{1 - \beta_{n+1}}||x_{n+1} - x_n|| + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right|(||f(x_n)|| + ||W_n y_n||) \\ &+ \frac{\gamma_{n+1}}{1 - \beta_{n+1}}||x_{n+1} - x_n|| + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \times \frac{M}{\tau}|r_{n+1} - r_n| + \frac{M\gamma_{n+1}}{1 - \beta_{n+1}}\prod_{i=1}^{n}\lambda_i \\ \leq{}& ||x_{n+1} - x_n|| + \left|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\right|(||f(x_n) + ||W_n y_n||) \\ &+ \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \times \frac{M}{\tau}|r_{n+1} - r_n| + \frac{M\gamma_{n+1}}{1 - \beta_{n+1}}\prod_{i=1}^{n}\lambda_i. \end{aligned} \tag{3.20}$$

This together with $\alpha_n \to 0$ and $r_{n+1} - r_n \to 0$ imply that $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Hence by Lemma 2.3, we obtain $\|z_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty}||x_{n+1} - x_n|| = \lim_{n\to\infty}(1 - \beta_n)||z_n - x_n|| = 0. \tag{3.21}$$

From (3.19) and $\lim_{n\to\infty}(r_{n+1}-r_n)=0$, we have $\lim_{n\to\infty}\|y_{n+1}-y_n\|=0$. Since $x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n W_n y_n$, we have

$$\begin{aligned}
\|x_n - W_n y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n y_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n) - W_n y_n\| + \beta_n\|x_n - W_n y_n\|,
\end{aligned} \tag{3.22}$$

that is,

$$\|x_n - W_n y_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - W_n y_n\|. \tag{3.23}$$

Hence we have $\lim_{n\to\infty}\|x_n - W_n y_n\|=0$. For $p\in\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{SEP}(h)$, note that S_r is firmly nonexpansive. Then we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|S_{r_n}x_n - S_{r_n}p\|^2 \\
&\le \langle S_{r_n}x_n - S_{r_n}p, x_n - p\rangle \\
&= \langle y_n - p, x_n - p\rangle \\
&= \frac{1}{2}\left(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2\right),
\end{aligned} \tag{3.24}$$

and hence

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2. \tag{3.25}$$

Therefore, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|W_n y_n - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\
&\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 \\
&\quad + \gamma_n\left(\|x_n - p\|^2 - \|x_n - y_n\|^2\right) \\
&\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n\|x_n - y_n\|^2.
\end{aligned} \tag{3.26}$$

Then we have

$$\begin{aligned}
\gamma_n\|x_n - y_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad \times (\|x_n - p\| - \|x_{n+1} - p\|) \\
&\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned} \tag{3.27}$$

It is easily seen that $\liminf_{n\to\infty}\gamma_n > 0$. So we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.28}$$

From $\|W_n y_n - y_n\| \le \|W_n y_n - x_n\| + \|x_n - y_n\|$, we also have $\|W_n y_n - y_n\|\to 0$. At that same time, we note that

$$\|W y_n - y_n\| \le \|W y_n - W_n y_n\| + \|W_n y_n - y_n\|. \tag{3.29}$$

It follows from (3.29) and Remark 3.2 that $\lim_{n\to\infty}\|Wy_n - y_n\| = 0$. Next, we show that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle \leq 0, \tag{3.30}$$

where $x^* = P_{F(W)\cap \mathrm{SEP}(h)}f(x^*)$. First, we can choose a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that

$$\lim_{j\to\infty}\langle f(x^*) - x^*, y_{n_j} - x^*\rangle = \limsup_{n\to\infty}\langle f(x^*) - x^*, y_n - x^*\rangle. \tag{3.31}$$

Since $\{y_{n_j}\}$ is bounded, there exists a subsequence $\{y_{n_{ji}}\}$ of $\{y_{n_j}\}$, which converges weakly to w. Without loss of generality, we can assume that $y_{n_j}\to w$ weakly. From $\|Wy_n - y_n\|\to 0$, we obtain $Wy_{n_j}\to w$ weakly. Now we show $w \in \mathrm{SEP}(h)$.

By $y_n = S_{r_n}x_n$, we have $h(y_n, x) + (1/r_n)\langle x - y_n, y_n - x_n\rangle \geq 0$ for all $x \in C$. From the monotonicity of h, we have $(1/r_n)\langle x - y_n, y_n - x_n\rangle \geq -h(y_n, x) \geq h(x, y_n)$, and hence

$$\left\langle x - y_{n_j}, \frac{y_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle \geq h(x, y_{n_j}). \tag{3.32}$$

Since $(y_{n_j} - x_{n_j})/r_{n_j}\to 0$ and $y_{n_j}\to w$ weakly, from the lower semicontinuity of $h(x,y)$ on the second variable y, we have $h(x,w) \leq 0$ for all $x \in C$. For t with $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)w$. Since $x \in C$ and $w \in C$, we have $x_t \in C$, and hence $h(x_t, w) \leq 0$. So from the convexity of equilibrium bifunction $h(x,y)$ on the second variable y, we have

$$0 = h(x_t, x_t) \leq th(x_t, x) + (1-t)h(x_t, w) \leq th(x_t, x). \tag{3.33}$$

Hence $h(x_t, x) \geq 0$. Therefore, we have $h(w, x) \geq 0$ for all $x \in C$, and hence $w \in \mathrm{SEP}(h)$.

We will show $w \in F(W)$. Assume $w \notin F(W)$. Since $y_{n_j}\to w$ weakly and $w \neq Ww$, from Opial's condition, we have

$$\begin{aligned}\liminf_{j\to\infty}\|y_{n_j} - w\| &< \liminf_{j\to\infty}\|y_{n_j} - Ww\| \\ &\leq \liminf_{j\to\infty}\left(\|y_{n_j} - Wy_{n_j}\| + \|Wy_{n_j} - Ww\|\right) \\ &\leq \liminf_{j\to\infty}\|y_{n_j} - w\|.\end{aligned} \tag{3.34}$$

This is a contradiction. So we get $w \in F(W) = \bigcap_{i=1}^{\infty}F(T_i)$. Therefore, $w \in \bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{SEP}(h)$. Since $x^* = P_{\bigcap_{i=1}^{\infty}F(T_i)\cap \mathrm{SEP}(h)}f(x^*)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle &= \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_j} - x^*\rangle \\ &= \lim_{j\to\infty}\langle f(x^*) - x^*, y_{n_j} - x^*\rangle \\ &= \langle f(x^*) - x^*, w - x^*\rangle \leq 0.\end{aligned} \tag{3.35}$$

First, we prove that $\{x_n\}$ converges strongly to $x^* \in \bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{SEP}(h)$. From (3.2), we have

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq ||\beta_n(x_n - x^*) + \gamma_n(W_n y_n - x^*)||^2 + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\
&\leq \{\beta_n||x_n - x^*|| + \gamma_n||W_n y_n - x^*)||\}^2 + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\quad + 2\alpha_n\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\
&\leq \{\beta_n||x_n - x^*|| + \gamma_n||y_n - x^*||\}^2 + 2\alpha\alpha_n||x_n - x^*||||x_{n+1} - x^*|| \\
&\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\leq (1-\alpha_n)^2||x_n - x^*||^2 + \alpha\alpha_n(||x_{n+1} - x^*||^2 + ||x_n - x^*||^2) \\
&\quad + 2\alpha_n\langle f(x^*) - x^*, x_{n+1} - x^*\rangle,
\end{aligned} \tag{3.36}$$

which implies that

$$\begin{aligned}
||x_{n+1} - x^*||^2 &\leq \frac{(1-\alpha_n)^2 + \alpha\alpha_n}{1 - \alpha\alpha_n}||x_n - x^*||^2 + \frac{2\alpha_n}{1 - \alpha\alpha_n}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&= \frac{1 - 2\alpha_n + \alpha\alpha_n}{1 - \alpha\alpha_n}||x_n - x^*||^2 + \frac{\alpha_n^2}{1 - \alpha\alpha_n}||x_n - x^*||^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha\alpha_n}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle \\
&\leq \left\{1 - \frac{2(1-\alpha)\alpha_n}{1 - \alpha\alpha_n}\right\}||x_n - x^*||^2 + \frac{2(1-\alpha)\alpha_n}{1 - \alpha\alpha_n} \\
&\quad \times \left\{\frac{M\alpha_n}{2(1-\alpha)} + \frac{1}{1-\alpha}\langle f(x^*) - x^*, x_{n+1} - x^*\rangle\right\} \\
&= (1 - \varphi_n)||x_n - x^*||^2 + \phi_n\varphi_n,
\end{aligned} \tag{3.37}$$

where $\varphi_n = 2(1-\alpha)\alpha_n/(1 - \alpha\alpha_n)$ and $\phi_n = M\alpha_n/2(1-\alpha) + 1/(1-\alpha)\langle f(x^*) - x^*, x_{n+1} - x^*\rangle$. It is easily seen that $\sum_{n=0}^{\infty}\varphi_n = \infty$ and $\limsup_{n\to\infty}\phi_n \leq 0$. Now applying Lemma 2.4 and (3.35) to (3.37), we conclude that $x_n \to x^*$ $(n \to \infty)$. Consequently, from (3.28), we have $y_n \to x^*$ $(n \to \infty)$. This completes the proof. □

Corollary 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $h : C \times C \to R$ be an equilibrium bifunction satisfying condition (A) such that* $\mathrm{SEP}(h) \neq \emptyset$. *Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in* $(0,1)$ *such that $\alpha_n + \beta_n + \gamma_n = 1$ and $\{r_n\} \subset (0,\infty)$ is a real sequence. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}(r_{n+1} - r_n) = 0$.

Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated iteratively by

$$\begin{gathered} h(y_n,x) + \frac{1}{r_n}\langle x - y_n, y_n - x_n\rangle \geq 0, \quad \forall x \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n. \end{gathered} \tag{3.38}$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by (3.38) converge strongly to $x^* \in \mathrm{SEP}(h)$, where $x^* = P_{\mathrm{SEP}(h)} f(x^*)$.

Proof. Take $T_i x = x$ for all $i = 1,2,\ldots$ and for all $x \in C$ in (3.1). Then $W_n x = x$ for all $x \in C$. The conclusion follows from Theorem 3.1. This completes the proof. □

Corollary 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into C such that $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Let f be a contraction of H into itself and given $x_0 \in H$ arbitrarily. Let $\{x_n\}$ be a sequence generated iteratively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C x_n. \tag{3.39}$$

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\bigcap_{i=1}^{\infty} F(T_i)} f(x^*)$.

Proof. Set $h(x,y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in N$. Then we have $y_n = P_C x_n$. From (3.2), we have

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n P_C x_n. \tag{3.40}$$

Then the conclusion follows from Theorem 3.5. This completes the proof. □

Acknowledgments

The authors thank the referee for his/her helpful comments and suggestions, which improved the presentation of this manuscript. The research was partially supported by the Grant NSC 96-2221-E-230-003.

References

[1] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[2] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[3] S. D. Flåm and A. S. Antipin, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, no. 1, pp. 29–41, 1997.

[4] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[5] A. Moudafi, "Viscosity approximation methods for fixed-points problems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 1, pp. 46–55, 2000.
[6] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.
[7] A. Tada and W. Takahashi, "Strong convergence theorem for an equilibrium problem and a nonexpansive mapping," in *Nonlinear Analysis and Convex Analysis*, W. Takahashi and T. Tanaka, Eds., pp. 609–617, Yokohama, Yokohama, Japan, 2007.
[8] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama, Yokohama, Japan, 2000.
[9] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.
[10] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.
[11] W. Takahashi, "Weak and strong convergence theorems for families of nonexpansive mappings and their applications," *Annales Universitatis Mariae Curie-Skłodowska. Sectio A*, vol. 51, no. 2, pp. 277–292, 1997.
[12] W. Takahashi and K. Shimoji, "Convergence theorems for nonexpansive mappings and feasibility problems," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1463–1471, 2000.
[13] K. Shimoji and W. Takahashi, "Strong convergence to common fixed points of infinite nonexpansive mappings and applications," *Taiwanese Journal of Mathematics*, vol. 5, no. 2, pp. 387–404, 2001.

Yonghong Yao: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
Email address: yuyanrong@tjpu.edu.cn

Yeong-Cheng Liou: Department of Information Management, Cheng Shiu University,
Kaohsiung 833, Taiwan
Email address: simplex_liou@hotmail.com

Jen-Chih Yao: Department of Applied Mathematics, National Sun Yat-Sen University,
Kaohsiung 804, Taiwan
Email address: yaojc@math.nsysu.edu.tw