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Research Article

Approximating Common Fixed Points of Lipschitzian Semigroup in Smooth Banach Spaces

Shahram Saeidi

Department of Mathematics, University of Kurdistan, Sanandaj 416, Kurdistan 66196-64583, Iran

Correspondence should be addressed to Shahram Saeidi, sh.saeidi@uok.ac.ir

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Let S be a left amenable semigroup, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty compact convex subset C of a smooth Banach space E into C with a uniform Lipschitzian condition, let $\{\mu_n\}$ be a strongly left regular sequence of means defined on an $\mathcal{S}$-stable subspace of $l^\infty(S)$, let f be a contraction on C, and let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, for all n. Let $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n$, for all $n \geq 1$. Then, under suitable hypotheses on the constants, we show that $\{x_n\}$ converges strongly to some z in $F(\mathcal{S})$, the set of common fixed points of $\mathcal{S}$, which is the unique solution of the variational inequality $\langle (f - I)z, J(y - z)\rangle \leq 0$, for all $y \in F(\mathcal{S})$.

1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. A mapping $T : C \to C$ is said to be

(i) *Lipschitzian* with Lipschitz constant $l > 0$ if

$$\|Tx - Ty\| \leq l\|x - y\|, \quad \forall x, y \in C; \tag{1.1}$$

(ii) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C; \tag{1.2}$$

(iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

Halpern [1] introduced the following iterative scheme for approximating a fixed point of a nonexpansive mapping T on C:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)Tx_n, \quad n = 1,2,\ldots, \tag{1.4}$$

where $x_1 = x$ is an arbitrary point in C and $\{\alpha_n\}$ is a sequence in $[0,1]$. Strong convergence of Halpern type iterative sequence has been widely studied: Wittmann [2] discussed such a sequence in a Hilbert space. Shioji and Takahashi [3] (see also [4]) extended Wittmann's result and proved strong convergence of $\{x_n\}$ defined by (1.4) in a uniformly convex Banach space with a uniformly Gateaux differentiable norm.

In particular, Xu [5] proposed the following viscosity iterative process (originally due to Moudafi [6]) in a uniformly smooth Banach space:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \quad n = 1,2,\ldots, \tag{1.5}$$

where, $f : C \to C$ is a contraction, and proved, under appropriate conditions, $\{x_n\}$ converges to a fixed point of T which is a solution of a variational inequality. Recently, many papers have been devoted to algorithms for finding such solutions, see, for example, [7–9].

It is an interesting problem to extend the above results to the nonexpansive semigroup case [10–18]. Lau, Miyake and Takahashi [19] considered the following iteration process;

$$x_{n+1} = \alpha_n x + (1-\alpha_n)T(\mu_n)x_n, \quad n = 1,2,\ldots, \tag{1.6}$$

for a semigroup $\mathcal{S} = \{T(s) : s \in S\}$ of nonexpansive mappings on a compact convex subset C of a smooth and strictly convex Banach space with respect to a left regular sequence $\{\mu_n\}$ of means defined on an appropriate invariant subspace of $l^\infty(S)$; for some related results we refer the readers to [20, 21].

The iterative methods for approximation of fixed points of asymptotically nonexpansive mappings have been studied by authors (see, e.g., [22–32] and references therein).

For a semigroup S, we can define a partial preordering $\prec$ on S by $a \prec b$ if and only if $aS \supset bS$. If S is a *left reversible semigroup* (i.e., $aS \cap bS \neq \emptyset$ for $a,b \in S$), then it is a directed set. (Indeed, for every $a,b \in S$, applying $aS \cap bS \neq \emptyset$, there exist $a',b' \in S$ with $aa' = bb'$; by taking $c = aa' = bb'$, we have $cS \subseteq aS \cap bS$, and then $a \prec c$ and $b \prec c$.)

If a semigroup S is left amenable, then S is left reversible [33].

Definition 1.1. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of a left reversible semigroup S as Lipschitzian mappings on C with Lipschitz constants $\{k(s) : s \in S\}$. We will say that $\mathcal{S}$ is an *asymptotically nonexpansive semigroup* on C, if there holds the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ on the Lipschitz constants. (Note that a left reversible semigroup is a directed set.)

It is worth mentioning that there is a notion of asymptotically nonexpansive defined dependent on left ideals in a semigroup in [34, 35].

In this paper, motivated by (1.5), (1.6) and the above-mentioned results, we introduce the following viscosity iterative scheme

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n, \quad \forall n \geq 1, \tag{1.7}$$

for an asymptotically nonexpansive semigroup $\mathcal{S} = \{T(s) : s \in S\}$ on a compact convex subset C of a smooth Banach space E with respect to a left regular sequence $\{\mu_n\}$ of means defined on an appropriate invariant subspace of $l^\infty(S)$, where f is a contraction on C, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$, for all n. Then, under appropriate conditions on constants, we prove that the sequence $\{x_n\}$ converges strongly to some z in $F(\mathcal{S})$, the set of common fixed points of $\mathcal{S}$, which is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z)\rangle \leq 0, \quad \forall y \in F(\mathcal{S}). \tag{1.8}$$

It is remarked that we have not assumed E to be strictly convex and our results are new even for nonexpansive mappings. Moreover, our results extend many previous results (e.g., [11, 19]).

2. Preliminaries

Let E be a Banach space and let E^* be the topological dual of E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^*\rangle$ or $x^*(x)$. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x^*\|^2 = \|x\|^2\}. \tag{2.1}$$

Using the Hahn-Banach theorem, it immediately follows that $J(x) \neq \varnothing$ for each $x \in E$. A Banach space E is said to be smooth if the duality mapping J of E is single valued. We know that if E is smooth, then J is norm to weak-star continuous; see [20, 21].

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real valued functions on S with supremum norm. For each $s \in S$, we define l_s and r_s on $l^\infty(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$. Let X be a subspace of $l^\infty(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp., right invariant), that is, $l_s(X) \subset X$ (resp., $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp., right invariant) if $\mu(l_s f) = \mu(f)$ (resp., $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp., right) *amenable* if X has a left (resp., right) invariant mean. X is amenable if X is both left and right amenable. A net $\{\mu_\alpha\}$ of means on X is said to be *strongly left regular* if

$$\lim_\alpha \|l_s^* \mu_\alpha - \mu_\alpha\| = 0, \tag{2.2}$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s. Let C be a nonempty closed and convex subset of E. Throughout this paper, S will always denote a semigroup with an identity e. S is called left reversible if any two right ideals in S have nonvoid intersection, that is, $aS \cap bS \neq \varnothing$ for $a, b \in S$. In this case, we can define a partial ordering $\prec$ on S by $a \prec b$ if and only if $aS \supset bS$. It is easy too see $t \prec ts$, $(\forall t, s \in S)$. Further, if $t \prec s$ then $pt \prec ps$ for all $p \in S$. If a semigroup S is left amenable, then S is left reversible. But the converse is false.

$\mathcal{S} = \{T(s) : s \in S\}$ is called a representation of S as Lipschitzian mappings on C if for each $s \in S$, the mapping $T(s)$ is Lipschitzian mapping on C with Lipschitz constant $k(s)$, and $T(st) = T(s)T(t)$ for $s,\ t \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, and by C_a the set of almost periodic elements in C, that is, all $x \in C$ such that $\{T(s)x : s \in S\}$ is relatively compact in the norm topology of E. We will call a subspace X of $l^\infty(S)$, *$\mathcal{S}$-stable* if the functions $s \mapsto \langle T(s)x, x^*\rangle$ and $s \mapsto \|T(s)x - y\|$ on S are in X for all $x, y \in C$ and $x^* \in E^*$. We know that if μ is a mean on X and if for each $x^* \in E^*$ the function $s \mapsto \langle T(s)x, x^*\rangle$ is contained in X and C is weakly compact, then there exists a unique point x_0 of E such that

$$\mu_s\langle T(s)x, x^*\rangle = \langle x_0, x^*\rangle, \tag{2.3}$$

for each $x^* \in E^*$. We denote such a point x_0 by $T(\mu)x$. Note that $T(\mu)z = z$, for each $z \in F(\mathcal{S})$; see [36–38]. Let D be a subset of B where B is a subset of a Banach space E and let P be a retraction of B onto D. Then P is said to be *sunny* [39] if for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$,

$$P(Px + t(x - Px)) = Px. \tag{2.4}$$

A subset D of B is said to be a sunny nonexpansive retract of B if there exists a sunny nonexpansive retraction P of B onto D. We know that if E is smooth and P is a retraction of B onto D, then P is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$,

$$\langle x - Px, J(z - Px)\rangle \leq 0. \tag{2.5}$$

For more details see [20, 21].

We will need the following lemma, which will appear in [32].

Lemma 2.1. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty weakly compact convex subset C of a Banach space E into C, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ on the Lipschitz constants of the mappings. Let X be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing 1, and μ be a left invariant mean on X. Then $F(\mathcal{S}) = F(T(\mu)) \cap C_a$.*

Corollary 2.2. *Let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. If $z \in C_a$ and $\liminf_{n\to\infty}\|T(\mu_n)z - z\| = 0$, then z is a common fixed point for $\mathcal{S}$.*

Proof. From $\liminf_{n\to\infty}\|T(\mu_n)z - z\| = 0$, there exists a subsequence $\{T(\mu_{n_k})z\}$ of $\{T(\mu_n)z\}$ that converges strongly to z. Since the set of means on X is compact in the weak-star topology, there exists a subnet $\{\mu_{n_{k_\alpha}} : \alpha \in \Lambda\}$ of $\{\mu_{n_k}\}$ such that $\{\mu_{n_{k_\alpha}}\}$ converges to μ in the weak-star topology. Then, it is easy to show that μ is a left invariant mean on X. On the other hand, for each $x^* \in E^*$, we have

$$\langle T(\mu_{n_{k_\alpha}})z, x^*\rangle = \mu_{n_{k_\alpha}}\langle T(\cdot)z, x^*\rangle \longrightarrow \mu\langle T(\cdot)z, x^*\rangle = \langle T(\mu)z, x^*\rangle. \tag{2.6}$$

Now, since $\{T(\mu_{n_k})z\}$ converges strongly to z, we have $\langle z, x^*\rangle = \langle T(\mu)z, x^*\rangle$ and hence $z = T(\mu)z$. It follows from Lemma 2.1 that z is a common fixed point of $\mathcal{S}$. □

Lemma 2.3. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty weakly compact convex subset C of a Banach space E into C,* *with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$ *on the Lipschitz constants of the mappings. Let* X *be a left invariant subspace of* $l^\infty(S)$ *containing* 1 *such that the mappings* $s \mapsto \langle T(s)x, x^* \rangle$ *be in* X *for all* $x \in X$ *and* $x^* \in E^*$, *and* $\{\mu_n\}$ *be a strongly left regular sequence of means on* X. *Then*

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T(\mu_n)x - T(\mu_n)y\| - \|x-y\|\right) \leq 0. \tag{2.7}$$

Proof. Consider an arbitrary $\varepsilon > 0$ and take $d = \operatorname{diam}(C)$. Since $\lim_s k(s) \leq 1$, there exists $s_0 \in S$ such that

$$\sup_{s\geq s_0} k(s) < 1 + \frac{\varepsilon}{2d}. \tag{2.8}$$

From $\lim_{n\to\infty}\|l_{s_0}^*\mu_n - \mu_n\| = 0$, we may choose a natural number N such that

$$\left\|l_{s_0}^*\mu_n - \mu_n\right\| < \frac{\varepsilon}{2d}, \quad \forall n \geq N. \tag{2.9}$$

Then, for each $x, y \in C$, $n \geq N$ and $x^* \in J(T(\mu_n)x - T(\mu_n)y)$ we have

$$\begin{aligned}
\left\|T(\mu_n)x - T(\mu_n)y\right\|^2 &= \left\langle T(\mu_n)x - T(\mu_n)y, x^*\right\rangle \\
&= (\mu_n)_s\left\langle T(s)x - T(s)y, x^*\right\rangle - \left(l_{s_0}^*\mu_n\right)_s\left\langle T(s)x - T(s)y, x^*\right\rangle \\
&\quad + \left(l_{s_0}^*\mu_n\right)_s\left\langle T(s)x - T(s)y, x^*\right\rangle \\
&\leq \left\|\mu_n - l_{s_0}^*\mu_n\right\| d\|x^*\| + (\mu_n)_s\left\langle T(s_0s)x - T(s_0s)y, x^*\right\rangle \\
&\leq \frac{\varepsilon}{2d} d\left\|T(\mu_n)x - T(\mu_n)y\right\| + \sup_{s\in S}\left\|T(s_0s)x - T(s_0s)y\right\|\left\|T(\mu_n)x - T(\mu_n)y\right\| \\
&\leq \frac{\varepsilon}{2}\left\|T(\mu_n)x - T(\mu_n)y\right\| + \sup_{s\in S} k(s_0s)\|x-y\|\left\|T(\mu_n)x - T(\mu_n)y\right\|.
\end{aligned} \tag{2.10}$$

Therefore,

$$\begin{aligned}
\left\|T(\mu_n)x - T(\mu_n)y\right\| &\leq \frac{\varepsilon}{2} + \sup_{s\in S} k(s_0s)\|x-y\| \\
&\leq \frac{\varepsilon}{2} + \sup_{s\geq s_0} k(s)\|x-y\| \leq \frac{\varepsilon}{2} + \left(1 + \frac{\varepsilon}{2d}\right)\|x-y\| \leq \varepsilon + \|x-y\|,
\end{aligned} \tag{2.11}$$

that is,

$$\sup_{x,y\in C}\left(\left\|T(\mu_n)x - T(\mu_n)y\right\| - \|x-y\|\right) \leq \varepsilon, \quad \forall n \geq N. \tag{2.12}$$

Since $\varepsilon > 0$ is arbitrary, the desired result follows. □

Remark 2.4. Taking in Lemma 2.3

$$c_n = \sup_{x,y\in C}\left(\left\|T(\mu_n)x - T(\mu_n)y\right\| - \|x-y\|\right), \quad \forall n, \tag{2.13}$$

we obtain $\limsup_{n\to\infty} c_n \le 0$. Moreover,

$$\left\|T(\mu_n)x - T(\mu_n)y\right\| \le \|x-y\| + c_n, \quad \forall x, y \in C. \tag{2.14}$$

Corollary 2.5. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty compact convex subset C of a Banach space E into C, with the uniform Lipschitzian condition $\lim_s k(s) \le 1$. Let X be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing 1, and μ be a left invariant mean on X. Then $T(\mu)$ is nonexpansive and $F(\mathcal{S}) \ne \emptyset$. Moreover, if E is smooth, then $F(\mathcal{S})$ is a sunny nonexpansive retract of C and the sunny nonexpansive retraction of C onto $F(\mathcal{S})$ is unique.*

Proof. From (2.14), by taking $\mu_n = \mu$ $(\forall n)$, it follows that T_μ is nonexpansive. So, from Lemma 2.1, we get $F(\mathcal{S}) = F(T_\mu) \ne \emptyset$. On the other hand, it is well-known that the fixed point set of a nonexpansive mapping on a compact convex subset of a smooth Banach space is a sunny nonexpansive retract of C and the sunny nonexpansive retraction of C onto $F(\mathcal{S})$ is unique [19, 20]. This concludes the result. □

We will need the following lemmas in what follows.

Lemma 2.6 (see [20, 21]). *Let X be a real Banach space and let J be the duality mapping. Then, for any given $x, y \in X$ and $j(x+y) \in J(x+y)$, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.15}$$

Lemma 2.7 (see [40]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \ n \ge 0, \tag{2.16}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 (see [41]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n$ and $\limsup_{n\to\infty}\beta_n < 1$. Suppose*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) z_n \tag{2.17}$$

for all integers $n \geq 0$ *and*

$$\limsup_{n\to\infty}\left(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\right)\leq 0. \tag{2.18}$$

Then $\lim_{n\to\infty}\|x_n - z_n\| = 0$.

3. The main theorem

We are now ready to establish our main theorem.

Theorem 3.1. *Let* S *be a left reversible semigroup and* $\mathcal{S} = \{T(s) : s \in S\}$ *be a representation of* S *as Lipschitzian mappings from a nonempty compact convex subset* C *of a smooth Banach space* E *into* C*, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$ *and* f *be an* α*-contraction on* C *for some* $0 < \alpha < 1$*. Let* X *be a left invariant* $\mathcal{S}$*-stable subspace of* $l^\infty(S)$ *containing* 1*,* $\{\mu_n\}$ *be a strongly left regular sequence of means on* X *such that* $\lim_{n\to\infty}\|\mu_{n+1} - \mu_n\| = 0$ *and* $\{c_n\}$ *be the sequence defined by* (2.13)*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences in* $(0,1)$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$;

(iii) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iv) $\limsup_{n\to\infty} c_n/\alpha_n \leq 0$; *(note that, by Remark 2.4,* $\limsup_{n\to\infty} c_n \leq 0$*)*

(v) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Let $\{x_n\}$ *be the following sequence generated by* $x_1 \in C$ *and* $\forall n \geq 1$*,*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n. \tag{3.1}$$

Then $\{x_n\}$ *converges strongly to* $z \in F(\mathcal{S})$ *which is the unique solution of the variational inequality*

$$\langle (f-I)z, J(y-z)\rangle \leq 0, \quad \forall y \in F(\mathcal{S}). \tag{3.2}$$

Equivalently, one has $z = Pfz$*, where* P *is the unique sunny nonexpansive retraction of* C *onto* $F(\mathcal{S})$*.*

Remark 3.2. For example, we may choose

$$\alpha_n := \begin{cases} \dfrac{1}{n} + \sqrt{c_n} & \text{if } c_n \geq 0, \\[2ex] \dfrac{1}{n} & \text{if } c_n < 0. \end{cases} \tag{3.3}$$

Proof. We divide the proof into several steps and prove the claim in each step.

Step 1. Claim. Let $\{\omega_n\}$ be a sequence in C. Then

$$\lim_{n\to\infty}\left\|T(\mu_{n+1})\omega_n - T(\mu_n)\omega_n\right\| = 0. \tag{3.4}$$

Put $D = \sup\{\|z\| : z \in C\}$. Then

$$\begin{aligned}\|T(\mu_{n+1})\omega_n - T(\mu_n)\omega_n\| &= \sup_{\|z\|=1} |\langle T(\mu_{n+1})\omega_n - T(\mu_n)\omega_n, z\rangle| \\ &= \sup_{\|z\|=1} |(\mu_{n+1})_s\langle T(s)\omega_n, z\rangle - (\mu_n)_s\langle T(s)\omega_n, z\rangle| \\ &\leq \|\mu_{n+1} - \mu_n\| \sup_{s\in S} \|T(s)\omega_n\| \leq \|\mu_{n+1} - \mu_n\| D \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.5}$$

Step 2. Claim. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Define a sequence $\{z_n\}$ by $z_n = (x_{n+1} - \beta_n x_n)/(1 - \beta_n)$ so that $x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$. We now compute

$$\begin{aligned}\|z_{n+1} - z_n\| &= \left\| \frac{1}{1-\beta_{n+1}}(x_{n+2} - \beta_{n+1}x_{n+1}) - \frac{1}{1-\beta_n}(x_{n+1} - \beta_n x_n) \right\| \\ &= \left\| \frac{1}{1-\beta_{n+1}}(\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} T(\mu_{n+1})x_{n+1}) \right. \\ &\quad \left. - \frac{1}{1-\beta_n}(\alpha_n f(x_n) + \gamma_n T(\mu_n)x_n) \right\| \\ &= \left\| \frac{1}{1-\beta_{n+1}}(\alpha_{n+1} f(x_{n+1}) + (1 - \alpha_{n+1} - \beta_{n+1}) T(\mu_{n+1})x_{n+1}) \right. \\ &\quad \left. - \frac{1}{1-\beta_n}(\alpha_n f(x_n) + (1 - \alpha_{n+1} - \beta_{n+1}) T(\mu_n)x_n) \right\| \\ &\leq \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_n\| \\ &\quad + \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - T(\mu_{n+1})x_{n+1}) - \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - T(\mu_{n+1})x_{n+1}) \right\|.\end{aligned} \tag{3.6}$$

Since C is bounded and $\limsup_{n\to\infty}\beta_n < 1$, we have for some big enough constant $K > 0$,

$$\begin{aligned}\|z_{n+1} - z_n\| &\leq \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + \|T(\mu_n)x_{n+1} - T(\mu_n)x_n\| + K(\alpha_{n+1} + \alpha_n) \\ &\leq \|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + \|x_{n+1} - x_n\| + c_n + K(\alpha_{n+1} + \alpha_n).\end{aligned} \tag{3.7}$$

Now, since $\alpha_n \to 0$ and by Step 1 and Lemma 2.3, we immediately conclude that

$$\begin{aligned}&\limsup_n (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \\ &\quad \leq \limsup_n (\|T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1}\| + c_n + K(\alpha_{n+1} + \alpha_n)) \leq 0.\end{aligned} \tag{3.8}$$

Applying Lemma 2.8, we get $\lim_n\|x_{n+1} - x_n\| = \lim_n(1 - \beta_n)\|x_n - z_n\| = 0$.

Step 3. Claim. The ω-limit set of $\{x_n\}$, $\omega(\{x_n\})$, is a subset of $F(\mathcal{S})$.

Let $y \in \omega(\{x_n\})$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging strongly to y. Note that

$$x_{n+1} - x_n = \alpha_n f(x_n) + (1-\beta_n)(T(\mu_n)x_n - x_n) - \alpha_n T(\mu_n)x_n. \tag{3.9}$$

So

$$\|x_n - T(\mu_n)x_n\| \le \frac{1}{1-\beta_n}(\|x_{n+1} - x_n\| + \alpha_n\|f(x_n) - T(\mu_n)x_n\|). \tag{3.10}$$

Hence, by (ii), (v) and Step 2, we have

$$\lim_{n\to\infty}\|x_n - T(\mu_n)x_n\| = 0. \tag{3.11}$$

From this and Lemma 2.3, we obtain

$$\begin{aligned}\limsup_{k\to\infty}\|y - T(\mu_{n_k})y\| &\le \limsup_{k\to\infty}(\|y - x_{n_k}\| + \|x_{n_k} - T(\mu_{n_k})x_{n_k}\| + \|T(\mu_{n_k})x_{n_k} - T(\mu_{n_k})y\|)\\ &\le \limsup_{k\to\infty}(2\|y - x_{n_k}\| + \|x_{n_k} - T(\mu_{n_k})x_{n_k}\| + c_{n_k}) \le 0.\end{aligned} \tag{3.12}$$

Therefore, applying Corollary 2.2, we get $y \in F(\mathcal{S})$.

Step 4. Claim. The sequence $\{x_n\}$ converges strongly to $z = Pfz$.

We know, from Corollary 2.5 and the proof of Corollary 2.2, that there exists a unique sunny nonexpansive retraction P of C onto $F(\mathcal{S})$. The Banach Contraction Mapping Principal guarantees that Pf has a unique fixed point z which by (2.5) is the unique solution of

$$\langle (f-I)z, J(y-z)\rangle \le 0, \quad \forall y \in F(\mathcal{S}). \tag{3.13}$$

We first show

$$\limsup_{n\to\infty}\langle (f-I)z, J(x_n - z)\rangle \le 0. \tag{3.14}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle (f-I)z, J(x_{n_k} - z)\rangle = \limsup_{n\to\infty}\langle (f-I)z, J(x_n - z)\rangle. \tag{3.15}$$

Without loss of generality, we can assume that $\{x_{n_k}\}$ converges to some $y \in C$. By Step 3, $y \in F(\mathcal{S})$. Smoothness of E and a combination of (3.13) and (3.15) give

$$\limsup_{n\to\infty}\langle (f-I)z, J(x_n - z)\rangle = \langle (f-I)z, J(y-z)\rangle \le 0, \tag{3.16}$$

as required. Now, taking

$$u_n = T(\mu_n)x_n, \quad \forall n \geq 1, \tag{3.17}$$

we have $\|u_n - z\| \leq \|x_n - z\| + c_n$. By using Lemma 2.6, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\| \left[\gamma_n(u_n - z) + \beta_n(x_n - z)\right] + \alpha_n(\gamma f(x_n) - z) \right\|^2 \\
&\leq \left\|\gamma_n(u_n - z) + \beta_n(x_n - z)\right\|^2 + 2\alpha_n\langle f(x_n) - z, J(x_{n+1} - z)\rangle \\
&\leq (1-\beta_n)\left\|\frac{\gamma_n}{1-\beta_n}(u_n - z)\right\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(z), J(x_{n+1} - z)\rangle + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\leq \frac{\gamma_n^2}{1-\beta_n}\|u_n - z\|^2 + \beta_n\|x_n - z\|^2 \\
&\quad + 2\alpha_n\alpha\|x_n - z\|\|x_{n+1} - z\| + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&\leq \frac{\gamma_n^2}{1-\beta_n}\|x_n - z\|^2 + \frac{c_n\gamma_n^2}{1-\beta_n} + \beta_n\|x_n - z\|^2 \\
&\quad + \alpha_n\alpha\left(\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right) + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle \\
&= \left(\frac{\gamma_n^2}{1-\beta_n} + \beta_n + \alpha_n\alpha\right)\|x_n - z\|^2 \\
&\quad + \alpha_n\alpha\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle + \frac{c_n\gamma_n^2}{1-\beta_n} \\
&= \left((1-\alpha_n\alpha) - 2\alpha_n + 2\alpha_n\alpha + \frac{\alpha_n^2}{1-\beta_n}\right)\|x_n - z\|^2 \\
&\quad + \alpha_n\alpha\|x_{n+1} - z\|^2 + 2\alpha_n\langle f(z) - z, J(x_{n+1} - z)\rangle + \frac{c_n\gamma_n^2}{1-\beta_n}.
\end{aligned} \tag{3.18}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \left(1 - \frac{2(1-\alpha)\alpha_n}{1-\alpha_n\alpha}\right)\|x_n - z\|^2 \\
&\quad + \frac{\alpha_n}{1-\alpha_n\alpha}\left(2\langle \gamma f(z) - z, J(x_{n+1} - z)\rangle + \frac{\alpha_n}{1-\beta_n}\|x_n - z\|^2 + \frac{c_n}{\alpha_n} \times \frac{\gamma_n^2}{1-\beta_n}\right).
\end{aligned} \tag{3.19}$$

Now, from conditions (ii)–(v), (3.14) and Lemma 2.7, we get $\|x_n - z\| \to 0$. □

Corollary 3.3. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings from a nonempty compact convex subset C of a smooth Banach space*

E into C and f be an α-contraction on C for some $0 < \alpha < 1$. Let X be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing 1 and $\{\mu_n\}$ be a strongly left regular sequence of means on X such that $\lim_{n\to\infty}\|\mu_{n+1} - \mu_n\| = 0$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1, \ \forall n,$

(ii) $\lim_{n\to\infty}\alpha_n = 0;$

(iii) $\sum_{n=1}^{\infty}\alpha_n = \infty;$

(iv) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1.$

Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and $\forall n \geq 1$,

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n. \tag{3.20}$$

Then $\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$ which is the unique solution of the variational inequality

$$\langle (f - I)z, J(y - z)\rangle \leq 0, \quad \forall y \in F(\mathcal{S}). \tag{3.21}$$

Equivalently, one has $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

Remark 3.4. If S is a countable left amenable semigroup, then there is a strong left regular sequence on $l^\infty(S)$ consisting finite means μ, that is, $\mu = \sum_{i=1}^{n}\lambda_i\delta_{x_i}$, $\lambda_i \geq 0$, $\sum_{i=1}^{n}\lambda_i = 1$. See [42, Corollary 3.7].

Remark 3.5. It is known that if S is a left reversible semigroup, then $WAP(S)$, the space of weakly almost periodic functions on S, has a left invariant mean. But the converse is not true (see [43]).

Problem. Can the hypothesis on S of Theorem 3.1 be replaced by $WAP(S)$ has a left invariant mean?

4. Applications

Corollary 4.1. *Let C be a compact convex subset of a smooth Banach space E and let S, T be asymptotically nonexpansive mappings of C into itself with $ST = TS$ and f be an α-contraction on C for some $0 < \alpha < 1$. Let $\{c_n\}$ be defined by*

$$c_n = \frac{d}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(1 - k_i l_j), \tag{4.1}$$

where, $d = \mathrm{diam}(C)$ and k_i and l_j are defined as

$$\|S^i x - S^i y\| \leq k_i\|x - y\|, \qquad \|T^j x - T^j y\| \leq l_j\|x - y\|, \tag{4.2}$$

for all $x, y \in C$, *and* $\lim_{i\to\infty} k_i = \lim_{j\to\infty} l_j = 1$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences in* $(0,1)$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$;

(iii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iv) $\limsup_{n\to\infty} c_n/\alpha_n \le 0$; *(note that* $\lim_{n\to\infty} c_n = 0$*)*

(v) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Let $x_1 = x \in C$ *and* $\{x_n\}$ *be a sequence defined by*

$$x_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \left(\frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x_n \right) \tag{4.3}$$

for each $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to* $z \in F(S) \cap F(T)$ *which is the unique solution of the variational inequality*

$$\langle (f - I)z, J(y - z) \rangle \le 0, \quad \forall y \in F(S) \cap F(T). \tag{4.4}$$

Equivalently, one has $z = Pfz$, *where* P *is the unique sunny nonexpansive retraction of* C *onto* $F(S) \cap F(T)$.

Proof. Let $T(i,j) = S^i T^j$ for each $i, j \in \mathbb{N} \cup \{0\}$. Then $\{T(i,j) : i, j \in \mathbb{N} \cup \{0\}\}$ is a semigroup of Lipschitzian mappings on C such that for all $x, y \in C$,

$$\|T(i,j)x - T(i,j)y\| \le k(i,j)\|x - y\| \tag{4.5}$$

where $k(i,j) = k_i l_j$. Hence $\lim_{i,j\to\infty} k(i,j) = 1$. On the other hand, for each $n \in \mathbb{N}$, define $\mu_n(f) = 1/n^2 \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(i,j)$ for each $f \in l^\infty((\mathbb{N} \cup \{0\})^2)$. Then, $\{\mu_n\}$ is a strongly regular sequence of means and $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ [9, 44]. Next, for each $x, y \in C$ and $n \in \mathbb{N}$, we have

$$\left\|T(\mu_n)x - T(\mu_n)y\right\| = \left\| \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x - \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j y \right\| \le \|x - y\| + c_n. \tag{4.6}$$

Now, apply Theorem 3.1 to conclude the result. □

Corollary 4.2. *Let* C *be a compact convex subset of a smooth Banach space* E *and let* $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ *be a strongly continuous semigroup of Lipschitzian mappings on* C *with the uniform Lipschitzian condition* $\lim_{t\to\infty} k(t) \le 1$ *and* $\{t_n\}$ *be an increasing sequence in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$. *Let* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *be sequences in* $(0,1)$ *such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1,\ \forall n$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$;

(iii) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iv) $\limsup_{n\to\infty} c_n/\alpha_n \leq 0$, *where*

$$c_n = \sup_{x,y\in C}\left\{\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x\,ds - \frac{1}{t_n}\int_0^{t_n} T(s)y\,ds\right\| - \|x-y\|\right\}; \tag{4.7}$$

(v) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Let $x_1 = x \in C$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n\,ds\right) \tag{4.8}$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$ which is the unique solution of the variational inequality

$$\langle (f-I)z, J(y-z)\rangle \leq 0, \quad \forall y \in F(\mathcal{S}). \tag{4.9}$$

Equivalently, one has $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

Proof. For $n \in \mathbb{N}$, define $\mu_n(f) = 1/t_n\int_0^{t_n} f(t)dt$ for each $f \in C(\mathbb{R}_+)$, where $f \in C(\mathbb{R}_+)$ denotes the space of all real valued bounded continuous functions on $\mathbb{R}_+$ with supremum norm. Then, $\{\mu_n\}$ is a strongly regular sequence of means and $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\| = 0$ [9, 44]. Further, for each $x \in C$, we have $T(\mu_n)x = 1/t_n\int_0^{t_n} T(s)x\,ds$. Therefore, it suffices to apply Theorem 3.1 to conclude the desired result. □

Corollary 4.3. *Let C be a compact convex subset of a smooth Banach space E and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of Lipschitzian mappings on C with the uniform Lipschitzian condition $\lim_{t\to\infty} k(t) \leq 1$ and $\{r_n\}$ be a decreasing sequence in $(0,\infty)$ such that $\lim_{n\to\infty} r_n = 0$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$;

(iii) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iv) $\limsup_{n\to\infty} c_n/\alpha_n \leq 0$, *where*

$$c_n = \sup_{x,y\in C}\left\{\left\|r_n\int_0^{\infty} \exp(-r_{k_n}t)T(t)x\,dt - r_n\int_0^{\infty}\exp(-r_{k_n}t)T(t)y\,dt\right\| - \|x-y\|\right\}; \tag{4.10}$$

(v) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Let $x_1 = x \in C$ and $\{x_n\}$ be a sequence defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n r_n\int_0^{\infty}\exp(-r_n s)T(s)x_n\,ds \tag{4.11}$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z \in F(\mathcal{S})$ which is the unique solution of the variational inequality

$$\langle (f-I)z, J(y-z)\rangle \le 0, \quad \forall y \in F(\mathcal{S}). \tag{4.12}$$

Equivalently, one has $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(\mathcal{S})$.

Proof. For $n \in \mathbb{N}$, define $\mu_n(f) = r_n\int_0^\infty \exp(-r_{k_n}t)f(t)dt$ for each $f \in C(\mathbb{R}_+)$. Then, $\{\mu_n\}$ is a strongly regular sequence of means and $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\| = 0$ [9, 44]. Further, for each $x \in C$, we have $T(\mu_n)x = r_n\int_0^\infty \exp(-r_nt)T(t)xdt$. Therefore, the result follows from Theorem 3.1. □

Corollary 4.4. *Let C be a compact convex subset of a smooth Banach space E and let S be an asymptotically nonexpansive mapping of C into itself and f be an α-contraction on C for some $0 < \alpha < 1$. Let $\{c_n\}$ be defined by*

$$c_n = \frac{d}{n}\sum_{i=0}^{n-1}(1-k_i), \tag{4.13}$$

where, $d = \operatorname{diam}(C)$ and k_i is defined as $\|S^ix - S^iy\| \le k_i\|x-y\|$, for all $x, y \in C$, and $\lim_{i\to\infty}k_i = 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0,1)$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$;

(iii) $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(iv) $\limsup_{n\to\infty} c_n/\alpha_n \le 0$;

(v) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Let $x_1 = x \in C$ and $\{x_n\}$ be a sequence defined by

$$x_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n\sum_{m=0}^{\infty}q_{n,m}T^m x_n \tag{4.14}$$

for each $n \in \mathbb{N}$ where $Q = \{q_{n,m}\}$ is a strongly regular matrix. Then $\{x_n\}$ converges strongly to $z \in F(S)$ which is the unique solution of the variational inequality

$$\langle (f-I)z, J(y-z)\rangle \le 0, \quad \forall y \in F(S). \tag{4.15}$$

Equivalently, one has $z = Pfz$, where P is the unique sunny nonexpansive retraction of C onto $F(S)$.

Proof. For each $n \in \mathbb{N}$, define

$$\mu_n(f) = \sum_{m=0}^{\infty}q_{n,m}f(m) \tag{4.16}$$

for each $f \in l^{\infty}(\mathbb{N} \cup \{0\})$. Since Q is a strongly regular matrix, for each m, we have $q_{n,m} \to 0$, as $n \to \infty$; see [37]. Then, it is easy to see that $\{\mu_n\}$ is a regular sequence of means, and $\|\mu_{n+1} - \mu_n\| \to 0$ [44]. Further, for each $x \in C$, we have $T(\mu_n)x = \sum_{m=0}^{\infty} q_{n,m} T^m x$. Now, apply Theorem 3.1 to conclude the result. □

For deducing some more applications, we refer to, for example, [44].

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