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Review Article

The Comparison of the Convergence Speed between Picard, Mann, Krasnoselskij and Ishikawa Iterations in Banach Spaces

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The purpose of this paper is to compare convergence speed of the Picard and Mann iterations on one hand, Krasnoselskij and Ishikawa iterations on the other hand, for the class of Zamfirescu operators. The results improve corresponding results of (Berinde 2004) and (Babu and Vara Prasad 2006).

1. Introduction

Let E be a real Banach space, D a closed convex subset of E, and $T : D \to D$ a self-map. Let $p_0, v_0, u_0, x_0 \in D$ be arbitrary. The sequence $\{p_n\}_{n=0}^{\infty} \subset D$ defined by

$$p_{n+1} = Tp_n, \quad n \geq 0, \tag{1.1}$$

is called the Picard iteration or Picard iterative procedure. For $\lambda \in (0,1)$, the sequence $\{v_n\}_{n=0}^{\infty} \subset D$ defined by

$$v_{n+1} = (1-\lambda)v_n + \lambda T v_n, \quad n \geq 0, \tag{1.2}$$

is called the Krasnoselskij iteration or Krasnoselskij iterative procedure. Let $\{a_n\}$ be a sequence of real numbers in $[0,1]$. The sequence $\{u_n\}_{n=0}^{\infty} \subset D$ defined by

$$u_{n+1} = (1-a_n)u_n + a_n T u_n, \quad n \geq 0, \tag{1.3}$$

is called the Mann iteration or Mann iterative procedure. The sequence $\{x_n\}_{n=0}^{\infty} \subset D$ defined by

$$x_0 \in D,$$
$$y_n = (1 - b_n)x_n + b_n T x_n, \quad n \geq 0, \tag{1.4}$$
$$x_{n+1} = (1 - a_n)x_n + a_n T y_n, \quad n \geq 0,$$

is called the Ishikawa iteration or Ishikawa iterative procedure, where $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers in $[0,1]$. Obviously, for $b_n = 0$ the Ishikawa iteration (1.4) can be reduced to (1.3); and for $\lambda = 1$ we obtain the Picard iteration. In the last twenty years, many authors have studied the convergence of the sequence of the Picard, Krasnoselskij, Mann, and Ishikawa iterations of a mapping T to a fixed point of T, under various contractive conditions. In such situations, it is of theoretical and practical importance to compare these iteration methods in order to establish which one converges faster if possible.

Definition 1.1 (see [1]). The operator $T : X \to X$ satisfies condition Zamfirescu if and only if there exist real numbers a, b, c satisfying $0 < a < 1$, $0 < b, c < 1/2$ such that for each pair x, y in X, at least one of the following conditions is true:

(1) $\|Tx - Ty\| \leq a\|x - y\|$;

(2) $\|Tx - Ty\| \leq b(\|x - Tx\| + \|y - Ty\|)$;

(3) $\|Tx - Ty\| \leq c(\|x - Ty\| + \|y - Tx\|)$.

Obviously, we could obtain that every Zamfirescu operator T satisfies the inequality

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\| \tag{1.5}$$

for all $x, y \in D$, where $\delta = \max\{a, b/(1-b), c/(1-c)\}$ with $0 < \delta < 1$.

In 1972, Zamfirescu [1] obtained a very interesting fixed point theorem for Zamfirescu operator.

Theorem Z (see [1]). *Let (X, d) be a complete metric space and $T : X \to X$ a Zamfirescu operator. Then, T has a unique fixed point q and the Picard iteration* (1.1) *converges to q.*

Later on, Berinde [3] improved and extended the above-mentioned theorem and the results in paper [2] with the following result.

Theorem B1 (see [3]). *Let E be an arbitrary Banach space, D a closed convex subset of E, and $T : D \to D$ an operator satisfying condition Z. Let $\{u_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (1.2) *for $u_0 \in D$, with $\{a_n\} \subset [0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, $\{u_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

Theorem B2 (see [3]). *Let E be an arbitrary Banach space, D a closed convex subset of E, and $T : D \to D$ an operator satisfying condition Z. Let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by* (1.3) *for $x_0 \in D$, with $\{a_n\}$ and $\{b_n\}$ being sequences of positive numbers in $[0,1]$ and $\{a_n\}$ satisfying $\sum_{n=0}^{\infty} a_n = \infty$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.*

In order to compare the fixed point iteration procedures $\{p_n\}$, $\{u_n\}$, and $\{x_n\}$ that converge to a certain fixed point of given operator T, Berinde [4] provided the following definitions.

Definition 1.2 (see [4]). Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists $l = \lim_{n\to\infty}(|a_n - a|/|b_n - b|)$. If $l = 0$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b. If $0 < l < \infty$, then it can be said that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 1.3 (see [4]). Suppose that for two fixed point iteration procedures $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p with the error estimates $\|u_n - p\| \le a_n$, $\|v_n - p\| \le b_n$, $n \ge 0$, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then it can be said that $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p.

The purpose of this paper is to improve the results in [4, 5] by giving a direct rate of convergence for some fixed point procedures.

2. Main results

In the sequel, suppose that δ is a constant from (1.5).

Theorem 2.1. *Let E be an arbitrary real Banach space, D a closed convex subset of E, and $T : D \to D$ a Zamfirescu operator. Let $\{p_n\}_{n=0}^{\infty}$ be defined by* (1.1) *for $x_0 \in D$, and let $\{u_n\}_{n=0}^{\infty}$ be defined by* (1.3) *for $y_0 \in D$ with $\{a_n\}$ in $[0, 1/(1+\delta))$ and satisfying* (i) $\sum_{n=0}^{\infty} a_n = \infty$, (ii) $a_n \to 0$ *as* $n \to \infty$. *Then, the Picard iteration converges faster than the Mann iteration to the fixed point of T.*

Proof. By [1, Theorem 2.3], T has a unique fixed point, denote it by q. Moreover, Picard's iteration $\{p_n\}_{n=0}^{\infty}$ defined by (1.1) converges to q, for any $p_0 \in E$, and

$$\|p_{n+1} - q\| = \|Tp_n - q\|. \tag{2.1}$$

Take $x = q$ and $y = p_n$ in (1.5), then we get

$$\|p_{n+1} - q\| \le \delta\|p_n - q\| \le \delta^{n+1}\|p_0 - q\|, \quad n \ge 0. \tag{2.2}$$

Now, by Mann's iteration in (1.3) and (1.5),

$$\begin{aligned}\|u_{n+1} - q\| &\ge (1 - a_n)\|u_n - q\| - a_n\|Tu_n - Tq\| \\ &\ge (1 - (1+\delta)a_n)\|u_n - q\| \\ &\ge (1 - (1+\delta)a_n)(1 - (1+\delta)a_{n-1}) \cdots (1 - (1+\delta)a_0)\|u_0 - q\|.\end{aligned} \tag{2.3}$$

From (2.2) and (2.3), it follows that $\|p_{n+1} - q\|/\|u_{n+1} - q\| \le \delta^{n+1}\|p_0 - q\|/(1 - (1+\delta)a_n)(1 - (1+\delta)a_{n-1}) \cdots (1 - (1+\delta)a_0)\|u_0 - q\| \to 0$ as $n \to \infty$. Indeed, we consider $\sum_{n=0}^{\infty}(\delta^{n+1}\|p_0 - q\|/(1 - (1+\delta)a_n)(1 - (1+\delta)a_{n-1}) \cdots (1 - (1+\delta)a_0)\|u_0 - q\|)$. Set $w_n = \delta^{n+1}\|p_0 - q\|/(1 - (1+\delta)a_n)(1 - (1+\delta)a_{n-1}) \cdots (1 - (1+\delta)a_0)\|u_0 - q\|$, then we obtain that $\lim_{n\to\infty}(w_{n+1}/w_n) = \delta < 1$. Applying the ratio test, we get $\sum_{n=0}^{\infty} w_n < \infty$, so $w_n \to 0$ as $n \to \infty$, that is, $\|p_n - q\| = o(\|u_n - q\|)$. By Definition 1.2, we obtain the conclusion of Theorem 2.1. □

Theorem 2.2. *Let E be an arbitrary Banach space, D a closed convex subset of E, and $T : D \to D$ a Zamfirescu operator. Let $\{v_n\}_{n=0}^{\infty}$ be defined by* (1.2) *for $v_0 \in D$, and let $\{x_n\}_{n=0}^{\infty}$ be defined by* (1.4) *for $x_0 \in D$ with $\{a_n\}$ and $\{b_n\}$ in $[0, 1/(1+\delta))$ and satisfying* (i) $\sum_{n=0}^{\infty} a_n = \infty$, (ii) $a_n, b_n \to 0$ *as $n \to \infty$. Let q be a fixed point of T in D. Then, the Krasnoselskij iteration converges faster than the Ishikawa iteration to the fixed point q of T.*

Proof. By Theorem B2 (see [3]), there exists a unique fixed point, denote it by q. For the Krasnoselskij iteration, by using (1.2) we have

$$\|v_{n+1}-q\| \leq (1-\lambda)\|v_n-q\| + \lambda\|Tv_n - Tq\|. \tag{2.4}$$

Take $x=q$ and $y=v_n$ in (1.5) to obtain

$$\|Tv_n - Tq\| \leq \delta\|v_n - q\|, \tag{2.5}$$

and then

$$\begin{aligned}\|v_{n+1}-q\| &\leq (1-(1-\delta)\lambda)\|v_n-q\| \\ &\leq (1-(1-\delta)\lambda)^{n+1}\|v_0-q\| \longrightarrow 0\end{aligned} \tag{2.6}$$

as $n\to\infty$. For the Ishikawa iterative procedure, by (1.4) we get

$$\|x_{n+1}-q\| \geq (1-a_n)\|x_n-q\| - a_n\|Ty_n-Tq\|. \tag{2.7}$$

Take $x=q$ and $y=y_n$ in (1.5) to obtain

$$\|Ty_n - Tq\| \leq \delta\|y_n-q\|, \tag{2.8}$$

and again using (1.4) and (1.5),

$$\begin{aligned}\|y_n-q\| &\leq (1-b_n)\|x_n-q\| + b_n\|Tx_n-Tq\| \\ &\leq (1-(1-\delta)b_n)\|x_n-q\|,\end{aligned} \tag{2.9}$$

and hence by (2.8), (2.9), and (2.7), we get

$$\begin{aligned}\|x_{n+1}-q\| &\geq (1-a_n-a_n\delta(1-(1-\delta)b_n))\|x_n-q\| \\ &\geq (1-(1+\delta)a_n)\|x_n-q\| \\ &\geq (1-(1+\delta)a_n)(1-(1+\delta)a_{n-1})\|x_{n-1}-q\| \\ &\geq (1-(1+\delta)a_n)(1-(1+\delta)a_{n-1})\cdots(1-(1+\delta)a_0)\|x_0-q\|.\end{aligned} \tag{2.10}$$

On repeating the proof course of Theorem 2.1, then $\|v_{n+1}-q\|/\|x_{n+1}-q\| \leq (1-(1-\delta)\lambda)^{n+1}\|v_0-q\|/(1-(1+\delta)a_n)(1-(1+\delta)a_{n-1})\cdots(1-(1+\delta)a_0)\|x_0-q\| \to 0$ as $n\to\infty$. Hence, $\|v_n-q\| = o(\|x_n-q\|)$. By Definition 1.2, we also obtain the conclusion of Theorem 2.2. □

Remark 2.3. Theorem 2.1 provides a direct comparison of the rate of convergence of Picard and Mann iterations in the class of Zamfirescu operators, while Theorem 2.2 obtains a similar result for Krasnoselskij and Ishikawa iterations. However, we do not have a direct comparison result of the rate of convergence in the case of Mann and Ishikawa iterations in the same class of mappings. So, the best result for these two fixed point iterations remains that of [5], obtained by means of the comparison sequences $\{a_n\}$ and $\{b_n\}$ and not in a direct way, as in the present paper (Theorems 2.1 and 2.2).

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