Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2008, Article ID 480187, 20 pages
doi:10.1155/2008/480187

Research Article

About Robust Stability of Dynamic Systems with Time Delays through Fixed Point Theory

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Received 22 September 2008; Revised 10 November 2008; Accepted 19 November 2008

Recommended by Andrzej Szulkin

This paper investigates the global asymptotic stability independent of the sizes of the delays of linear time-varying systems with internal point delays which possess a limiting equation via fixed point theory. The error equation between the solutions of the limiting equation and that of the current one is considered as a perturbation equation in the fixed- point and stability analyses. The existence of a unique fixed point which is later proved to be an asymptotically stable equilibrium point is investigated. The stability conditions are basically concerned with the matrix measure of the delay-free matrix of dynamics to be negative and to have a modulus larger than the contribution of the error dynamics with respect to the limiting one. Alternative conditions are obtained concerned with the matrix dynamics for zero delay to be negative and to have a modulus larger than an appropriate contributions of the error dynamics of the current dynamics with respect to the limiting one. Since global stability is guaranteed under some deviation of the current solution related to the limiting one, which is considered as nominal, the stability is robust against such errors for certain tolerance margins.

1. Introduction

Time-delay dynamic systems are an interesting field of research in dynamic systems and functional differential equations. Their intrinsic related theoretical interest is due to the fact that the necessary formalism lies in that of functional differential equations, being infinite dimensional. Another reason for their interest relies on the wide range of their applicability in modelling a number of physical systems like, for instance, transportation systems, queuing systems, teleoperated systems, war/peace models, biological systems, finite impulse response filtering, and so on [1–4]. Important particular interest has been devoted to stability, stabilization, and model-matching of control systems where the object to be controlled possesses delayed dynamics and the controller is synthesized incorporating delayed dynamics or its structure may be delay-free (see, e.g., [1, 4–14]). The properties are formulated as either being independent of or dependent on the sizes of the delays. An intrinsic problem which generated analysis complexity is the presence of infinitely many characteristic zeros because of the functional nature of the dynamics. This fact generates difficulties in the closed-loop pole-placement problem compared to the delay-free case [14], as well as in the stabilization problem [2, 4–6, 8–11, 13, 15–20], including the case of singular time-delay systems where the solution is sometimes nonunique and impulsive because of the dynamics associated to a nilpotent matrix [15]. The properties of the associated evolution operators have been investigated in [2, 6, 11]. This paper is devoted to obtain results relying on a comparison and an asymptotic comparison of the solutions between a nominal (unperturbed) functional differential equation involving wide classes of delays and a perturbed version (describing the current dynamics) with some smallness in the limit assumptions on the perturbed functional differential equation. The nominal equation is defined as the limiting equation of the perturbed one since the parameters of the last one converge asymptotically to those of its limiting counterpart. The problem of interest arises since very often the perturbations related to a nominal model in dynamic systems occur during the transients while they are asymptotically vanishing in the steady state or, in the most general worst case, they grow at a smaller rate than the solution of the nominal differential equation. In this context, the nominal differential equation may be viewed as the limiting equation of the perturbed one. The comparison between the solutions of the limiting differential equation and those of the perturbed one based on Perron-type results has been studied classically for ordinary differential equations and more recently for the case of functional equations [10, 21, 22]. Particular functional equations of interest are those involving both point and distributed delays potentially including the last ones Volterra-type terms [2, 5–7, 23]. On the other hand, fixed point theory [2, 21, 24] is a very powerful mathematical tool to be used in many applications where stability is required. At a theoretical level, fixed point theory is being of an increasing interest along the last years. For instance, the concept of weak contractiveness is discussed in [25] where the contraction constant is allowed to be unity but a negative vanishing term associated with some continuous nondecreasing function is also allowed. Weak contractiveness still ensures the existence of a unique fixed point. The existence of a unique fixed point has also been proved for asymptotic contractions [26]. Also, the existence of a nonempty fixed point set in a self-map of X, where(X, d) is a complete metric space allows guaranteeing the T-stability of iteration procedures [27]. In this paper, linear time-varying functional differential equations with point constant delays are investigated. Based on the contraction mapping principle, it is first proved the existence of a unique fixed point. The related proofs are based on the convergence of the parameters of the current equation to their counterparts of the limiting equation. The existence of such a fixed point requires that a relevant matrix of the limiting equation (either that of the delay-free dynamics or that of the zero-delay dynamics) be a stability matrix. Furthermore, an inequality concerning the parameters of the absolute value of such a matrix with a measure of all the remaining dynamics (formulated in terms of norms) has to be fulfilled. Once the existence of a unique fixed point has been proved, simple extra conditions ensure that such a point is a globally stable zero equilibrium point of the state-trajectory solution. This leads immediately to prove the global asymptotic stability independent of the sizes of the delays of the dynamic system. The analysis is then extended to the case of closed-loop systems obtained via state or output linear feedback from the original uncontrolled dynamic system. A method to synthesize both the time-invariant parts and the incremental ones of the controller matrices is given so that the existence of a fixed point of the closed-loop system is guaranteed. The obtained results are of robust stability type since the global asymptotic stability is guaranteed under a certain deviation from the current solution with respect to the limiting one, which is considered the nominal dynamics.

1.1. Notation

$\mathbb{C}, \mathbb{R}$, and $\mathbb{Z}$ are the sets of complex, real, and integer numbers, respectively.

$\mathbb{R}_+$ and $\mathbb{Z}_+$ are the sets of positive real and integer numbers, respectively; $\mathbb{C}_+$ is the set of complex numbers with positive real part.

$\mathbb{C}_{0+} := \mathbb{C}_+ \cup \{i\omega : \omega \in \mathbb{R}\}$, where i is the complex unity, $\mathbb{R}_{0+} := \mathbb{R}_+ \cup \{0\}$, and $\mathbb{Z}_{0+} := \mathbb{Z}_+ \cup \{0\}$.

$\mathbb{R}_-$ and $\mathbb{Z}_-$ are the sets of negative real and integer numbers, respectively; $\mathbb{C}_-$ is the set of complex numbers with negative real part.

$\mathbb{C}_{0-} := \mathbb{C}_- \cup \{i\omega : \omega \in \mathbb{R}\}$, where i is the complex unity, $\mathbb{R}_{0-} := \mathbb{R}_- \cup \{0\}$, and $\mathbb{Z}_{0-} := \mathbb{Z}_- \cup \{0\}$.

$\overline{N} := \{1, 2, \ldots, N\} \subset \mathbb{Z}_{0+}$, "$\vee$" is the logic disjunction, and "$\wedge$" is the logic conjunction.

$[t/h]$ is the integer part of the rational quotient t/h.

$\sigma(M)$ denotes the spectrum of the real or complex square matrix M (i.e., its set of distinct eigenvalues).

$\|\ \|$ denotes any vector or induced matrix norm. Also, $\|m\|_p$ and $\|M\|_p$ are the ℓ_p-norms of the vector m or (induced) real or complex matrix M, and $\mu_p(M)$ denote the ℓ_p measure of the square matrix M [4]. The matrix measure $\mu_p(M)$ is defined as the existing limit $\mu_p(M) := \lim_{\varepsilon \to 0^+}(\|I_n + \varepsilon X\|_p - \varepsilon)/\varepsilon$ which has the property $\max(-\|M\|_p, \max_{i \in \overline{n}} \operatorname{re} \lambda_i(M)) \le \mu_p(M) \le \|M\|_p$ for any square n-matrix M of spectrum $\sigma(M) = \{\lambda_i(M) \in \mathbb{C} : 1 \le i \le n\}$. An important property for the investigation of this paper is that $\mu_2(M) < 0$ if M is a stability matrix, that is, if $\operatorname{re} \lambda_i(M) < 0;\ 1 \le i \le n$.

$\|\ \|_\infty$ denotes the supremum norm on $\mathbb{R}_{0+}$, or its induced supremum metric, for functions or vector and matrix functions without specification of any pointwise particular vector or matrix norm for each $t \in \mathbb{R}_{0+}$. If pointwise vector or matrix norms are specified, the corresponding particular supremum norms are defined by using an extra subscript. Thus, $\|m\|_{p\infty} := \sup_{t \in \mathbb{R}_{0+}} \|m(t)\|_p$ and $\|M\|_{p\infty} := \sup_{t \in \mathbb{R}_{0+}} \|M(t)\|_p$ are, respectively, the supremum norms on for vector and matrix functions of domains in $\mathbb{R}_{0+} \times \mathbb{R}^n$, respectively, in $\mathbb{R}_{0+} \times \mathbb{R}^{n \times m}$ defined from their ℓ_p pointwise respective norms for each $t \in \mathbb{R}_{0+}$.

I_n is the nth identity matrix.

$K_p(M)$ is the condition number of the matrix M with respect to the ℓ_p-norm.

2. Linear systems with point constant delays and the contraction mapping theorem

Consider the following time-varying linear system subject to r constant point delays:

$$\dot{x}(t) = \sum_{i=0}^{r} \widehat{A}_i(t) x(t - r_i) = \sum_{i=0}^{r} A_i x(t - r_i) + \sum_{i=0}^{r} \widetilde{A}_i(t) x(t - r_i), \tag{2.1}$$

where r_i ($i \in \overline{r}$) are the r (in general incommensurate delays) $0 = r_0 < r_i$ ($i \in \overline{r}$) subject to the system piecewise continuous bounded matrix functions of dynamics $\widehat{A}_i : \mathbb{R}_{0+} \to \mathbb{R}^{n \times n}$ ($i \in \overline{r} \cup \{0\}$) which are decomposable as a (nonunique) sum of a constant matrix plus a matrix function of time $\widehat{A}_i(t) = A_i + \widetilde{A}_i(t), \forall t \in \mathbb{R}_{0+}$. Equation (2.1) is assumed subject to any piecewise continuous real vector function of initial conditions $\varphi : [-r_r, 0] \to \mathbb{R}^n$ with $\varphi(0) = x(0) = x_0$, that is, $\varphi \in BPC^{(0)}([-r_r, 0]), \mathbb{R}^n)$. Thus, it has a unique solution $[-r_r, 0] \cup \mathbb{R}_{0+}$, satisfying $x \equiv \varphi$, $\forall t \in [-r_r, 0]$ and the differential system (2.1), $\forall t \in \mathbb{R}_+$ for any bounded piecewise continuous set $\widetilde{A}_i : [-r_i, 0] \to \mathbb{R}^{n\times n}$ $(i \in \overline{r} \cup \{0\})$ what follows from Picard-Lindeloff's theorem, [4, 11, 15]. Such a unique solution is

$$x(t) = e^{A_0 t}\left[x_0 + \int_0^t e^{-A_0\tau}\widetilde{A}_0(\tau)x(\tau)d\tau + \sum_{i=1}^{r}\left(\int_0^{r_i} e^{-A_0\tau}\widehat{A}_i(\tau)\varphi(\tau - r_i)d\tau + \int_{r_i}^{t} e^{-A_0\tau}\widehat{A}_i(\tau)x(\tau - r_i)d\tau\right)\right]. \tag{2.2}$$

According to Lyapunov's stability theory, global stability means that the state-trajectory solution is uniformly bounded for any bounded function of initial conditions. Global asymptotic stability implies also that there is a unique asymptotically stable equilibrium point which is then a global attractor. See, for instance, [2, 4–11, 13, 16, 21, 24, 28]. Generic relations of stability with fixed point theory have been reported in [2, 21, 24, 27, 29, 30]. It turns out that a system whose state-trajectory solutions are all bounded and converge to a unique point is globally asymptotically stable to its equilibrium in Lyapunov's sense, provided that such equilibrium is unique. The following simple result is well known. Assume the system (2.1) with $A_i = 0\,(\forall i \in \overline{r}), \widetilde{A}_i(t) = 0\,(\forall i \in \overline{r} \cup \{0\})$, then the resulting linear time-invariant delay-free system (2.1) is globally asymptotically stable if A_0 is a stability matrix so that if $\mu_2(A_0) < 0$. Nonasymptotic stability is guaranteed if $\mu_2(A_0) \le 0$. The subsequent result is concerned with global stability independent of the sizes of the delays and it is obtained from the contraction mapping theorem for the case when (2.1) has a limiting equation with a unique asymptotically stable equilibrium point. It is assumed that the matrices defining the delayed dynamics have sufficiently small norms and that the norm of the error matrix of the delay-free dynamics with respect to its limiting value is also sufficiently small.

Theorem 2.1. *The following properties hold.*

(i) *Assume that A_0 is a stability matrix of ℓ_2-matrix measure $\mu_2(A_0)$ and that $\sup_{t\in\mathbb{R}_0^+}(\|\widetilde{A}_0(t)\|_2 + \sum_{i=1}^{r}\|\widehat{A}_i(t + r_i)\|_2) < \rho_0/K_0 = (|\mu_2(A_0)| - \widetilde{\rho}_0)/K_0$ for some real constants $K_0 \ge 1$ and $\widetilde{\rho}_0 \in (0, |\mu_2(A_0)|)$ and any real constant $\rho_0 \in (0, |\mu_2(A_0)|)$ such that the C_0-semigroup of the infinitesimal generator A_0 satisfies $\|e^{A_0 t}\|_2 \le K_0 e^{-\rho_0 t}$. Assume also that $\lim_{t\to\infty}\widehat{A}_i(t) = A_i, \forall i \in \overline{r} \cup \{0\}$ and that $(\sum_{i=0}^{r} A_i)$ is nonsingular. Then, the system (3.1) is globally asymptotically stable independent of the sizes of the delays.*

(ii) *If all the eigenvalues of A_0 are distinct, then global asymptotic stability independent of the sizes of the delays delay holds if $\sup_{t\in\mathbb{R}_0^+}(\|\widetilde{A}_0(t)\|_2 + \sum_{i=1}^{r}\|\widehat{A}_i(t + r_i)\|_2) < |\mu_2(A_0)|/K_0$, since $\rho_0 \in (0, |\mu_2(A_0)|]$, with the remaining conditions being identical.*

Proof. (i) The pointwise difference between the two solutions $x(t)$ and $z(t)$ of (2.1) subject to respective initial conditions $\varphi_x : [-r_r, 0] \to \mathbb{R}^n$ and $\varphi_z : [-r_r, 0] \to \mathbb{R}^n$ is

$$x(t) - z(t) = e^{A_0 t}\left[(x_0 - z_0) + \int_0^t e^{-A_0\tau}\widetilde{A}_0(\tau)(x(\tau) - z(\tau))d\tau + \sum_{i=1}^{r}\left(\int_0^{r_i} e^{-A_0\tau}\widehat{A}_i(\tau)(\phi_x(\tau - r_i) - \phi_z(\tau - r_i))d\tau + \int_{r_i}^{t} e^{-A_0\tau}\widehat{A}_i(\tau)(x(\tau - r_i) - z(\tau - r_i))d\tau\right)\right]. \tag{2.3}$$

Define the complete metric space $(M, \|\cdot\|_\infty)$ with the supremum metric $\|\cdot\|_\infty$ and

$$M := \{\phi \in BC^{(0)}(\mathbb{R}, \mathbb{R}^n) : \phi \equiv \varphi \in BPC^{(0)}([-r_r, 0]), \mathbb{R}^n)\}, \tag{2.4}$$

where $BC^{(0)}(\mathbb{R}, \mathbb{R}^n)$ is the set of bounded continuous n-vector functions on $\mathbb{R}$. Now, define $P : M \to M$ as the subsequent bounded continuous function:

$$\begin{aligned}(P\phi)(t) := e^{A_0 t}\Bigg[\varphi(0) + \int_0^t e^{-A_0\tau}\widetilde{A}_0(\tau)\phi(\tau)d\tau \\ + \sum_{i=1}^{r}\left(\int_0^{r_i} e^{-A_0\tau}\widehat{A}_i(\tau)\varphi(\tau - r_i)d\tau + \int_{r_i}^{t} e^{-A_0\tau}\widehat{A}_i(\tau)\phi(\tau - r_i)d\tau\right)\Bigg].\end{aligned} \tag{2.5}$$

Since $e^{A_0 t}$ is an infinitesimal generator of the C_0-semigroup of the infinitesimal generator A_0, there exist real constants $K_0 \geq 1$ (which is norm dependent) and ρ_0, satisfying $0 > -\rho_0 \geq \mu_2(A_0) := \max_{i\in\overline{\vartheta}}(\mathrm{Re}\lambda_i : \lambda_i \in \sigma(A_0))$, since A_0 is a stability matrix, such that for any matrix norm $\|e^{A_0 t}\| \leq K_0 e^{-\rho_0 t}; \forall t \in \mathbb{R}_{0+}$.Then, one gets from (2.4)-(2.5) that the supremum metric, induced by the supremum norm, is then the supremum norm

$$\begin{aligned}&\|(P\phi)(t) - (P\eta)(t)\| \\ &\quad = \left\| e^{A_0 t}\left[\int_0^t e^{-A_0\tau}\widetilde{A}_0(\tau)(\phi(\tau) - \eta(\tau))d\tau + \sum_{i=1}^{r}\int_{r_i}^{t} e^{-A_0\tau}\widehat{A}_i(\tau)(\phi(\tau - r_i) - \eta(\tau - r_i))d\tau\right]\right\| \\ &\quad \leq \frac{K_0}{\rho_0}\left(\|\widetilde{A}_0\|_\infty + \sum_{i=1}^{r}\|\widehat{A}_i\|_\infty\right)\|\phi - \eta\|_\infty; \qquad \forall \phi, \eta \in M,\ \forall t \geq r_r\end{aligned} \tag{2.6}$$

for any vector of matrix norms on $\mathbb{R}_{0+}$. Now, P is a contraction if $(K_0/\rho_0)(\|\widetilde{A}_0\|_\infty + \sum_{i=1}^{r}\|\widehat{A}_i\|_\infty) < 1$ and then there is a unique point $\phi \in M$ such that $P\varphi = \varphi$ from the contraction mapping theorem [21, 24]. P is also a contraction if $(K_0/\rho_0)\sup_{t\in\mathbb{R}_{0+}}(\|\widetilde{A}_0(t)\|_2 + \sum_{i=1}^{r}\|\widehat{A}_i(t)\|_2) < 1$ holds. The above conditions may be also tested with any supremum norm associated with the supremum metric. For instance, the ℓ_2 supremum real vector function norm $\|v\|_{2\infty} = \sup_{t\in\mathbb{R}_{0+}}\|v(t)\|_2 = \sup_{t\in\mathbb{R}_{0+}}\sqrt{(v^T(t)v(t))}$ for any $v : \mathbb{R}_{0+} \to \mathbb{R}^m$ and its induced real matrix function norm $\|G\|_{2\infty} = \sup_{t\in\mathbb{R}_{0+}}\|G(t)\|_2 = \sup_{t\in\mathbb{R}_{0+}}\sqrt{\lambda_{\max}(G^T(t)G(t))} = \sup_{t\in\mathbb{R}_{0+}}\max_{0\neq\|v\|_2\leq 1}(\|G(t)v\|/\|v\|_2)$ for $G : \mathbb{R}_{0+} \to \mathbb{R}^{n\times m}$provided that such norms exist, where $\lambda_{\max}(Q)$ denotes the maximum (real) eigenvalue of the square symmetric matrix (Q). Note that P is a contraction if $(K_0/\rho_0)(\|\widetilde{A}_0\|_\infty + \sum_{i=1}^{r}\|\widehat{A}_i\|_\infty) < 1$ holds for any $\rho_0 \in \mathbb{R}_+$ satisfying

$$0 > -\rho_0 = \widetilde{\rho}_0 - |\mu_2(A_0)| > \mu_2(A_0) = \max_{i\in\overline{\vartheta}}(\mathrm{Re}\lambda_i : \lambda_i \in \sigma(A_0)) \tag{2.7}$$

for some $\mathbb{R}_+ \ni \widetilde{\rho}_0 \in (0, |\mu_2(A_0)|)$, where $\sigma(A_0)$ is the spectrum of A_0 of cardinal $\vartheta := \mathrm{card}\,\sigma(A_0) \leq n$ and any given vector norm and corresponding induced matrix norm. The limiting equation of (3.1) is $\dot{x}^*(t) = \sum_{i=0}^{r}A_i x^*(t - r_i)$. Since $(\sum_{i=0}^{r}A_i)$ is nonsingular,

$\ker(\sum_{i=0}^{r} A_i) = \{0 \in \mathbb{R}^n\}$. Thus, the unique fixed point $x^* = 0$ in $\mathbb{R}^n$ of the limiting equation, and then that of (3.1) whose uniqueness follows from the contraction mapping theorem, is $\{0\}$. As a result, the unique fixed point $\{0\}$ is a global attractor so that (2.1) is globally asymptotically stable. Property (i) has been proven.

(ii) If all the eigenvalues of A_0 are distinct, that is, card $\sigma(A_0) = n$, then Property (i) holds for all $\rho_0 \in (0, |\mu_2(A_0)|]$ and $\widetilde{\rho}_0 \in [0, |\mu_2(A_0)|)$. □

A stronger result than Theorem 2.1 with the replacement $(|\mu_2(A_0)| - \widetilde{\rho}_0)/K_0 \to |\mu_2(A_0)|$ in the relevant first inequality is now given. In other words, K_0 may be taken as unity and $\widetilde{\rho}_0$ may be zeroed.

Corollary 2.2. *Assume that A_0 is a stability matrix of ℓ_2-matrix measure $\mu_2(A_0)$ and that $\sup_{t\in\mathbb{R}_{0+}}(\|\widetilde{A}_0(t)\|_2 + \sum_{i=1}^{r}\|\widehat{A}_i(t+r_i)\|_2) < |\mu_2(A_0)|$. Assume also that $\lim_{t\to\infty}\widehat{A}_i(t) = A_i, \forall i \in \overline{r}\cup\{0\}$ and that $(\sum_{i=0}^{r} A_i)$ is nonsingular. Then, the system* (2.1) *is globally asymptotically stable independent of the sizes of the delays.*

Proof. Rearrange $K_0 e^{-\rho_0 t} = (K_0 e^{-\varepsilon_0 t})e^{-(\rho_0-\varepsilon_0)t}$ for any $\mathbb{R} \ni \varepsilon_0 \in (0, \rho_0)$. Then, $K_0 e^{-\rho_0 t} \le e^{-(\rho_0-\varepsilon_0)t}, \forall t \ge t_0 = \ln(K_0/\varepsilon_0)$, where t_0 depends on ε_0 and K_0. Redefine the bounded continuous function $P_a(t_0) : M_a(t_0) \to M_a(t_0)$, replacing $P : M \to M$ of Theorem 2.1, with the set in (2.4) being redefined as

$$M_a(t_0) := \{\phi \in BC^{(0)}(\mathbb{R}, \mathbb{R}^n) : \phi \equiv \varphi \in BPC^{(0)}([-r_r, 0]), \mathbb{R}^n),\ \phi \equiv \varphi \in BC^{(0)}((0, t_0), \mathbb{R}^n)\}, \tag{2.8}$$

so that the fixed point is looked for any potential perturbation in $[t_0, \infty)$ and not in $[-r_r, t_0)$. First, note that $P_a(t_0)$ is still continuous everywhere in its definition domain and also uniformly bounded since A_0 being a stability matrix and $P : M \to M$ being bounded imply

$$\|(P_a(t_0)\phi)(t)\|_2 \le \|(P\phi)(t)\|_2 + K_0\int_0^{t_0}\left\|\sum_{i=1}^{r} e^{-\rho_0(t-\tau)}\widehat{A}_i(\tau+r_i)\phi(\tau)\right\|_2 d\tau \le \|(P\phi)(t)\|_2 + K_1 < \infty, \tag{2.9}$$

$\forall t \ge t_0$ for some finite $K_1 = K_1(t_0) \in \mathbb{R}_+$ since t_0 is finite. Thus, (3.11) may be replaced with

$$\begin{aligned} &\|(P_a(t_0)\phi)(t) - (P_a(t_0)\eta)(t)\|_2 \\ &\quad = \left\|e^{A_0 t}\left[\int_{t_0}^{t} e^{-A_0\tau}\widetilde{A}_0(\tau)(\phi(\tau)-\eta(\tau))d\tau + \sum_{i=1}^{r}\int_{t_0+r_i}^{t} e^{-A_0\tau}\widehat{A}_i(\tau)(\phi(\tau-r_i)-\eta(\tau-r_i))d\tau\right]\right\|_2 \\ &\quad \le \frac{1}{\rho_0-\varepsilon_0}\left(\|\widetilde{A}_0\|_{2\infty} + \sum_{i=1}^{r}\|\widehat{A}_i\|_{2\infty}\right)\|\phi-\eta\|_{2\infty}; \quad \forall \phi, \eta \in M_a(t_0),\ \forall t \ge \max(t_0, r_r), \end{aligned} \tag{2.10}$$

so that $P_a(t_0)$ is a contraction if $(1/(|\mu_2(A_0)| - \varepsilon_{10} - \varepsilon_0))(\|\widetilde{A}_0\|_2 + \sum_{i=1}^{r}\|\widehat{A}_i\|_2) = (1/(\rho_0 - \varepsilon_0))(\|\widetilde{A}_0\|_2 + \sum_{i=1}^{r}\|\widehat{A}_i\|_2) < 1$, since ρ_0 may be chosen either fulfilling $\rho_0 < |\mu_2(A_0)|$ (the stability abscissa of A_0 is associated with a multiple eigenvalue), but arbitrarily close to $\mu_2(A_0)$, or $\rho_0 \le |\mu_2(A_0)|$ (the dominant eigenvalue of A_0 is single). As a result, $P_a(t_0)$ has a unique fixed point in $M_a(t_0)$. Since any positive and arbitrarily close to zero real constant $\varepsilon_{10} + \varepsilon_0$ may be used, $P_a(t_0)$ is a contraction if $(1/|\mu_2(A_0)|)(\|\widetilde{A}_0\|_2 + \sum_{i=1}^{r}\|\widehat{A}_i\|_2) < 1$. In addition, since (3.1) converges to a limiting equation and since $\ker(\sum_{i=0}^{r}A_i) = \{0\}$, the unique fixed point is zero which is a global asymptotic attractor independent of the sizes of the delays. Therefore, no state-trajectory solution may converge to a distinct point or to be oscillatory since the attractor is global and asymptotic, and no state-trajectory solution may be unbounded (since $P_a(t_0) : M_a(t_0) \to M_a(t_0)$ is bounded). Therefore, the constraint $(1/|\mu_2(A_0)|)(\|\widetilde{A}_0\|_\infty + \sum_{i=1}^{r}\|\widehat{A}_i\|_\infty) < 1$ leads to a contraction and then to a fixed point for the mapping $P : M \to M$ and the result has been proven. □

Similar results to Theorem 2.1 and Corollary 2.2 may be obtained by comparing the dynamic time-delay system (3.1) with the obtained one for zero delays. The system (2.1) is equivalently written as

$$\dot{x}(t) = \left(\sum_{i=0}^{r}\widehat{A}_i(t)\right)x(t) + \sum_{i=1}^{r}\widehat{A}_i(t)\left(x(t-r_i) - x(t)\right). \tag{2.11}$$

By stating the analogy with (2.2), the state-trajectory solution of (2.11), being equivalent to (2.2), is given by

$$x(t) = e^{At}\Bigg[x_0 + \int_0^t e^{-A\tau}\widetilde{A}(\tau)x(\tau)d\tau + \sum_{i=1}^{r}\left(\int_0^{r_i} e^{-A\tau}\widehat{A}_i(\tau)\left(\varphi(\tau-r_i) - \varphi(\tau)\right)d\tau + \int_{r_i}^{t} e^{-A\tau}\widehat{A}_i(\tau)\left(x(\tau-r_i) - x(\tau)\right)d\tau\right)\Bigg], \tag{2.12}$$

where the delay-free system is given by $\dot{z}(t) = (\sum_{i=0}^{r}\widehat{A}_i(t))z(t)$ of limiting counterpart $\dot{z}^*(t) = (\sum_{i=0}^{r}A_i)z^*(t)$, with

$$A := \sum_{i=0}^{r}A_i, \qquad \widetilde{A}(t) := \sum_{i=0}^{r}\widehat{A}_i(t) - A. \tag{2.13}$$

Use again the complete metric space $(M, \|\cdot\|_\infty)$ with the supremum metric $\|\cdot\|_\infty$ and M defined in (2.4) and replace the continuous mapping (2.5), using (2.12), by $P_\alpha : M \to M$ defined as

$$(P_\alpha\phi)(t) := e^{At}\Bigg[\varphi(0) + \int_0^t e^{-A\tau}\widetilde{A}_0(\tau)\phi(\tau)d\tau + \sum_{i=1}^{r}\left(\int_0^{r_i} e^{-A\tau}\widehat{A}_i(\tau)\left(\varphi(\tau-r_i)-\varphi(\tau)\right)d\tau + \int_{r_i}^{t} e^{-A\tau}\widehat{A}_i(\tau)\left(x(\tau-r_i)-x(\tau)\right)d\tau\right)\Bigg]. \tag{2.14}$$

The constraint (2.6) changes to

$$\|(P_\alpha\phi)(t)-(P_\alpha\eta)(t)\|_\infty \le \frac{K}{\rho}\left(\|\widetilde{A}_0\|_\infty + 2\sum_{i=1}^{r}\|\widehat{A}_i\|_\infty\right)\|\phi-\eta\|_\infty; \quad \forall\phi,\eta\in M, \tag{2.15}$$

where $\mathbb{R}\ni K\ge 1$ (norm-dependent) and $\rho\in\mathbb{R}_+$ (provided that A is a stability matrix) are such that, for instance, for the supremum on $\mathbb{R}_{0+}$ of the ℓ_2 vector (and induced matrix) norm, $\|e^{At}\|_2\le Ke^{-\rho t}, \forall t\in\mathbb{R}_{0+}$ so that (2.15) becomes

$$\sup_{t\in\mathbb{R}_{0+}}\|(P_\alpha\phi)(t)-(P_\alpha\eta)(t)\|_2 \le \frac{K}{\rho}\sup_{t\in\mathbb{R}_{0+}}\left(\|\widetilde{A}_0(t)\|_2+2\sum_{i=1}^{r}\|\widehat{A}_i(t)\|_2\right)\sup_{t\in\mathbb{R}_{0+}}\|\phi(t)-\eta(t)\|_2; \quad \forall\phi,\eta\in M. \tag{2.16}$$

Thus, Theorem 2.1 and Corollary 2.2 are modified as follows.

Theorem 2.3. *The following properties hold.*

(i) *Assume that A is a stability matrix of ℓ_2-matrix measure $\mu_2(A)$ and that $\sup_{t\in\mathbb{R}_0^+}(\|\widetilde{A}(t)\|_2+2\sum_{i=1}^{r}\|\widehat{A}_i(t+r_i)\|_2)<\rho/K=(|\mu_2(A)|-\widetilde{\rho})/K$ for some real constants $K\ge 1$ and $\widetilde{\rho}\in(0,|\mu_2(A)|)$ and any real constant $\rho\in(0,|\mu_2(A)|)$ such that the C_0-semigroup of the infinitesimal generator A satisfies $\|e^{At}\|_2\le Ke^{-\rho t}$. Assume also that $\lim_{t\to\infty}\widehat{A}_i(t)=A_i, \forall i\in\overline{r}\cup\{0\}$. Then, the system* (2.1) *is globally asymptotically stable independent of the sizes of the delays.*

(ii) *If all the eigenvalues of A are distinct, then global asymptotic stability independent of the sizes of the delays delay holds if $\sup_{t\in\mathbb{R}_0^+}(\|\widetilde{A}(t)\|_2+2\sum_{i=1}^{r}\|\widehat{A}_i(t+r_i)\|_2)<|\mu_2(A)|/K$, since $\rho\in(0,|\mu_2(A)|]$, with the remaining conditions being identical.*

Corollary 2.4. *Assume that A is a stability matrix of ℓ_2-matrix measure $\mu_2(A)$ and that $\sup_{t\in\mathbb{R}_0^+}(\|\widetilde{A}(t)\|_2+2\sum_{i=1}^{r}\|\widehat{A}_i(t+r_i)\|_2)<|\mu_2(A)|$. Assume also that $\lim_{t\to\infty}\widehat{A}_i(t)=A_i, \forall i\in\overline{r}\cup\{0\}$. Then, the system* (2.1) *is globally asymptotically stable independent of the sizes of the delays.*

Note that the requirement that $(\sum_{i=0}^{r}A_i)$ is nonsingular is imposed in Theorem 2.3 and Corollary 2.4, since $A=(\sum_{i=0}^{r}A_i)$ is directly nonsingular as it is a stability matrix. Note also that $A=(\sum_{i=0}^{r}A_i)$ being a stability matrix is also a direct consequence of Theorem 2.1 and Corollary 2.2, which give a result of asymptotic stability independent of the delays thus being valid for zero delays. However, such condition of nonsingularity of A (and even the strongest one of A being a stability matrix) is neither required to apply of the contraction mapping principle [21, 24], nor a direct consequence of it in Theorem 2.1 and Corollary 2.2. As a result, it cannot be invoked prior to stability but only being a consequence after stability has been proven.

Remark 2.5. Note that concerning the system matrices of the delay-free limiting systems and $\dot{z}^*(t)=Az^*(t)$, with $A=\sum_{i=0}^{r}A_i$, and of zero $\dot{z}_0^*(t)=A_0z_0^*(t)$ delayed dynamics and zero delays, respectively, one has the respective ℓ_2-matrix measures $\mu_2(A)=(1/2)\max(\operatorname{Re}\lambda(A+A^T));\mu_2(A_0)=(1/2)\max(\operatorname{Re}\lambda(A_0+A_0^T))$. Provided they are stable, those limiting systems possess the respective Lyapunov's functions $V^*(t) = z^*(t)z^*(t)$ and $V_0^*(t) = z_0^*(t)z_0^*(t)$ with respective time-derivatives:

$$\begin{aligned}\dot{V}^*(t) &= 2\dot{z}^{*^T}(t)z^*(t) = z^{*^T}(t)(A + A^T)z^*(t) \le 2\mu_2(A)V^*(t),\\ \dot{V}_0^*(t) &= 2\dot{z}_0^{*^T}(t)z_0^*(t) = z_0^{*^T}(t)(A_0 + A_0^T)z_0^*(t) \le 2\mu_2(A_0)V_0^*(t),\end{aligned} \tag{2.17}$$

so that

$$\frac{d}{dt}\|z^*(t)\|_2^2 \le 2\mu_2(A)\|z^*(t)\|_2^2, \qquad \frac{d}{dt}\|z_0^*(t)\|_2^2 \le 2\mu_2(A_0)\|z_0^*(t)\|_2^2. \tag{2.18}$$

As a result, a stronger result than Theorem 2.1(i) holds by replacing $(|\mu_2(A_0)| - \tilde{\rho}_0)/K_0 \to |\mu_2(A_0)|$ and also a stronger result than Theorem 2.1(ii) holds by replacing $|\mu_2(A_0)|/K_0 \to |\mu_2(A_0)|$. In the same way, a stronger result than Theorem 2.3(i) holds by replacing $(|\mu_2(A)| - \tilde{\rho})/K \to |\mu_2(A)|$ and a stronger result than Theorem 2.3(ii) holds by replacing $|\mu_2(A)|/K \to |\mu_2(A)|$. Then, Corollaries 2.2 and 2.4 follow directly as stronger results than Theorem 2.1, respectively, Theorem 2.3 via a very short modified proof by using simple Lyapunov's theory. In other words, global asymptotic stability of the current system holds under asymptotic stability of the respective auxiliary limiting delay-free systems by taking $K_0 = K = 1, \rho_0 = |\mu_2(A_0)|, \rho = |\mu_2(A)|$ and $\tilde{\rho}_0 = \tilde{\rho} = 0$ in the relevant inequalities of norms of Theorems 2.1 and 2.3 just as proven in Corollaries 2.2 and 2.4.

3. Feedback linear systems with point constant delays and the contraction mapping theorem

The fixed point theory and associated stability results of Section 2 are used and extended directly to state-feedback controlled systems as follows. Instead of the dynamic system (2.1), consider the controlled dynamic system:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{r}\overline{A}_i(t)x(t - r_i) + \overline{B}(t)u(t)\\ &= \sum_{i=0}^{r}(\overline{A}_i^* + \tilde{\overline{A}}_i(t))x(t - r_i) + (\overline{B}^* + \tilde{\overline{B}}(t))u(t) \qquad (3.1)\\ &= \sum_{i=0}^{r}(\overline{A}_i^* + \tilde{\overline{A}}_i(t))x(t) + \sum_{i=1}^{r}(\overline{A}_i^* + \tilde{\overline{A}}_i(t))(x(t - r_i) - x(t)) + (\overline{B}^* + \tilde{\overline{B}}(t))u(t), \qquad (3.2)\end{aligned}$$

where $\overline{A}_i : \mathbb{R}_{0+} \to \mathbb{R}^{n\times n}$ $(i \in \overline{r} \cup \{0\})$ and $\overline{B} \in \mathbb{R}^{n\times n}$ are piecewise continuous bounded matrix functions, $\overline{A}_i^* \in \mathbb{R}^{n\times n}$ $(i \in \overline{r} \cup \{0\}), \overline{B}^* \in \mathbb{R}^{n\times m}$ and the control $u : \mathbb{R}_{0+} \to \mathbb{R}^m$ is generated according to the state-feedback linear control law:

$$u(t) = \sum_{i=0}^{r}\overline{K}_i(t)x(t - r_i) = \sum_{i=0}^{r}(\overline{K}_i^* + \tilde{\overline{K}}_i(t))x(t - r_i), \tag{3.3}$$

where $\overline{K}_i : \mathbb{R}_{0+} \to \mathbb{R}^{n\times n}$ $(i \in \overline{r} \cup \{0\})$ are piecewise continuous bounded matrix functions and $\overline{K}_i^* \in \mathbb{R}^{n\times m}$ $(i \in \overline{r} \cup \{0\})$. The substitution of (3.3) into (3.1) leads to a closed-loop system identical to (2.1) through the identities:

$$\begin{aligned} A_i &= \overline{A}_i^* + \overline{B}^* \overline{K}_i^*, \\ \widehat{A}_i(t) &= A_i + \widetilde{A}_i(t) = \overline{A}_i(t) + \overline{B}(t)\overline{K}_i(t), \\ \widetilde{A}_i(t) &= \widetilde{\overline{A}}_i(t) + \overline{B}^* \widetilde{\overline{K}}_i(t) + \widetilde{\overline{B}}(t)\left(\overline{K}_i^* + \widetilde{\overline{K}}_i(t)\right) = \left(\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\overline{K}_i^*\right) + \overline{B}(t)\widetilde{\overline{K}}_i(t); \quad \forall i \in \overline{r} \cup \{0\}. \end{aligned} \tag{3.4}$$

Important properties of dynamic systems are those of controllability, observability, stabilizability, and detectability. For the linear time-invariant dynamic systems

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad y(t) = Cx(t) \tag{3.5}$$

of state $x(t)$ of n state variables and control $u(t)$ and output $y(t)$ of respective dimensions m and p, those properties are easily tested through the appropriate Popov-Belevitch-Hutus rank tests [31]. Thus,

(1) the dynamic system is controllable (or, simply, the pair (A, B) is controllable) if and only if rank $[sI_n - AB] = n$; $\forall s \in \mathbb{C}$. An equivalent test is that (A, B) is controllable if and only if rank $[BAB \cdots A^{n-1}B] = n$. The meaning of this property is that for any bounded $x^* \in \mathbb{R}^n$, there exists a piecewise continuous control $u : [0, t_f] \cap \mathbb{R}_{0+} \to \mathbb{R}^m$ such that $x(t_f) = x^*$ for some finite t_f. An equivalent property is the existence of a controller of gain $K \in \mathbb{R}^{m\times n}$ such that a linear state-feedback control defined by $u(t) = Kx(t)$ makes the matrix $\overline{A} = A + BK$ feedback obtained system $\dot{x}(t) = (A + BK)x(t)$ to possess a prescribed spectrum $\sigma(\overline{A})$.

(2) The dynamic system is stabilizable (or, simply, the pair (A, B) is stabilizable) if and only if rank $[sI_n - AB] = n$; $\forall s \in \mathbb{C}_{0+}$. Its meaning is that there exists $K \in \mathbb{R}^{m\times n}$ such that the matrix of dynamics $\overline{A}$ of the closed-loop feedback system is a stability matrix; that is, $\sigma(\overline{A}) \cap \mathbb{C}_{0+} = \varnothing$ and any state-trajectory solution with bounded initial conditions is uniformly bounded and converges asymptotically to the zero equilibrium, as a result. By comparing the controllability and stabilizability tests, it turns out that controllability implies stabilizability but the converse is not true in general.

(3) The dynamic system is observable (or, simply, the pair (A, C) is observable) if and only if the pair (A^T, C^T) is controllable. If A, B, and C are admitted to be complex matrices, then transposes are replaced with conjugate transposes. Observability is related to the ability of calculating the past-state vector from output measurements (usually $p < n$). Similarly, the dynamic system is detectable (or, simply, the pair (A, C) is detectable) if and only if the pair (A^T, C^T) is stabilizable.

The above concepts are extendable with more involved tests to time-varying and nonlinear dynamic systems. Related results have also been investigated related to fixed point theory (see, e.g., [32, 33]). Recent stability results based on fixed point theory are provided in [34, 35]. The following result follows directly from the controllability property of linear systems. It will be then used for obtaining small left-hand side terms in the norms inequalities of Theorems 2.1 and 2.3 via feedback under assumptions of controllability of relevant matrix pairs.

Lemma 3.1. *The following properties hold.*

(i) *Assume that the pair* $(\overline{A}_i^*, \overline{B}^*)$ *is controllable for any given* $i \in \overline{r} \cup \{0\}$. *Then, for any prescribed set of nonnecessarily distinct complex numbers* $S_{Ai} = \{\rho_{ji} \in \mathbb{C} : j \in \overline{n}\}$, *there exists a controller matrix* $\overline{K}_i^*$ *such that the spectrum of the obtained* A_i *via* (3.4) *is* $\sigma(A_i) = S_{Ai}$. *As a result,* $\mu_2(A_i) = (1/2)\lambda_{\max}(A_i + A_i^T)$ *is also predefined according to the prescribed set* S_{Ai}.

(ii) *If the pair* $(\overline{A}_0^*, \overline{B}^*)$ *is controllable, then there exists a controller matrix* $\overline{K}_0^*$ *such that the* $\sigma(A_0) = S_{A0}$ *for any given prescribed set of complex numbers* S_{A0}. *As a result,* $\mu_2(A_0) = (1/2)\lambda_{\max}(A_0 + A_0^T)$ *is also predefined according to* $\sigma(A_0)$.

If the pair $(\sum_{i=0}^r \overline{A}_i^*, \overline{B}^*)$ *is controllable then there exists a controller matrix* $\overline{K}^* = \sum_{i=0}^r \overline{K}_i^*$ *such that the* $\sigma(\sum_{i=0}^r \overline{A}_i^*) = S_A$ *for any given prescribed set of complex numbers* S_A. *As a result,* $\mu_2(\sum_{i=0}^r \overline{A}_i^*) = (1/2)\lambda_{\max}(\sum_{i=0}^r \overline{A}_i^* + \sum_{i=0}^r \overline{A}_i^{*T})$ *becomes also predefined accordingly.*

Corollary 2.2, through Lemma 3.1(i), leads directly to the subsequent result.

Theorem 3.2. *Assume that*

(1) $\lim_{t\to\infty} \widehat{A}_i(t) = A_i, \forall i \in \overline{r} \cup \{0\}$ *and that* $(\sum_{i=0}^r A_i)$ *is nonsingular,*

(2) $(\overline{A}_i^*, \overline{B}^*); \forall i \in \overline{r} \cup \{0\}$ *are controllable pairs, and*

(3) $\sup_{t\in\mathbb{R}_{0+}} \max(\|\widetilde{\overline{A}}_i(t)\|_2, \|\widetilde{\overline{B}}_i(t)\|_2) \le \zeta$ *for some* $\zeta \in \mathbb{R}_+$.

Then, there exist (in general, nonunique) constant controller gains $\overline{K}_i(t) = \overline{K}_i^*, i \in \overline{r} \cup \{0\}, \forall t \in \mathbb{R}_{0+}$ *such that the closed-loop system* (3.1)–(3.3) *is globally asymptotically stable independent of the sizes of the delays.*

Proof. Theorem 2.1 holds if $\mu_2(A_0) < 0 \wedge Z_0 < |\mu_2(A_0)|\ \forall t \in \mathbb{R}_{0+}$, where

$$Z_0 := \sup_{t\in\mathbb{R}_0^+}\Bigg(\left\|\widetilde{\overline{A}}_0(t) + \widetilde{\overline{B}}(t)\overline{K}_0^* + \overline{B}(t)\widetilde{\overline{K}}_0(t)\right\|_2 + \sum_{i=1}^r \left(\left\|\overline{A}_i^* + \overline{B}^*\overline{K}_i^*\right\|_2 + \left\|\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\overline{K}_i^* + \overline{B}(t)\widetilde{\overline{K}}_i(t)\right\|_2 \right)\Bigg). \tag{3.6}$$

Note the following.

(1) Since $(\overline{A}_0^*, \overline{B}^*)$ is controllable, there exists $\overline{K}_0^*$ such that $A_0 = \overline{A}_0^* + \overline{B}^*\overline{K}_0^*$ is a stability matrix and with prescribed spectrum $\sigma(A_0)$, then with prescribed matrix measure $\mu_2(A_0) < 0$.

(2) Since $(\overline{A}_i^*, \overline{B}^*)$ is controllable for $i \in \overline{r}$, there exists $\overline{K}_i^*$ such that $A_i = \overline{A}_i^* + \overline{B}^*\overline{K}_i$ has any prescribed spectrum $\sigma(A_i)$. Then, fix $\sigma(A_i) = \{\lambda_{ji} \in \mathbb{C} : \lambda_{ji} \ne \lambda_{ki}$ if $j \ne k$, $|\lambda_{ji}| \le \zeta_0;\ \forall j \in \overline{n}\}; \forall i \in \overline{r}$. Since the eigenvalues of A_i are distinct, it always exists a nonsingular real n-matrix $T_i\ (i \in \overline{r})$ such that $\|A_i\|_2 = \|\overline{A}_i^* + \overline{B}^*\overline{K}_i^*\|_2 = \|T_i^{-1}\Lambda_{i\zeta}T_i\|_2 \le \zeta_0 K_2(T_i); \forall i \in \overline{r}$, where $\Lambda_{i\varepsilon} := \mathrm{Diag}\,(\lambda_{1i}.\lambda_{2i}, \dots, \lambda_{ni}); \forall i \in \overline{r}$. As a result, one gets from (3.6) that for some real n-matrices $T_i\ (i \in \overline{r}), Z_0 \le \overline{Z}_0 := (r + 1 + \sum_{i=1}^r \|\overline{K}_i^*\|_2)\zeta + \zeta_0\sum_{i=1}^r (K_2(T_i))$, provided that $\sup_{t\in\mathbb{R}_{0+}} \max(\|\widetilde{\overline{A}}_i(t)\|_2, \|\widetilde{\overline{B}}_i(t)\|_2) \le \varepsilon$ under the incremental controller gains choice $\widetilde{\overline{K}}_i(t) = 0; \forall i \in \overline{r}$. The proof follows from Theorem 2.1(i), since $\overline{Z}_0$ is independent of $\overline{K}_0^*$ and thus

on T_0, so that $\mu_2(A_0)$ is independent of $\overline{Z}_0$ so that it can be fixed fulfilling $|\mu_2(A_0)| > \overline{Z}_0$ by appropriately selecting $\overline{K}_0^*$ for any given $\zeta \in \mathbb{R}_+$.

Theorem 3.2 is useful to guarantee closed-loop stabilization of the system (2.1) under controllability conditions of the time-invariant dynamics by first stabilizing the delay-free dynamics of the limiting equation via linear feedback with sufficiently large stability abscissa. The result is achievable irrespective of the norms of the incremental matrices of dynamics of (2.1) with respect to its limiting equation. Theorem 3.2 is now extended by replacing the time-invariant controllers by time-varying ones. □

Corollary 3.3. *Suppose that the assumptions (1)–(3) of Theorem 3.2 hold. Then, there exist nonzero (nonunique) controller gain matrix functions* $\overline{K}_i(t) = \overline{K}_i^* + \widetilde{\overline{K}}_i(t), i \in \overline{r} \cup \{0\}, \forall t \in \mathbb{R}_{0+}$ *such that the closed-loop system* (3.1)–(3.3) *is globally asymptotically stable independent of the sizes of the delays.*

Furthermore, if $\operatorname{rank} \overline{B}(t) = m, \forall t \in \mathbb{R}_{0+}$, *then the controller gains* $\overline{K}_i(t) = \overline{K}_i^* + \widetilde{\overline{K}}_i(t)$ $i \in \overline{r} \cup \{0\}$ *defined with any members of sets of constant controller gains* $\overline{K}_i^*, i \in \overline{r} \cup \{0\}$ *chosen according to Theorem 3.2 and incremental controller gains* $\widetilde{\overline{K}}_i(t) = -(\overline{B}^T(t)\overline{B}(t))^{-1}\overline{B}^T(t)(\widetilde{\overline{A}}_i(t) + \widetilde{B}(t)\overline{K}_i^*); \forall i \in \overline{r} \cup \{0\}$ *guarantee the global asymptotic stability independent of the sizes of the delays of the closed-loop system* (3.1)-(3.2).

Proof. It follows from Theorem 3.2, provided that the incremental controller gains $\widetilde{\overline{K}}_i : \mathbb{R}_{0+} \to \mathbb{R}^{n\times m}$ satisfy $|\mu_2(A_0)| > \overline{Z} := \overline{Z}_0 + \sup_{t\in\mathbb{R}_{0+}} (\|\overline{B}(t)\| \sum_{i=0}^r \|\widetilde{\overline{K}}_i(t)\|)_2$ after replacing $Z_0 \to Z$ and $\overline{Z}_0 \to \overline{Z}$. Such nonzero controller gains always exist since $|\mu_2(A_0)| > \overline{Z}_0$ from Theorem 3.2. To simplify the subsequent notation define the matrix function $G : \mathbb{R}_{0+} \to \mathbb{R}^{n\times 2(r+1)n} \times \mathbb{R}^{n\times 2m} \times \mathbb{R}^{n\times(r+1)m}$ by $G(t) := [\overline{A}_0^*, \dots, \overline{A}_r, \widetilde{\overline{A}}_0, \dots, \widetilde{\overline{A}}_r, \overline{B}^*, \widetilde{\overline{B}}, \overline{K}_0^T, \dots, \overline{K}_r^T]$. Now, taking into account (3.6), define the nonnegative real functional $z_0 : \mathbb{R}_{0+} \times \mathbb{R}^{n\times 2(r+1)n} \times \mathbb{R}^{n\times 2m} \times \mathbb{R}^{n\times 2(r+1)m} \to \mathbb{R}_{0+}$ by

$$z_0\left(t, G, \widetilde{\overline{K}}_0^T, \dots, \widetilde{\overline{K}}_r^T\right) := \Bigg(\|\widetilde{\overline{A}}_0(t) + \widetilde{\overline{B}}(t)\overline{K}_0^* + \overline{B}(t)\widetilde{\overline{K}}_0(t)\|_2 + \sum_{i=1}^r \left(\|\overline{A}_i^* + \overline{B}^*\overline{K}_i^*\|_2 + \|\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\overline{K}_i^* + \overline{B}(t)\widetilde{\overline{K}}_i(t)\|_2 \right) \Bigg). \tag{3.7}$$

Note by construction, (3.6) and Theorem 3.2, that $z_0(t, 0, \widetilde{\overline{K}}_0^T, \dots, \widetilde{\overline{K}}_r^T) \le Z_0 \le \overline{Z}_0 < |\mu_2(A_0)|; \forall t \in \mathbb{R}_{0+}$. Thus, it follows from Theorem 3.2 that there is an open ball $\mathbb{B}$ of $\mathbb{R}^{n\times 2(r+1)m}$ centered at cero fulfilling $z_0(t, G, X) < |\mu_2(A_0)|, \forall X \in \mathbb{B}$ so that the closed-loop system (3.1)–(3.3) is globally asymptotically stable independent of the delays for sets of nonzero incremental controller gains since the same property is fulfilled for sets of constant controller gains.

If $\operatorname{rank} \overline{B}(t) = m, \forall t \in \mathbb{R}_{0+}$, then the incremental controller gains $\widetilde{\overline{K}}_i^0(t) = -(\overline{B}^T(t)\overline{B}(t))^{-1}\overline{B}^T(t)(\widetilde{\overline{A}}_i(t) + \widetilde{B}(t)\overline{K}_i^*); \forall i \in \overline{r} \cup \{0\}$ fulfill

$$\begin{aligned} &\|\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\overline{K}_i^* + \overline{B}(t)\widetilde{\overline{K}}_i^0(t)\|_2 \\ &\quad = \operatorname*{Inf}_{\widetilde{\overline{K}}_i \in \mathbb{R}^{m\times n}} \|\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\overline{K}_i^* + \overline{B}(t)\widetilde{\overline{K}}_i(t)\|_2 \|(I_n - \overline{B}(t)(\overline{B}^T(t)\overline{B}(t))^{-1}\overline{B}^T(t))(\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\overline{K}_i^*)\|_2, \end{aligned} \tag{3.8}$$

from least squares minimization. As a result, the $(r+1)$ matrix function on incremental controllers $(\tilde{\overline{K}}_0^{0^T}, \tilde{\overline{K}}_1^{0^T}, \dots, \tilde{\overline{K}}_r^{0^T}) \in \mathbb{B}$ and, furthermore, $\mu_2(A_0) < 0$ and

$$\left|\mu_2(A_0)\right| > z_0\left(t, G, \tilde{\overline{K}}_0^{0^T}, \tilde{\overline{K}}_1^{0^T}, \dots, \tilde{\overline{K}}_r^{0^T}\right) = \sum_{i=0}^{r}\left(\left\|\left(I_n - \overline{B}(t)\left(\overline{B}^T(t)\overline{B}(t)\right)^{-1}\overline{B}^T(t)\right)\left(\tilde{\overline{A}}_i(t) + \tilde{\overline{B}}(t)\overline{K}_i^*\right)\right\|_2\right) + \sum_{i=1}^{r}\left\|\overline{A}_i^* + \overline{B}^*\overline{K}_i^*\right\|_2 \tag{3.9}$$

guarantee the global asymptotic stability independent of the sizes of the delays of the closed-loop system (3.1)–(3.3). □

Note that if $m \geq n$ and $\operatorname{rank}\overline{B}(t) = m; \forall t \in \mathbb{R}_{0+}$, then $|\mu_2(A_0)| \geq \sum_{i=1}^{r}\|\overline{A}_i^* + \overline{B}^*\overline{K}_i^*\|_2$ and $\mu_2(A_0) < 0$ guarantee the closed-loop stability from Corollary 3.3. If $m < n$ and $\operatorname{rank}(\overline{B}(t), \tilde{\overline{A}}_i(t) + \tilde{\overline{B}}(t)\overline{K}_0^*) = \operatorname{rank}\overline{B}(t) = m; \forall i \in \overline{r} \cup \{0\}; \forall t \in \mathbb{R}_{0+}$, then from Kronecker-Capelli's theorem, see, for instance, [11, 15], there exist infinitely many solutions of the incremental controller gains which make $(I_n - \overline{B}(t)(\overline{B}^T(t)\overline{B}(t))^{-1}\overline{B}(t))(\tilde{\overline{A}}_i(t) + \tilde{\overline{B}}(t)\overline{K}_i^*) = 0$ so that the closed-loop stability is guaranteed under $|\mu_2(A_0)| \geq \sum_{i=1}^{r}\|\overline{A}_i^* + \overline{B}^*\overline{K}_i^*\|_2$ with $\mu_2(A_0) < 0$. On the other hand, Corollary 3.3 allows obtaining a subsequent direct result under a weaker Condition 2 of Theorem 3.2. In particular, only the controllability of $(\overline{A}_0^*, \overline{B}^*)$, and not that of the remaining pairs $(\overline{A}_i^*, \overline{B}^*); \forall i \in \overline{r}$, is requested for selecting an appropriate negative value of $\mu_2(A_0)$ and the static controller gains are chosen for least-squares minimization of the associated term in z_0.

Corollary 3.4. *Assume that*

(1) $\lim_{t\to\infty}\hat{A}_i(t) = A_i; \forall i \in \overline{r} \cup \{0\}$ *and that* $(\sum_{i=0}^{r} A_i)$ *is nonsingular,*

(2) $(\overline{A}_0^*, \overline{B}^*)$ *is a controllable pair,*

(3) $\sup_{t\in\mathbb{R}_{0+}} \max(\|\tilde{\overline{A}}_i(t)\|_2, \|\tilde{\overline{B}}_i(t)\|_2) \leq \zeta$ *for some* $\zeta \in \mathbb{R}_+$,

(4) $\operatorname{rank}\overline{B}^* = \operatorname{rank}\overline{B}(t) = m;\ \forall t \in \mathbb{R}_{0+}$.

Then, the closed-loop system (3.1)–(3.3) *is globally asymptotically stable independent of the sizes of the delays provided that the controller gains are synthesized as follows:*

(5) $\overline{K}_i^* = -(\overline{B}^{*T}\overline{B}^*)^{-1}\overline{B}^{*T}\overline{A}_i^*;\ \forall i \in \overline{r}$,

(6) $\tilde{\overline{K}}_i^0(t) = -(\overline{B}^T(t)\overline{B}(t))^{-1}\overline{B}^T(t)(\tilde{\overline{A}}_i(t) + \tilde{B}(t)\overline{K}_i^*);\ \forall i \in \overline{r} \cup \{0\}$,

(7) $\overline{K}_0^*$ *is synthesized so that such that* $\sigma(A_0)$ *satisfies the constraints* $\mu_2(A_0) < 0$,

$$\begin{aligned}|\mu_2(A_0)| > \sum_{i=1}^{r}\Bigg(&\left\| \left(I_n - \overline{B}^*\left(\overline{B}^{*T}\overline{B}^*\right)^{-1}\overline{B}^{*T}\right)\overline{A}_i^*\right\|_2 \\ &+ \sum_{i=0}^{r}\Big(\Big\| \left(I_n - \overline{B}(t)\left(\overline{B}^T(t)\overline{B}(t)\right)^{-1}\overline{B}^T(t)\right) \\ &\quad \times \left(\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\left(I_n - \overline{B}^*\left(\overline{B}^{*T}\overline{B}^*\right)^{-1}\overline{B}^{*T}\right)\overline{A}_i^*\right)\Big\|_2\Big)\Bigg).\end{aligned} \tag{3.10}$$

In the same way as Theorem 3.2 is obtained from Corollary 2.2 (a refinement of Theorem 2.1), Theorem 2.3 leads to the subsequent result which is obtained based on a comparison of the delayed dynamics with the delay-free limiting dynamics.

Theorem 3.5. *Assume that*

(1) $\lim_{t\to\infty}\widehat{A}_i(t) = A_i, \forall i \in \overline{r} \cup \{0\}$ *and that* $(\sum_{i=0}^{r} A_i)$ *is a stability matrix,*

(2) $(\sum_{i=0}^{r}\overline{A}_i^*, \overline{B}^*)$ *is a controllable pair,*

(3) $\sup_{t\in\mathbb{R}_{0+}}\max(\|\widetilde{\overline{A}}_i(t)\|_2, \|\widetilde{\overline{B}}_i(t)\|_2) \le \zeta$ *for some* $\zeta \in \mathbb{R}_+$.

Then, there exist sets of nonunique controller gain matrix functions $\overline{K}_i : \mathbb{R}_{0+} \to \mathbb{R}^{m\times n}$ *defined by* $\overline{K}_i(t) = \overline{K}_i^* + \widetilde{\overline{K}}_i(t), i \in \overline{r} \cup \{0\}, \forall t \in \mathbb{R}_{0+}$ *such that the closed-loop system* (3.1)–(3.3) *is globally asymptotically stable independent of the sizes of the delays and* $[(\sum_{i=1}^{r}\overline{A}_i^*) + \overline{B}^*(\sum_{i=1}^{r}\overline{K}_i^*)]$ *has an arbitrary spectrum of distinct eigenvalues of modulus not larger than a prescribed upper bound* $\zeta \in \mathbb{R}_+$.

Proof. The substitution of (3.3) into (3.2) yields

$$\begin{aligned}\dot{x}(t) = &\sum_{i=0}^{r}\left(\overline{A}_i^* + \overline{B}^*\overline{K}_i^*\right)x(t) + \sum_{i=0}^{r}\left(\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}_i(t)\overline{K}_i^* + \overline{B}(t)\widetilde{\overline{K}}_i(t)\right)x(t) \\ &+ \sum_{i=1}^{r}\left(\overline{A}_i^* + \overline{B}^*\overline{K}_i^* + \widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}_i(t)\overline{K}_i^* + \overline{B}(t)\widetilde{\overline{K}}_i(t)\right)\left(x(t-r_i) - x(t)\right),\end{aligned} \tag{3.11}$$

$((\sum_{i=0}^{r}\overline{A}_i^*), \overline{B}^*)$ being controllable $\Rightarrow \exists T$ (a nonsingular real n-matrix) such that $\|(\sum_{i=0}^{r}\overline{A}_i^*) + \overline{B}^*(\sum_{i=0}^{r}\overline{K}_i^*)\|_2 = \|T^{-1}\Lambda_\upsilon T\|_2 \le \upsilon K_2(T) \wedge \sigma(A) = \sigma(\Lambda_\upsilon) = \{\lambda_i : \lambda_i \ne \lambda_j\ (i \ne j) \wedge |\lambda_i| \le \upsilon, \forall i \in \overline{n}\}$ for any given $\upsilon \in \mathbb{R}_+$. Define

$$\begin{aligned}Z_A := \sup_{t\in\mathbb{R}_0^+}\Bigg(&\left\|\sum_{i=0}^{r}\left(\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}_i(t)\overline{K}_i^* + \overline{B}(t)\widetilde{\overline{K}}_i(t)\right)\right\|_2 \\ &+ 2\left\|\left(\sum_{i=1}^{r}\overline{A}_i^*\right) + \overline{B}^*\left(\sum_{i=1}^{r}\overline{K}_i^*\right)\right\|_2 + 2\sum_{i=1}^{r}\left(\left\|\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\overline{K}_i^* + \overline{B}(t)\widetilde{\overline{K}}_i(t)\right\|_2\right)\Bigg) \\ \le \overline{Z}_A := 2K_2(T)\upsilon + &\sup_{t\in\mathbb{R}_{0+}}\left(r + \left\|\sum_{i=0}^{r}\overline{K}_i^*\right\|_2\right)\zeta + 3\sup_{t\in\mathbb{R}_{0+}}\left(\left\|\overline{B}(t)\right\|_2\left(\sum_{i=1}^{r}\left\|\widetilde{\overline{K}}_i(t)\right\|_2\right)\right).\end{aligned} \tag{3.12}$$

Then, stability of the closed-loop system (3.1)–(3.3) holds if $(\sum_{i=0}^{r}\overline{A}_i^*)$ is a stability matrix and

(a) the set of static controller gain matrices $\overline{K}_i^* : \mathbb{R}_{0+} \to \mathbb{R}^{m\times n}; \forall i \in r \cup \{0\}$ satisfies that $\mathrm{sp}\,((\sum_{i=1}^{r}\overline{A}_i^*) + \overline{B}^*(\sum_{i=1}^{r}\overline{K}_i^*)) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ consists of n distinct complex numbers of modulus not larger than any prescribed $\upsilon \in \mathbb{R}_+$ and the nonsingular matrix T defines the similarity transformation $T[(\sum_{i=1}^{r}\overline{A}_i^*) + \overline{B}^*(\sum_{i=1}^{r}\overline{K}_i^*)]T^{-1} = \Lambda_\upsilon = \mathrm{Diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

(b) The set of incremental controller gain matrix functions $\tilde{\overline{K}}_i : \mathbb{R}_{0+} \to \mathbb{R}^{m\times n}; \forall i \in r \cup \{0\}$is chosen so that $|\mu_2(\sum_{i=0}^{r}\overline{A}_i^*)| > \overline{Z}_A$ with $\overline{Z}_A$ being defined in (3.12).

□

Corollaries to Theorem 3.5 might be obtained directly based on the ideals of Corollaries 3.3-3.4 for Theorem 3.2.

4. Further extensions

The following definitions and associate properties are well known in control theory of linear and time-invariant dynamic systems.

(1) A pair of complex matrices $(\Phi, H), \Phi \in \mathbb{C}^{n\times n},\ H \in \mathbb{R}^{n\times m}$ is said to be stabilizable (or asymptotically controllable) if $\exists K \in \mathbb{R}^{m\times n}$ such that $\sigma(\Phi + HK) \subset \mathbb{C}_-$.

(2) Stabilizability of (Φ, H) is a weaker property than the controllability of such a pair what means that $\exists K \in \mathbb{R}^{m\times n}$ such that $\sigma(\Phi + HK) = \Lambda_K$ for each prescribed set of n numbers (possibly repeated) $\Lambda_K \subset \mathbb{C}$. An equivalent characterization of the controllability of (Φ, H) is rank $(H, \Phi H, \ldots, \Phi^{n-1}H) = n$.

(3) If an open-loop system is stabilizable but not controllable, all its uncontrollable open-loop modes are invariant and stable under any state-feedback law.

This section gives extensions of some results of Section 3 for the case when some controllability conditions are lost but stabilizability still holds and some further extensions for the case of output feedback controllers. The following result is weaker but more general than Theorem 3.2.

Corollary 4.1. *Assume that*

(1) $\lim_{t\to\infty}\widehat{A}_i(t) = A_i, \forall i \in \overline{r} \cup \{0\}$ *and that* $(\sum_{i=0}^{r}A_i)$ *is nonsingular,*

(2) $(\overline{A}_0^*, \overline{B}^*)$ *is a stabilizable, but not controllable, pair and* $(\overline{A}_i^*, \overline{B}^*); \forall i \in \overline{r}$ *are all controllable pairs,*

(3) $\sup_{t\in\mathbb{R}_{0+}} \max(\|\tilde{\overline{A}}_i(t)\|_2, \|\tilde{\overline{B}}_i(t)\|_2) \leq \zeta$ *for some* $\zeta \in \mathbb{R}_+$.

Then, there exist (in general, nonunique) constant controller gains $\overline{K}_i(t) = \overline{K}_i^*, i \in \overline{r} \cup \{0\}, \forall t \in \mathbb{R}_{0+}$ *such that the closed-loop system* (3.1)–(3.3) *is globally asymptotically stable independent of the sizes of the delays provided that* $\overline{Z}_0 < |\rho_{0uc}|, \forall t \in \mathbb{R}_{0+}$, *where* Z_0 *is defined in* (3.6) *and* $-\rho_{0uc} < 0$ *is the stability abscissa of the uncontrollable dominant eigenvalue of* $\overline{A}_0^*$. *The stability property also holds if* $\overline{Z}_0 < |\rho_{0uc}|$ *with* $\overline{Z}_0$ *being defined in the proof of Theorem 3.2.*

Proof. It is similar to that of Theorem 3.2 by noting that $\sup_{\overline{K}_0^*\in S\subset\mathbb{R}^{m\times n}}\mu_2(A_0) \leq -\rho_{0uc} < 0$, where $S \subset \mathbb{R}^{m\times n}$ is the in general open and not simply connected domain of stabilizable static controller gains $\overline{K}_0^*$ of the pair $(\overline{A}_0^*, \overline{B}^*)$. □

Theorem 3.2 may also be extended straightforwardly by replacing $(\overline{A}_0^*, \overline{B}^*)$ controllable by $(\overline{A}_0^*, \overline{B}^*)$ stabilizable, $Z_0 \rightarrow Z$, and $\mu_2(A_0) \rightarrow -\rho_{0uc}$. Corollaries 3.3-3.4 are also directly extendable based on Corollary 4.1. On the other hand, Theorem 3.5 extends directly to the subsequent result which is weaker in the sense that the matrix measure cannot be prefixed since controllability of $(\sum_{i=0}^{r}\overline{A}_i^*, \overline{B}^*)$ is replaced by its stabilizability.

Corollary 4.2. *Assume that*

(1) $\lim_{t\to\infty}\widehat{A}_i(t) = A_i, \forall i \in \overline{r} \cup \{0\}$ *and that* $(\sum_{i=0}^{r} A_i)$ *is a stability matrix,*

(2) $(\sum_{i=0}^{r}\overline{A}_i^*, \overline{B}^*)$ *is a stabilizable pair,*

(3) $\sup_{t\in\mathbb{R}_{0+}} \max(\|\widetilde{\overline{A}}_i(t)\|_2, \|\widetilde{\overline{B}}_i(t)\|_2) \leq \zeta$ *for some* $\zeta \in \mathbb{R}_+$.

Then, there exist sets of nonunique controller gain matrix functions $\overline{K}_i : \mathbb{R}_{0+} \rightarrow \mathbb{R}^{m\times n}$ *defined by* $\overline{K}_i(t) = \overline{K}_i^* + \widetilde{\overline{K}}_i(t), i \in \overline{r} \cup \{0\}, \forall t \in \mathbb{R}_{0+}$ *such that the closed-loop system* (3.1)–(3.3) *is globally asymptotically stable independent of the sizes of the delays provided that* $Z_A < |\rho_{uc}|$, *with* Z_A *defined in* (3.12) *and* $-\rho_{uc} < 0$ *is the stability abscissa of the uncontrollable (stable and invariant under state feedback) dominant eigenvalue of* $\sum_{i=0}^{r}\overline{A}_i^*$.

Now, assume that the control law (3.3) is replaced with

$$u(t) = \sum_{i=0}^{r}\overline{K}_i(t)\overline{C}x(t - r_i) = \sum_{i=0}^{r}\left(\overline{K}_i^* + \widetilde{\overline{K}}_i(t)\right)\overline{C}x(t - r_i) \tag{4.1}$$

for some set of output matrices $\overline{C} \in \mathbb{R}^{p\times m}$ with $p \leq m$. The interpretation of (4.1) is that the controller has not access to all the state components of the system but only to some linear combinations of them, namely, the output vector defined by $y(t) = \overline{C}x(t)$. This situation is very realistic under the constraint $\max(m, p) < n$, that is, the numbers of input and output components are less than the number of state components. The following further definitions and related properties features are well known from basic control theory [28].

(4) Observability is a dual property to controllability in the sense that the pair (Ω, P), $P \in \mathbb{R}^{p\times n}$, $\Omega \in \mathbb{R}^{n\times n}$, is said to be observable if the pair (Ω^T, P^T) is controllable and conversely.

(5) The triple (P, Ω, K) is said to be controllable and observable if (P, Ω) is observable and (Ω, K) is controllable. If the (P, Ω, K) is controllable and observable, then there exists $L \in \mathbb{R}^{m\times p}$ such that $\sigma(\Omega + KLP)$ has $\alpha = \max(m, p)$ values arbitrarily close to any prescribed subset of C of cardinal α with possibly repeated members provided that K and P are full rank. The remaining $(\alpha - n)$ members of $\sigma(\Omega + KLP)$ cannot be allocated arbitrarily close to prefixed values.

Detectability is a dual property to stabilizability in the sense that (P, Ω) is detectable if (Ω^T, P^T) is stabilizable.

The above properties lead to the fact that the control output feedback law (3.1) is unable to reallocate all the eigenvalues of A_0, respectively, those of $(\sum_{i=0}^{r}\overline{A}_i^*)$ to exact prescribed positions if the triples $(\overline{A}_0^*, \overline{B}^*, \overline{C}^*)$, respectively, $(\sum_{i=0}^{r}\overline{A}_i^*, \overline{B}^*, \overline{C}^*)$, are controllable and observable even if $\max(m, p) \geq n$ and $\overline{B}^*$ and $\overline{C}^*$ are both full rank. However, under this constraint, all the eigenvalues of the matrix $\overline{A}_0^* + \overline{B}^*\overline{K}_0^*\overline{C}^*$, respectively, of the matrix $(\sum_{i=0}^r \overline{A}_i^*) + \overline{B}^*\overline{K}_0^*\overline{C}^*$ can be allocated arbitrarily close to any prefixed set of n complex numbers, by some choice of the static controller gain $\overline{K}_0^* \in \mathbb{R}^{m\times p}$. Also, if $\max(m,p) < n, \overline{B}^*$, and $\overline{C}^*$ are full rank and $(\overline{A}_0^*, \overline{B}^*, \overline{C}^*)$, respectively, $(\sum_{i=0}^r \overline{A}_i^*, \overline{B}^*, \overline{C}^*)$ are controllable and observable triples, then $(n - \max(m,p))$ of the eigenvalues of $\overline{A}_0^* + \overline{B}^*\overline{K}_0^*\overline{C}^*$, respectively, of $(\sum_{i=0}^r \overline{A}_i^*) + \overline{B}^*\overline{K}_0^*\overline{C}^*$, may be allocated arbitrarily close to prescribed complex sets by some $\overline{K}_0^* \in \mathbb{R}^{m\times p}$.

Corollary 4.1 is reformulated as follows for the case of linear output feedback (4.1) by taking into account the above properties of linear time-invariant output feedback.

Corollary 4.3. *Assume that*

(1) $\lim_{t\to\infty} \widehat{A}_i(t) = A_i, \forall i \in \overline{r} \cup \{0\}$ *and that* $(\sum_{i=0}^r A_i)$ *is nonsingular,*

(2) $(\overline{A}_0^*, \overline{B}^*, \overline{C}^*)$ *is a stabilizable and detectable triple, and* $(\overline{A}_i^*, \overline{B}^*, \overline{C}^*); \forall i \in \overline{r}$ *are all controllable triples,*

(3) $\sup_{t\in\mathbb{R}_{0+}} \max(\|\widetilde{\overline{A}}_i(t)\|_2, \|\widetilde{\overline{B}}_i(t)\|_2) \le \zeta$ *for some* $\zeta \in \mathbb{R}_+$,

(4)

$$Z_{0m} := \sup_{t\in\mathbb{R}_0^+}\left(\|\widetilde{\overline{A}}_0(t) + \widetilde{\overline{B}}(t)\overline{K}_0^*\overline{C}^* + \overline{B}(t)\widetilde{\overline{K}}_0(t)\overline{C}^*\|_2 + \sum_{i=1}^r \left(\|\overline{A}_i^* + \overline{B}^*\overline{K}_i^*\overline{C}^*\|_2 + \|\widetilde{\overline{A}}_i(t) + \widetilde{\overline{B}}(t)\overline{K}_i^*\overline{C}^* + \overline{B}(t)\widetilde{\overline{K}}_i(t)\overline{C}^*\|_2 \right)\right). \tag{4.2}$$

Then, there exist (in general, nonunique) constant controller gains $\overline{K}_i(t) = \overline{K}_i^*, i \in \overline{r} \cup \{0\}, \forall t \in \mathbb{R}_{0+}$ *such that the closed-loop system* (3.1)–(3.3) *is globally asymptotically stable independent of the sizes of the delays provided that* $Z_{0m} < |\mu_2(\overline{A}_i^0 + \overline{B}^*\overline{K}_i^0\overline{C}^*)|$. *If the above Condition (2), replaced with* $(\overline{A}_0^*, \overline{B}^*, \overline{C}^*)$, *is a controllable and observable triple instead of stabilizable and detectable and, furthermore,* $\operatorname{rank}\overline{B}^* = m, \operatorname{rank}\overline{C}^* = p$ *and* $\max(p,m) \ge n$ *then constant controller gains* $\overline{K}_i(t) = \overline{K}_i^* \in \mathbb{R}^{p\times m}, i \in \overline{r} \cup \{0\}, \forall t \in \mathbb{R}_{0+}$ *can be found so that the closed-loop stability is guaranteed if* $Z_{0m} < |\mu_2(\overline{A}_i^0 + \overline{B}^*\overline{K}_i^0\overline{C}^*)|$ *for a prefixed* $\mu_2(\overline{A}_i^0 + \overline{B}^*\overline{K}_i^0\overline{C}^*)$.

Corollary 4.2 is reformulated as follows for the case of linear output feedback (4.1).

Corollary 4.4. *Assume that*

(1) $\lim_{t\to\infty} \widehat{A}_i(t) = A_i, \forall i \in \overline{r} \cup \{0\}$ *and that* $(\sum_{i=0}^r A_i)$ *is a stability matrix,*

(2) $(\sum_{i=0}^r \overline{A}_i^*, \overline{B}^*)$ *is a stabilizable pair,*

(3) $\sup_{t\in\mathbb{R}_{0+}} \max(\|\widetilde{\overline{A}}_i(t)\|_2, \|\widetilde{\overline{B}}_i(t)\|_2) \le \zeta$ *for some* $\zeta \in \mathbb{R}_+$,

(4)

$$
\begin{aligned}
Z_{Am} &:= \sup_{t\in\mathbb{R}_0^+}\Bigg(\left\|\sum_{i=0}^{r}\left(\overline{\widetilde{A}}_i(t)+\overline{\widetilde{B}}_i(t)\overline{K}_i^*\overline{C}^*+\overline{B}(t)\overline{\widetilde{K}}_i(t)\overline{C}^*\right)\right\|_2 \\
&\quad +2\left\|\left(\sum_{i=1}^{r}\overline{A}_i^*\right)+\overline{B}^*\left(\sum_{i=1}^{r}\overline{K}_i^*\right)\overline{C}^*\right\|_2+2\sum_{i=1}^{r}\left(\|\overline{\widetilde{A}}_i(t)+\overline{\widetilde{B}}(t)\overline{K}_i^*\overline{C}^*+\overline{B}(t)\overline{\widetilde{K}}_i(t)\overline{C}^*\|_2\right)\Bigg) \\
&\le \overline{Z}_{Am} := 2\left\|\left(\sum_{i=0}^{r}\overline{A}_i^*\right)+\overline{B}^*\left(\sum_{i=0}^{r}\overline{K}_i^*\right)\overline{C}^*\right\|_2+\sup_{t\in\mathbb{R}_{0+}}\left(r+\left\|\sum_{i=0}^{r}\overline{K}_i^*\right\|_2\overline{C}^*\right)\zeta \\
&\quad +3\sup_{t\in\mathbb{R}_{0+}}\left(\|\overline{B}(t)\|_2\left(\sum_{i=1}^{r}\|\overline{\widetilde{K}}_i(t)\|_2\overline{C}^*\right)\right).
\end{aligned} \tag{4.3}
$$

Then, there exist sets of nonunique controller gain matrix functions $\overline{K}_i : \mathbb{R}_{0+} \to \mathbb{R}^{m\times n}$ *defined by* $\overline{K}_i(t) = \overline{K}_i^* + \overline{\widetilde{K}}_i(t), i \in \overline{r}\cup\{0\}, \forall t \in \mathbb{R}_{0+}$ *such that the closed-loop system* (3.1)–(3.3) *is globally asymptotically stable independent of the sizes of the delays provided that* $Z_{Am} < |\mu_2((\sum_{i=0}^{r}\overline{A}_i^*) + \overline{B}^*(\sum_{i=0}^{r}\overline{K}_i^*)\overline{C}^*)|$. *The stability property of the closed-loop system also holds if* $\overline{Z}_{Am} < |\mu_2((\sum_{i=0}^{r}\overline{A}_i^*) + \overline{B}^*(\sum_{i=0}^{r}\overline{K}_i^*)\overline{C}^*)|$. *If the above Condition (2), replaced with* $((\sum_{i=0}^{r}\overline{A}_i^*), \overline{B}^*, \overline{C}^*)$, *is a controllable and observable triple instead of stabilizable and detectable and, furthermore,* $\operatorname{rank}\overline{B}^* = m, \operatorname{rank}\overline{C}^* = p$ *and* $\max(p,m) \ge n$ *then constant controller gains* $\overline{K}_i(t) = \overline{K}_i^* \in \mathbb{R}^{p\times m}, i \in \overline{r}\cup\{0\}, \forall t \in \mathbb{R}_{0+}$ *can be found so that the closed-loop stability is guaranteed if* $Z_{0m} < |\mu_2((\sum_{i=0}^{r}\overline{A}_i^*) + \overline{B}^*\overline{K}_i^{0}\overline{C}^*)|$ *for a prefixed matrix measure.*

Acknowledgments

The author is very grateful to the Spanish Ministry of Education for its partial support of this work through Project DPI2006-00714. He is also grateful to the Basque Government for its support through GIC07143-IT-269-07, SAIOTEK SPED06UN10, and SPE07UN04. The author is grateful to the reviewers who helped very much in clarifying the manuscript content in the revised version.

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