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Research Article

Strong Convergence of an Iterative Method for Equilibrium Problems and Variational Inequality Problems

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We introduce an iterative method for finding a common element of the set of solutions of equilibrium problems, the set of solutions of variational inequality problems, and the set of fixed points of finite many nonexpansive mappings. We prove strong convergence of the iterative sequence generated by the proposed iterative algorithm to the unique solution of a variational inequality, which is the optimality condition for the minimization problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Suppose that C is nonempty, closed convex subset of H and F is a bifunction from $C \times C$ to R, where R is the set of real number. The equilibrium problem is to find a $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of such solutions is denoted by $\mathrm{EP}(f)$. Numerous problems in physics, optimization, and economics reduce to find a solution of equilibrium problem. Some methods have been proposed to solve the equilibrium problems in Hilbert space, see, for instance, Blum and Oettli [1], Combettes and Hirstoaga [2], and Moudafi [3].

A mapping $A : C \to H$ is called monotone if $\langle Au - Av, u - v \rangle \geq 0$. A is called relaxed (u, v)-cocoercive, if there exist constants $u > 0$ and $v > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq -u\|Ax - Ay\|^2 + v\|x - y\|^2, \quad \forall x, y \in C, \tag{1.2}$$

when $u = 0$, A is called v-strong monotone; when $v = 0$, A is called relaxed u-cocoercive. Let $A : C \to H$ be a monotone operator, the variational inequality problem is to find a point $u \in C$, such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.3}$$

The set of solutions of variational inequality problem is denoted by VI(C, A). The variational inequality problem has been extensively studied in literatures, see, for example, [4, 5] and references therein.

Let B be a strong positive bounded linear operator on H with coefficient $\overline{\gamma}$, that is, there exists a constant $\overline{\gamma} > 0$ such that $\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2$, for all $x \in H$. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(T)} \langle Ax, x\rangle - \langle x, b\rangle, \tag{1.4}$$

where T is a nonexpansive mapping on H and b is a point on H.

A mapping T from C into itself is called nonexpansive, if $\|Tx-Ty\| \leq \|x-y\|$, $\forall x, y \in C$. The set of fixed points of T is denoted by $F(T)$. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings and $F = \bigcap_{i=1}^N F(T_i) \neq \varnothing$, define the mappings

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1 - \lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1 - \lambda_{n,2})I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1 - \lambda_{n,N-1})I, \\ W_n = U_{n,N} &= \lambda_{n,N}T_NU_{n,N-1} + (1 - \lambda_{n,N})I, \end{aligned} \tag{1.5}$$

where $\{\lambda_{n,i}\}_{i=1}^N \subset (0, 1]$ for all $n \geq 1$. Such a mapping W_n is called W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\{\lambda_{n,i}\}_{i=1}^N$. We know that W_n is nonexpansive and $F(W_n) = \bigcap_{i=1}^N F(T_i)$, see [6].

Let $S : C \to C$ be a nonexpansive mapping and $f : C \to C$ is a contractive with coefficient $\alpha \in [0, 1)$. Marino and Xu [7] considered the following general iterative scheme:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (1 - \alpha_n B)Sx_n. \tag{1.6}$$

They proved that $\{x_n\}$ converges strongly to $z = P_{F(S)}(I - B + \gamma f)(z)$, where $P_{F(S)}$ is the metric projection from H onto $F(S)$.

By combining equilibrium problems and (1.6), Plutbieng and Pumpaeng [8] proposed the following algorithm:

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Su_n. \tag{1.7}$$

They proved that if the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy some appropriate conditions, then sequence $\{x_n\}$ convergence to the unique solution z of the variational inequality

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F(S) \cap \text{EP(F)}. \tag{1.8}$$

Motivated by [8], Colao et al. [9] introduced an iterative method for equilibrium problem and finite family of nonexpansive mappings

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + ((1 - \beta)I - \alpha_n B)W_n u_n, \tag{1.9}$$

and proved that $\{x_n\}$ converges strongly to a point $x^* \in F \cap \text{EP(F)}$ and x^* also solves the variational inequality (1.8). For equilibrium problems, also see [10, 11].

On the other hand, let $A : C \to C$ be a α-cocoercive mapping, for finding common element of the solution of variational inequality problems and the set of fixed point of nonexpansive mappings, Takahashi and Toyoda [12] introduced iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(I - \lambda_n A)x_n. \tag{1.10}$$

They proved that $\{x_n\}$ converges weakly to $z \in F(S) \cap \text{VI(C, A)}$. Inspired by (1.10) and [13], Y. Yao and J.-C. Yao [14] given the following iterative process:

$$y_n = P_C(I - \lambda_n A)x_n,$$
$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(I - \lambda_n A)y_n, \tag{1.11}$$

and proved that $\{x_n\}$ converges strongly to $z \in F(S) \cap \text{VI(C, A)}$. By combining viscosity approximation method and (1.10), Chen et al. [15] introduced the process

$$x_{n+1} = \alpha_n f(x_n) + \beta_n SP_C(I - \lambda_n A)x_n, \tag{1.12}$$

and studied the strong convergence of sequence $\{x_n\}$ generated by (1.12). Motivated by (1.6), (1.11), and (1.12), Qin et al. [16] introduced the following general iterative process

$$\begin{aligned} y_n &= P_C(I - s_n A)x_n, \\ x_{n+1} &= \alpha_n \gamma f(W_n x_n) + (I - \alpha_n B)W_n P_C(I - t_n A)y_n, \end{aligned} \tag{1.13}$$

and established a strong convergence theorem of $\{x_n\}$ to an element of $\bigcap_{i=1}^{N} F(T_i) \cap \mathrm{VI(C, A)}$.

The purpose of this paper is to introduce the iterative process: $x_1 \in H$ and

$$\begin{aligned} &F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ &y_n = b_n u_n + (1 - b_n)W_n P_C(I - s_n A)u_n, \\ &x_{n+1} = \alpha_n \gamma f(W_n x_n) + \beta x_n + ((1-\beta)I - \alpha_n B)W_n P_C(I - t_n A)y_n, \end{aligned} \tag{1.14}$$

where W_n is defined by (1.5), A is (u, v)-cocoercive, and B is a bounded linear operator. We should show that the sequences $\{x_n\}$ converge strongly to an element of $\bigcap_{i=1}^{N} F(T_i) \cap \mathrm{VI(C, A)} \cap \mathrm{EP(F)}$. Our result extends the corresponding results of Qin et al. [16] and Colao et al. [9], and many others.

2. Preliminaries

Let H be a real Hilbert space and C a nonempty, closed convex subset of H. We denote strong convergence of $\{x_n\}$ to x by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. Let $P_C : C \to H$ is a mapping such that for every point $x \in H$, there exists a unique $P_C x \in C$ satisfying $\|x - P_C x\| \leq \|x - y\|$, for all $y \in C$. P_C is called the metric projection of H onto C. It is known that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \quad \forall x \in H,\ y \in C, \tag{2.1}$$

$$\langle x - y, P_c x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2}$$

Let $A : C \to H$ be a monotone mapping of C into H, then $u \in \mathrm{VI(C, A)}$ if and only if $u = P_C(u - \lambda A u)$, for all $\lambda > 0$. The following result is useful in the rest of this paper.

Lemma 2.1 (see [17]). *Assume $\{a_n\}$ is a sequence of nonegative real number such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad \forall n \geq 0, \tag{2.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\delta_n\}$ is a sequence in R such that

(1) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (see [18]). *Let $\{x_n\}$, $\{u_n\}$ be bounded sequences in Banach space E satisfying $x_{n+1} = \tau_n x_n + (1-\tau_n)u_n$ (for all $n \geq 0$) and $\liminf_{n\to\infty}(\|u_{n+1}-u_n\| - \|x_{n+1}-x_n\|) \leq 0$. Let τ_n be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\tau_n \leq \limsup_{n\to\infty}\tau_n < 1$. Then, $\lim_{n\to\infty}\|x_n - u_n\| = 0$.*

Lemma 2.3. *For all $x, y \in H$, there holds the inequality*

$$\|x+y\| \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{2.4}$$

Lemma 2.4 (see [7]). *Assume that A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to R$, we assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone: $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for all $x \in C, F(x, \cdot\,)$ is convex and lower semicontinuous.

The following result is in Blum and Oettli [1].

Lemma 2.5 (see [1]). *Let C be a nonempty closed convex subset of a Hilbert space E, let F be a bifunction from $C \times C$ into R satisfying (A1)–(A4), let $r > 0$, and let $x \in H$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

We also know the following lemmas.

Lemma 2.6 (see [19]). *Let C be a nonempty closed convex subset of Hilbert space H, let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4), let $r > 0$, and let $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}, \tag{2.6}$$

for all $x \in H$. Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive-type mapping, that is, for all $x, y \in H$,*

$$\left\|T_r x - T_r y\right\|^2 \leq \left\langle T_r x - T_r y, x - y\right\rangle; \tag{2.7}$$

(3) $F(T_r) = \mathrm{EP(F)}$;

(4) $\mathrm{EP(F)}$ *is closed and convex.*

A monotone operator $T : H \to 2^H$ is said to be maximal monotone if its graph $G(T) = \{(u,v) : v \in Tu\}$ is not properly contained in the graph of any other monotone operators. Let A be a monotone mapping of C into H and let $N_C(v)$ be the normal cone for C at a point $v \in C$, that is

$$N_C(v) = \{x \in H : \langle v - y, x\rangle \geq 0, \forall y \in C\}. \tag{2.8}$$

Define

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \varnothing, & v \notin C. \end{cases} \tag{2.9}$$

It is known that in this case T is maximal monotone, and $0 \in Tv$ if and only if $v \in \mathrm{VI(C, A)}$.

3. Strong Convergence Theorem

Theorem 3.1. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from C into itself and $F : C \times C \to R$ a bifunction satisfying (A1)–(A4). Let $A : C \to H$ be relaxed (u,v)-cocoercive and μ-Lipschitzian. Let $f : C \to C$ be an α-contraction with $0 \leq \alpha < 1$ and B a strong positive linear bounded operator with coefficient $\overline{\gamma} > 0$, γ is a constant with $0 < \gamma < \overline{\gamma}/\alpha$. Let sequences $\{\alpha_n\}$, $\{b_n\}$ be in $(0,1)$ and $\{r_n\}$ be in $(0,\infty)$, β is a constant in $(0,1)$. Assume $C_0 = \bigcap_{i=1}^N F(T_i) \cap \mathrm{VI(C,A)} \cap \mathrm{EP(F)} \neq \varnothing$ and*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$, $\liminf_{n\to\infty} r_n > 0$;

(iii) $\{s_n\}, \{t_n\} \in [a,b]$ *for some* a, b *with* $0 \leq a \leq b \leq 2(v - u\mu^2)/\mu^2$ *and* $\lim_{n\to\infty}|s_{n+1} - s_n| = \lim_{n\to\infty}|t_{n+1} - t_n| = 0$;

(iv) $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = \lim_{n\to\infty}|b_{n+1} - b_n| = 0$.

Then the sequence $\{x_n\}$ generated by (1.14) converges strongly to $x^* \in C_0$ and x^* solves the variational inequality $x^* = P_{C_0}(I - (B - \gamma f))x^*$, that is,

$$\langle \gamma f x^* - Bx^*, x - x^*\rangle \leq 0, \quad \forall x \in C_0. \tag{3.1}$$

Proof. Without loss of generality, we can assume $\alpha_n \leq (1-\beta)\|B\|^{-1}$. Then from Lemma 2.4 we know

$$\|(1-\beta)I - \alpha_n B\| = (1-\beta)\left\|I - \frac{\alpha_n}{1-\beta}B\right\| \leq (1-\beta)\left(1 - \frac{\alpha_n}{1-\beta}\overline{\gamma}\right) = 1 - \beta - \alpha_n\overline{\gamma}. \tag{3.2}$$

Since A is relaxed (u,v)-cocoercive and μ-Lipschitzian and (iii) holds, we know from [14] that for all $x, y \in C$ and $n \geq 1$, the following holds:

$$\begin{aligned} \|(I - s_n A)x - (I - s_n A)y\| &\leq \|x - y\|, \\ \|(I - t_n A)x - (I - t_n A)y\| &\leq \|x - y\|. \end{aligned} \tag{3.3}$$

We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Take $p \in C_0$, notice that $u_n = T_{r_n}x_n$ and form Lemma 2.6(2) that T_{r_n} is nonexpansive, we have

$$\|u_n - p\| = \|T_{r_n}x_n - T_{r_n}p\| \leq \|x_n - p\|. \tag{3.4}$$

Since $p = W_nP_C(p - s_nAp)$, we have

$$\begin{aligned}
\|y_n - p\| &= \|b_nu_n + (1-b_n)W_nP_C(u_n - s_nAu_n) - p\| \\
&\leq b_n\|u_n - p\| + (1-b_n)\|W_nP_C(u_n - s_nAu_n) - p\| \\
&\leq b_n\|u_n - p\| + (1-b_n)\|u_n - s_nAu_n - (p - s_nAp)\| \\
&\leq b_n\|u_n - p\| + (1-b_n)\|u_n - p\| \\
&\leq \|x_n - p\|.
\end{aligned} \tag{3.5}$$

Then we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(W_nx_n) + \beta x_n + ((1-\beta)I - \alpha_nB)W_nP_C(I - t_nA)y_n - p\| \\
&= \|\alpha_n(\gamma f(W_nx_n) - Bp) + \beta(x_n - p) \\
&\quad + ((1-\beta)I - \alpha_nB)W_n(P_C(I - t_nA)y_n - p)\| \\
&= \alpha_n\|\gamma f(W_nx_n) - Bp\| + \beta\|x_n - p\| + (1 - \beta - \alpha_n\overline{\gamma})\|y_n - p\|.
\end{aligned} \tag{3.6}$$

Thus From (3.5) we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n\gamma\|f(W_nx_n) - f(p)\| + \alpha_n\|\gamma f(p) - Bp\| + \beta\|x_n - p\| \\
&\quad + (1 - \beta - \alpha_n\overline{\gamma})\|x_n - p\| \\
&\leq \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + (1 - \alpha_n\overline{\gamma})\|x_n - p\| \\
&= (1 - \alpha_n(\overline{\gamma} - \alpha\gamma))\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| \\
&= (1 - \alpha_n(\overline{\gamma} - \alpha\gamma))\|x_n - p\| + \alpha_n(\overline{\gamma} - \alpha\gamma)\cdot\frac{\|\gamma f(p) - Bp\|}{\overline{\gamma} - \alpha\gamma} \\
&\leq \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - Bp\|}{\overline{\gamma} - \alpha\gamma}\right\} \\
&\leq \max\left\{\|x_1 - p\|, \frac{\|\gamma f(p) - Bp\|}{\overline{\gamma} - \alpha\gamma}\right\},
\end{aligned} \tag{3.7}$$

hence $\{x_n\}$ is bounded, so is $\{u_n\}$, $\{y_n\}$.

Step 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Let $x_{n+1}=\beta x_n+(1-\beta)z_n$, for all $n\geq 0$, where

$$z_n=\frac{1}{1-\beta}\left[\alpha_n\gamma f(W_nx_n)+((1-\beta)I-\alpha_nB)W_nP_C(y_n-t_nAy_n)\right]. \tag{3.8}$$

Then we have

$$\begin{aligned}\|z_{n+1}-z_n\|&=\frac{1}{1-\beta}\|\gamma(\alpha_{n+1}f(W_{n+1}x_{n+1})-\alpha_nf(W_nx_n))\\&\quad+((1-\beta)I-\alpha_{n+1}B)W_{n+1}P_C(y_{n+1}-t_{n+1}Ay_{n+1})\\&\quad-((1-\beta)I-\alpha_nB)W_nP_C(y_n-t_nAy_n)\|\\&=\left\|\frac{\gamma}{1-\beta}(\alpha_{n+1}f(W_{n+1}x_{n+1})-\alpha_nf(W_nx_n))\right.\\&\quad+[W_{n+1}P_C(y_{n+1}-t_{n+1}Ay_{n+1})-W_nP_C(y_n-t_nAy_n)]\\&\quad-\frac{1}{1-\beta}[\alpha_{n+1}BW_{n+1}P_C(y_{n+1}-t_{n+1}Ay_{n+1})\\&\qquad\left.-\alpha_nBW_nP_C(y_n-t_nAy_n)]\right\|\\&\leq\|W_{n+1}P_C(y_{n+1}-t_{n+1}Ay_{n+1})-W_{n+1}P_C(y_n-t_{n+1}Ay_n)\|\\&\quad+\|W_{n+1}P_C(y_n-t_{n+1}Ay_n)-W_{n+1}P_C(y_n-t_nAy_n)\|\\&\quad+\|W_{n+1}P_C(y_n-t_nAy_n)-W_nP_C(y_n-t_nAy_n)\|+K_1,\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}K_1&=\frac{\alpha_{n+1}}{1-\beta}(\gamma\|f(W_{n+1}x_{n+1})\|+\|BW_{n+1}P_C(y_{n+1}-t_{n+1}Ay_{n+1})\|)\\&\quad+\frac{\alpha_n}{1-\beta}(\gamma\|f(W_nx_n)\|+\|BW_nP_C(y_n-t_nAy_n)\|).\end{aligned} \tag{3.10}$$

Next we estimate $\|W_{n+1}P_C(y_{n+1}-t_{n+1}Ay_{n+1})-W_{n+1}P_C(y_n-t_{n+1}Ay_n)\|$, $\|W_{n+1}P_C(y_n-t_{n+1}Ay_n)-W_{n+1}P_C(y_n-t_nAy_n)\|$ and $\|W_{n+1}P_C(y_n-t_nAy_n)-W_nP_C(y_n-t_nAy_n)\|$. At first

$$\|W_{n+1}P_C(y_n-t_{n+1}Ay_n)-W_{n+1}P_C(y_n-t_nAy_n)\|\leq\|t_nAy_n-t_{n+1}Ay_n\|=|t_{n+1}-t_n|\cdot\|Ay_n\|. \tag{3.11}$$

Put $v_n = P_C(y_n - t_n A y_n)$, we have

$$\begin{aligned}
&\|W_{n+1}P_C(y_n - t_nAy_n) - W_nP_C(y_n - t_nAy_n)\| \\
&\quad = \|W_{n+1}v_n - W_nv_n\| \\
&\quad = \|U_{n+1,N}v_n - U_{n,N}v_n\| \\
&\quad = \|\lambda_{n+1,N}T_NU_{n+1,N-1}v_n + (1-\lambda_{n+1,N})v_n - \lambda_{n,N}T_NU_{n,N-1}v_n - (1-\lambda_{n,N})v_n\| \\
&\quad \le \|\lambda_{n+1,N}T_NU_{n+1,N-1}v_n - \lambda_{n,N}T_NU_{n,N-1}v_n\| + |\lambda_{n+1,N} - \lambda_{n,N}| \cdot \|v_n\| \\
&\quad \le \lambda_{n+1,N}\|U_{n+1,N-1}v_n - U_{n,N-1}v_n\| + |\lambda_{n+1,N} - \lambda_{n,N}| \cdot \|T_NU_{n,N-1}v_n\| \\
&\qquad + |\lambda_{n+1,N} - \lambda_{n,N}| \cdot \|v_n\| \\
&\quad \le \|U_{n+1,N-1}v_n - U_{n,N-1}v_n\| + |\lambda_{n+1,N} - \lambda_{n,N}|(\|T_NU_{n,N-1}v_n\| + \|v_n\|).
\end{aligned} \tag{3.12}$$

By recursion we get

$$\|W_{n+1}P_C(y_n - t_nAy_n) - W_nP_C(y_n - t_nAy_n)\| \le M \cdot \sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|, \tag{3.13}$$

for some $M > 0$. Similarly, we also get

$$\|W_{n+1}P_C(u_n - s_nAu_n) - W_nP_C(u_n - s_nAu_n)\| \le M \cdot \sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|. \tag{3.14}$$

Since

$$\begin{gathered}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\
F(u_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in C.
\end{gathered} \tag{3.15}$$

Put $y = u_{n+1}$ in the first inequality and $y = u_n$ in the second one, we have

$$\begin{gathered}
F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \ge 0, \\
F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \ge 0.
\end{gathered} \tag{3.16}$$

Adding both inequality, by (A2) we have

$$\left\langle u_{n+1} - u_n, \frac{1}{r_n}(u_n - x_n) - \frac{1}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \ge 0, \tag{3.17}$$

therefore, we have

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + (x_{n+1} - x_n) + \frac{r_{n+1} - r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0, \tag{3.18}$$

which implies that

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, (x_{n+1} - x_n) + \frac{r_{n+1} - r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \cdot \left\{ \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|u_{n+1} - x_{n+1}\| \right\}. \end{aligned} \tag{3.19}$$

Hence we have

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|u_{n+1} - x_{n+1}\|, \tag{3.20}$$

so, by (3.20) and the property $\|(I - t_nA)x - (I - t_nA)y\| \leq \|x - y\|$, we arrive at

$$\begin{aligned} &\|W_{n+1}P_C(y_{n+1} - t_{n+1}Ay_{n+1}) - W_{n+1}P_C(y_n - t_{n+1}Ay_n)\| \\ &\quad\leq \|y_{n+1} - y_n\| \\ &\quad= \|b_{n+1}u_{n+1} + (1 - b_{n+1})W_{n+1}P_C(I - s_{n+1}A)u_{n+1} \\ &\qquad - b_nu_n - (1 - b_n)W_nP_C(I - s_nA)u_n\| \\ &\quad= \|b_{n+1}(u_{n+1} - u_n) + (b_{n+1} - b_n)u_n \\ &\qquad + (1 - b_{n+1})[W_{n+1}P_C(u_{n+1} - s_{n+1}Au_{n+1}) - W_{n+1}P_C(u_n - s_{n+1}Au_n)] \\ &\qquad + (1 - b_{n+1})[W_{n+1}P_C(u_n - s_{n+1}Au_n) - W_{n+1}P_C(u_n - s_nAu_n)] \\ &\qquad \times (1 - b_{n+1})[W_{n+1}P_C(u_n - s_nAu_n) - W_nP_C(u_n - s_nAu_n)] \\ &\qquad + (b_n - b_{n+1})W_nP_C(u_n - s_nAu_n)\| \\ &\quad\leq b_{n+1}\|u_{n+1} - u_n\| + |b_{n+1} - b_n| \cdot \|u_n\| \\ &\qquad + (1 - b_{n+1})\|u_{n+1} - u_n\| + (1 - b_{n+1})|s_n - s_{n+1}| \cdot \|Au_n\| \\ &\qquad + (1 - b_{n+1})\|W_{n+1}P_C(u_n - s_nAu_n) - W_nP_C(u_n - s_nAu_n)\| \\ &\qquad + \|b_n - b_{n+1}\| \cdot \|W_nP_C(u_n - s_nAu_n)\| \\ &\quad= \|u_{n+1} - u_n\| + (1 - b_{n+1})\|W_{n+1}P_C(u_n - s_nAu_n) - W_nP_C(u_n - s_nAu_n)\| + K_2, \end{aligned} \tag{3.21}$$

where

$$K_2 := (1-b_{n+1})\left|s_n - s_{n+1}\right| \cdot \left\|Au_n\right\| + \left\|b_{n+1} - b_n\right\|\left(\left\|W_n P_C(u_n - s_n A u_n)\right\| + \left\|v_n\right\|\right). \tag{3.22}$$

Therefore, by (3.14) and (3.20) we get

$$\begin{aligned} &\left\|W_{n+1}P_C(y_{n+1} - t_{n+1}Ay_{n+1}) - W_{n+1}P_C(y_n - t_{n+1}Ay_n)\right\| \\ &\quad \le \left\|x_{n+1} - x_n\right\| + \frac{|r_{n+1} - r_n|}{r_{n+1}}\left\|u_{n+1} - x_{n+1}\right\| \\ &\quad + (1-b_{n+1})M \cdot \sum_{i=1}^{N}\left|\lambda_{n+1,i} - \lambda_{n,i}\right| + K_2. \end{aligned} \tag{3.23}$$

Now submitting (3.11), (3.13), and (3.23) into (3.9), we have

$$\begin{aligned} \left\|z_{n+1} - z_n\right\| &= \left\|x_{n+1} - x_n\right\| + \frac{|r_{n+1} - r_n|}{r_{n+1}}\left\|u_{n+1} - x_{n+1}\right\| \\ &\quad + (1-b_{n+1})M \cdot \sum_{i=1}^{N}\left|\lambda_{n+1,i} - \lambda_{n,i}\right| + K_2 \\ &\quad + \left|t_{n+1} - t_n\right| \cdot \left\|Ay_n\right\| + M \cdot \sum_{i=1}^{N}\left|\lambda_{n+1,i} - \lambda_{n,i}\right| + K_1. \end{aligned} \tag{3.24}$$

Thus conditions (ii), (iii), and (iv) imply that

$$\limsup_{n\to\infty}\left(\left\|z_{n+1} - z_n\right\| - \left\|x_{n+1} - x_n\right\|\right) \le 0. \tag{3.25}$$

Then, Lemma 2.2 yields

$$\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = \lim_{n\to\infty}(1-\beta)\left\|x_n - z_n\right\| = 0. \tag{3.26}$$

Step 3. $\lim_{n\to\infty}\|Ay_n - Ap\| = \lim_{n\to\infty}\|Au_n - Ap\| = 0$ for $p \in C_0$.

Note that

$$\begin{aligned}
\|y_n - p\|^2 &= \|b_n u_n + (1-b_n) W_n P_C (I - s_n A) u_n - p\|^2 \\
&\le b_n \|u_n - p\|^2 + (1-b_n) \|(u_n - s_n A u_n) - (p - s_n A p)\|^2 \\
&= b_n \|u_n - p\|^2 + (1-b_n) \\
&\quad \times \left\{ \|u_n - p\|^2 - 2 s_n \langle u_n - p, A u_n - A p\rangle + s_n^2 \|A u_n - A p\|^2 \right\} \\
&\le \|u_n - p\|^2 + (1-b_n) \\
&\quad \times \left\{ 2 s_n u \|A u_n - A p\|^2 - 2 s_n v \|u_n - p\|^2 + s_n^2 \|A u_n - A p\|^2 \right\} \\
&= \|u_n - p\|^2 + (1-b_n) \left(2 s_n u + s_n^2 - \frac{2 s_n v}{\mu^2} \right) \|A u_n - A p\|^2,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\|v_n - p\|^2 &= \|P_C (y_n - t_n A y_n) - P_C (p - t_n A p)\|^2 \\
&\le \|(y_n - p) - t_n (A y_n - A p)\|^2 \\
&= \|y_n - p\|^2 - 2 t_n \langle y_n - p, A y_n - A p\rangle + t_n^2 \|A y_n - A p\|^2 \\
&\le \|y_n - p\|^2 + 2 t_n u \|A y_n - A p\| - 2 t_n v \|y_n - p\|^2 + t_n^2 \|A y_n - A p\|^2 \\
&\le \|y_n - p\|^2 + \left(2 t_n u + t_n^2 - \frac{2 t_n v}{\mu^2} \right) \|A y_n - A p\|,
\end{aligned} \tag{3.28}$$

hence by $2t_n u + t_n^2 - (2t_n v/\mu^2) < 0$, we know

$$\|v_n - p\|^2 \le \|y_n - p\|^2 \le \|u_n - p\|^2 + (1-b_n) \left(2 s_n u + s_n^2 - \frac{2 s_n v}{\mu^2} \right) \|A u_n - A p\|^2. \tag{3.29}$$

By Lemma 2.3, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n \gamma f(W_n x_n) + \beta x_n + ((1-\beta) I - \alpha_n B) W_n v_n - p\|^2 \\
&= \|[(1-\beta)(W_n v_n - p) + \beta (x_n - p)] + \alpha_n [\gamma f(W_n x_n) - B W_n v_n]\|^2 \\
&\le \|(1-\beta)(W_n v_n - p) + \beta (x_n - p)\|^2 + 2\alpha_n \langle \gamma f(W_n x_n) - B W_n v_n, x_{n+1} - p\rangle \\
&\le (1-\beta) \|W_n v_n - p\|^2 + \beta \|x_n - p\|^2 + 2\alpha_n \langle \gamma f(W_n x_n) - B W_n v_n, x_{n+1} - p\rangle \\
&\le (1-\beta) \|v_n - p\|^2 + \beta \|x_n - p\|^2 + 2\alpha_n M_1,
\end{aligned} \tag{3.30}$$

for some $M_1 > 0$. Submitting (3.28) into (3.30), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\beta)\left[\|y_n-p\|^2+\left(2t_nu+t_n^2-\frac{2t_nv}{\mu^2}\right)\|Ay_n-Ap\|\right]\\&\quad+\beta\|x_n-p\|^2+2\alpha_nM_1\\&\le \|x_n-p\|^2+(1-\beta)\left(2t_nu+t_n^2-\frac{2t_nv}{\mu^2}\right)\|Ay_n-Ap\|+2\alpha_nM_1,\end{aligned}\tag{3.31}$$

which implies

$$\begin{aligned}&(1-\beta)\left(\frac{2t_nv}{\mu^2}-2t_nu-t_n^2\right)\|Ay_n-Ap\|\\&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_nM_1\\&\quad=\|x_n-x_{n+1}+x_{n+1}-p\|^2-\|x_{n+1}-p\|^2+2\alpha_nM_1\\&\quad=\|x_n-x_{n+1}\|^2+2\langle x_n-x_{n+1},x_{n+1}-p\rangle+2\alpha_nM_1\\&\quad\le\|x_n-x_{n+1}\|^2+2\|x_n-x_{n+1}\|\cdot\|x_{n+1}-p\|+2\alpha_nM_1,\end{aligned}\tag{3.32}$$

hence from conditions (i), (iii), and (3.26), we have

$$\lim_{n\to\infty}\|Ay_n-Ap\|=0.\tag{3.33}$$

Similarly, submitting (3.29) into (3.30), we also have

$$\lim_{n\to\infty}\|Au_n-Ap\|=0.\tag{3.34}$$

Step 4. $\lim_{n\to\infty}\|W_nv_n-v_n\|=0$.

By (2.2) we have

$$\begin{aligned}\|v_n-p\|^2&=\|P_C(y_n-t_nAy_n)-P_C(p-t_nAp)\|^2\\&\le\langle(y_n-t_nAy_n)-(p-t_nAp),v_n-p\rangle\\&=\frac{1}{2}\{\|y_n-t_nAy_n-(p-t_nAp)\|^2+\|v_n-p\|^2\\&\qquad-\|y_n-v_n-t_n(Ay_n-Ap)\|^2\}\\&\le\frac{1}{2}\{\|y_n-p\|^2+\|v_n-p\|^2-\|y_n-v_n-t_n(Ay_n-Ap)\|^2\}\\&=\frac{1}{2}\{\|y_n-p\|^2+\|v_n-p\|^2-\|y_n-v_n\|^2-t_n^2\|Ay_n-Ap\|^2\\&\qquad+2t_n\langle y_n-v_n,Ay_n-Ap\rangle\},\end{aligned}\tag{3.35}$$

hence

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|y_n - v_n\|^2 - t_n^2\|Ay_n - Ap\|^2 + 2t_n\|y_n - v_n\| \cdot \|Ay_n - Ap\|. \tag{3.36}$$

Submitting (3.36) into (3.30), we have

$$\|x_{n+1} - p\|^2 \le (1-\beta)\{\|x_n - p\|^2 - \|y_n - v_n\|^2 - t_n^2\|Ay_n - Ap\|^2 + 2t_n\|y_n - v_n\| \cdot \|Ay_n - Ap\|\} + \beta\|x_n - p\|^2 + 2\alpha_n M_1. \tag{3.37}$$

This implies that

$$(1-\beta)\|y_n - v_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - t_n^2\|Ay_n - Ap\|^2 + 2t_n\|y_n - v_n\| \cdot \|Ay_n - Ap\| + 2\alpha_n M_1. \tag{3.38}$$

Hence by Step 3 we get

$$\lim_{n\to\infty}\|y_n - v_n\| = 0. \tag{3.39}$$

Put $y_n = b_n u_n + (1-b_n)W_n q_n$, where $q_n = P_C(I - s_n A)u_n$. We have

$$\begin{aligned}
\|W_n q_n - p\|^2 &\le \|P_C(I - s_nA)u_n - P_C(I - s_nA)p\|^2 \\
&\le \langle u_n - s_nAu_n - (p - s_nAp), W_nq_n - p\rangle \\
&= \frac{1}{2}\{\|u_n - s_nAu_n - (p - s_nAp)\|^2 + \|W_nq_n - p\|^2 \\
&\quad - \|u_n - W_nq_n - s_n(Au_n - Ap)\|^2\} \\
&\le \frac{1}{2}\{\|u_n - p\|^2 + \|W_nq_n - p\|^2 - \|u_n - W_nq_n\|^2 \\
&\quad - s_n^2\|Ay_n - Ap\|^2 + 2s_n\langle u_n - W_nq_n, Au_n - Ap\rangle\}.
\end{aligned} \tag{3.40}$$

Hence

$$\|W_nq_n - p\|^2 \le \|u_n - p\|^2 - \|u_n - W_nq_n\|^2 - s_n^2\|Ay_n - Ap\|^2 + 2s_n\|u_n - W_nq_n\| \cdot \|Au_n - Ap\|, \tag{3.41}$$

so, we get

$$\begin{aligned}\|y_n - p\|^2 &\le b_n\|u_n - p\|^2 + (1-b_n)\|W_nq_n - p\|^2 \\ &\le \|u_n - p\|^2 - (1-b_n)\|u_n - W_nq_n\|^2 \\ &\quad + (1-b_n)\left[-s_n^2\|Ay_n - Ap\|^2 + 2s_n\|u_n - W_nq_n\| \cdot \|Au_n - Ap\|\right].\end{aligned} \tag{3.42}$$

Submitting into (3.30), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 \le (1-\beta)\{&\|u_n - p\|^2 - (1-b_n)\|u_n - W_nq_n\|^2 \\ &+ (1-b_n)\left[-s_n^2\|Ay_n - Ap\|^2 + 2s_n\|u_n - W_nq_n\| \cdot \|Au_n - Ap\|\right]\} \\ &+ \beta\|x_n - p\|^2 + 2\alpha_nM_1 \\ &\le \|x_n - p\|^2 - (1-\beta)(1-b_n)\|u_n - W_nq_n\|^2 + K_3,\end{aligned} \tag{3.43}$$

where

$$K_3 := (1-b_n)\left[-s_n^2\|Ay_n - Ap\|^2 + 2s_n\|u_n - W_nq_n\| \cdot \|Au_n - Ap\|\right] + 2\alpha_nM_1, \tag{3.44}$$

which implies

$$\lim_{n\to\infty}\|u_n - W_nq_n\| = 0. \tag{3.45}$$

So, we have

$$\lim_{n\to\infty}\|y_n - u_n\| = \lim_{n\to\infty}(1-b_n)\|u_n - W_nq_n\| = 0, \tag{3.46}$$

which together with (3.39) gives

$$\lim_{n\to\infty}\|u_n - v_n\| = 0. \tag{3.47}$$

Since $u_n = T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\| \\ &\le \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle \\ &= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|T_{r_n}x_n - x_n\|^2\},\end{aligned} \tag{3.48}$$

which implies

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2, \tag{3.49}$$

which together with (3.30) gives

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq (1-\beta)\|v_n - p\|^2 + \beta\|x_n - p\|^2 + 2\alpha_n M_1 \\
&\leq (1-\beta)\left\{\|v_n - u_n\|^2 + \|u_n - p\|^2 + 2\langle v_n - u_n, u_n - p\rangle\right\} \\
&\quad + \beta\|x_n - p\|^2 + 2\alpha_n M_1 \\
&\leq (1-\beta)\|v_n - u_n\|^2 + (1-\beta)\left(\|x_n - p\|^2 - \|u_n - x_n\|^2\right) \\
&\quad + 2(1-\beta)\|v_n - u_n\| \cdot \|u_n - p\| + \beta\|x_n - p\|^2 + 2\alpha_n M_1.
\end{aligned} \tag{3.50}$$

So

$$\begin{aligned}
(1-\beta)\|u_n - x_n\|^2 &\leq (1-\beta)\|v_n - u_n\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2(1-\beta)\|v_n - u_n\| \cdot \|u_n - p\| + 2\alpha_n M_1.
\end{aligned} \tag{3.51}$$

Now (3.47) and condition (i) imply that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.52}$$

Since

$$\begin{aligned}
\|x_n - W_n v_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_n v_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(W_n x_n) + \beta x_n + ((1-\beta)I - \alpha_n B)W_n v_n - W_n v_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n[\gamma f(W_n x_n) - BW_n v_n] + \beta(x_n - W_n v_n)\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n(\gamma\|f(W_n x_n)\| + \|BW_n v_n\|) + \beta\|x_n - W_n v_n\|.
\end{aligned} \tag{3.53}$$

Then we get

$$\|x_n - W_n v_n\| \leq \frac{1}{1-\beta}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta}(\gamma\|f(W_n x_n)\| + \|BW_n v_n\|), \tag{3.54}$$

hence

$$\lim_{n\to\infty}\|x_n - W_n v_n\| = 0. \tag{3.55}$$

Note that

$$\|W_n v_n - v_n\| \le \|W_n v_n - x_n\| + \|x_n - u_n\| + \|u_n - v_n\|, \tag{3.56}$$

thus from (3.47)–(3.55), we have

$$\lim_{n\to\infty} \|v_n - W_n v_n\| = 0. \tag{3.57}$$

Step 5. $\limsup_{n\to\infty}\langle \gamma f(x^*) - Bx^*, x_{n+1} - x^*\rangle \le 0$, where x^* is the unique solution of variational inequality $\langle \gamma f x^* - Bx^*, x - x^*\rangle \le 0$, for all $x \in C_0$.

Take a subsequence $\{x_{n_j}\}$ of $\{x_{n+1}\}$, such that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Bx^*, x_{n+1} - x^*\rangle = \lim_{j\to\infty}\langle \gamma f(x^*) - Bx^*, x_{n_j} - x^*\rangle. \tag{3.58}$$

Since $\{x_n\}$ is bounded, without loss of generality, we assume $\{x_{n_j}\}$ itself converges weakly to a point p. We should prove $p \in C_0 = \bigcap_{i=1}^{N} F(T_i) \cap \mathrm{VI(C, A)} \cap \mathrm{EP(F)}$.

First, let

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \varnothing, & v \notin C, \end{cases} \tag{3.59}$$

with the same argument as used in [14], we can derive $p \in T^{-1}0$, since T is maximal monotone, we know $p \in \mathrm{VI(C, A)}$.

Next, from (A2), for all $y \in C$ we have

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge F(y, u_n), \tag{3.60}$$

in particular

$$\left\langle y - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}}\right\rangle \ge F(y, u_{n_j}). \tag{3.61}$$

Condition (A4) implies that F is weakly semicontinuous, then from (3.52) and let $j \to \infty$ we have

$$F(y, p) \le 0, \quad \forall y \in C. \tag{3.62}$$

Replacing y by $y_t := ty + (1-t)p$ with $t \in [0,1]$, using (A1) and (A4), we get

$$0 = F(y_t, y_t) = tF(y_t, y) + (1-t)F(y_t, p) \le tF(y_t, y). \tag{3.63}$$

Divide by t in both side yields $F(ty+(1-t)p,y)\geq 0$, let $t\to 0^+$, by (A3) we conclude $F(p,y)\geq 0$, for all $y\in C$. Therefore, $p\in \mathrm{EP(F)}$.

Finally, from $\|x_n - v_n\|\to 0$ we know that $v_{n_j}\rightharpoonup p$ $(j\to\infty)$. Assume $p\notin\bigcap_{i=1}^N F(T_i)$, that is, $p\neq W_n p$, for all $n\in N$. Since Hilbert space satisfies Opial's condition, we have

$$\begin{aligned}
\liminf_{j\to\infty}\|v_{n_j}-p\| &\leq \liminf_{j\to\infty}\|v_{n_j}-W_{n_j}p\| \\
&\leq \liminf_{j\to\infty}\left(\|v_{n_j}-W_{n_j}v_{n_j}\|+\|W_{n_j}v_{n_j}-W_{n_j}p\|\right) \\
&\leq \liminf_{j\to\infty}\|v_{n_j}-p\|,
\end{aligned}\tag{3.64}$$

this is a contradiction, thus $p\in\bigcap_{i=1}^N F(T_i)$, therefore, $p\in C_0=\bigcap_{i=1}^N F(T_i)\cap \mathrm{VI(C,A)}\cap\mathrm{EP(F)}$. So we know

$$\begin{aligned}
\limsup_{n\to\infty}\langle\gamma f(x^*)-Bx^*, x_{n+1}-x^*\rangle &= \lim_{j\to\infty}\langle\gamma f(x^*)-Bx^*, x_{n_j}-x^*\rangle \\
&= \langle\gamma f(x^*)-Bx^*, p-x^*\rangle \leq 0.
\end{aligned}\tag{3.65}$$

Step 6. The sequence $\{x_n\}$ converges strongly to x^*.

From the definition of $\{x_n\}$ and Lemmas 2.3, and 2.4, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\|\left[\left((1-\beta)I-\alpha_n B\right)(W_n v_n - x^*)+\beta(x_n-x^*)\right]+\alpha_n\left(\gamma f(W_n x_n)-Bx^*\right)\right\|^2 \\
&\leq \left\|\left[\left((1-\beta)I-\alpha_n B\right)(W_n v_n - x^*)+\beta(x_n-x^*)\right]\right\|^2 \\
&\quad + 2\alpha_n\langle\gamma f(W_n x_n)-Bx^*, x_{n+1}-x^*\rangle \\
&\leq (1-\beta)\left\|\frac{(1-\beta)I-\alpha_n B}{1-\beta}(W_n v_n-x^*)\right\|^2+\beta\|x_n-x^*\|^2 \\
&\quad + 2\alpha_n\gamma\langle f(W_n x_n)-f(x^*), x_{n+1}-x^*\rangle + 2\alpha_n\langle\gamma f(x^*)-Bx^*, x_{n+1}-x^*\rangle \\
&\leq \frac{\left((1-\beta)-\alpha_n\overline{\gamma}\right)^2}{1-\beta}\|W_n v_n-x^*\|^2+\beta\|x_n-x^*\|^2 \\
&\quad + \alpha\alpha_n\gamma\left(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\right)+2\alpha_n\langle\gamma f(x^*)-Bx^*, x_{n+1}-x^*\rangle \\
&\leq \left(\frac{\left((1-\beta)-\alpha_n\overline{\gamma}\right)^2}{1-\beta}+\beta+\alpha\alpha_n\gamma\right)\|x_n-x^*\|^2 \\
&\quad + \alpha\alpha_n\gamma\|x_{n+1}-x^*\|^2+2\alpha_n\langle\gamma f(x^*)-Bx^*, x_{n+1}-x^*\rangle \\
&= \left(1-(2\overline{\gamma}-\alpha\gamma)\alpha_n+\frac{\alpha_n^2\overline{\gamma}^2}{1-\beta}\right)\|x_n-x^*\|^2 \\
&\quad + \alpha\alpha_n\gamma\|x_{n+1}-x^*\|^2+2\alpha_n\langle\gamma f(x^*)-Bx^*, x_{n+1}-x^*\rangle,
\end{aligned}\tag{3.66}$$

which implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \left(\frac{1-2(\overline{\gamma}-\alpha\gamma)\alpha_n}{1-\alpha\alpha_n\gamma}+\frac{\alpha_n^2\overline{\gamma}^2}{(1-\beta)(1-\alpha\alpha_n\gamma)}\right)\|x_n-x^*\|^2\\&\quad+\frac{2\alpha_n}{1-\alpha\alpha_n\gamma}\langle\gamma f(x^*)-Bx^*,x_{n+1}-x^*\rangle\\&=\left(1-\frac{2\alpha_n(\overline{\gamma}-\alpha\gamma)}{1-\alpha\alpha_n\gamma}\right)\|x_n-x^*\|^2\\&\quad+\frac{\alpha_n}{1-\alpha\alpha_n\gamma}\left\{\frac{\alpha_n\overline{\gamma}^2}{1-\beta}\|x_n-x^*\|^2+2\langle\gamma f(x^*)-Bx^*,x_{n+1}-x^*\rangle\right\}.\end{aligned}\tag{3.67}$$

Since from condition (i) we have $\sum_{n=1}^{\infty}(2\alpha_n(\overline{\gamma}-\alpha\gamma)/1-\alpha\alpha_n\gamma)=\infty$ and

$$\limsup_{n\to\infty}\frac{\alpha_n\overline{\gamma}^2}{1-\beta}\|x_n-x^*\|^2+2\langle\gamma f(x^*)-Bx^*,x_{n+1}-x^*\rangle\le 0,\tag{3.68}$$

so, by Lemma 2.1, we conclude $\|x_n\to x^*\|$. This completes the proof. □

Putting $F\equiv 0$ and $b_n=\beta=0$ for all $n\ge 1$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from C into itself. Let $A:C\to H$ be relaxed (u,v)-cocoercive and μ-Lipschitzian. Let $f:C\to C$ be an α-contraction with $0\le\alpha<1$ and B a strong positive linear bounded operator with coefficient $\overline{\gamma}>0$, γ be a constant with $0<\gamma<\overline{\gamma}/\alpha$. Let sequences $\{\alpha_n\}$, $\{b_n\}$ in $(0,1)$ and β be a constant in $(0,1)$. Assume $C_0=\bigcap_{i=1}^N F(T_i)\cap \mathrm{VI}(C,A)\neq\emptyset$ and*

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $\{s_n\}$, $\{t_n\}\in[a,b]$ *for some a, b with $0\le a\le b\le(2(v-u\mu^2))/\mu^2$ and* $\lim_{n\to\infty}|s_{n+1}-s_n|=\lim_{n\to\infty}|t_{n+1}-t_n|=0$;

(iii) $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=0$.

Then the sequence $\{x_n\}$ generated by $x_1\in C$ and

$$\begin{aligned}y_n&=b_nx_n+(1-b_n)W_nP_C(I-s_nA)x_n,\\x_{n+1}&=\alpha_n\gamma f(W_nx_n)+\beta x_n+((1-\beta)I-\alpha_nB)W_nP_C(I-t_nA)y_n,\end{aligned}\tag{3.69}$$

converges strongly to $x^*\in C_0$ and x^* solves the variational inequality $x^*=P_{C_0}(I-(B-\gamma f))x^*$, that is,

$$\langle\gamma fx^*-Bx^*,x-x^*\rangle\le 0,\quad\forall x\in C_0.\tag{3.70}$$

Putting $P_C(I-s_nA)=P_C(I-t_nA)=I$ and $N=1$, $T_1=S$, $\beta=b_n=0$ in Theorem 3.1, we obtain the following corollary.

Corollary 3.3. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. S a finite family of nonexpansive mappings from C into itself and $F : C \times C \to R$ a bifunction satisfying (A1)–(A4). Let $f : C \to C$ be an α-contraction with $0 \le \alpha < 1$ and B a strong positive linear bounded operator with coefficient $\overline{\gamma} > 0$, γ is a constant with $0 < \gamma < \overline{\gamma}/\alpha$. Let sequences $\{\alpha_n\}$, $\{b_n\}$ in $(0,1)$ and $\{r_n\}$ in $(0,\infty)$. Assume $C_0 = F(S) \cap \mathrm{EP(F)} \ne \emptyset$ and*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$, $\liminf_{n\to\infty} r_n > 0$.

Then the sequence $\{x_n\}$ generated by $x_1 \in C$ and

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in H,$$
$$x_{n+1} = \alpha_n\gamma f(Sx_n) + (1 - \alpha_n B)Su_n, \tag{3.71}$$

converges strongly to $x^* \in C_0$ and x^* solves the variational inequality $x^* = P_{C_0}(I - (B - \gamma f))x^*$, that is,

$$\langle \gamma f x^* - Bx^*, x - x^*\rangle \le 0, \quad \forall x \in C_0. \tag{3.72}$$

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