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Research Article

Fixed Point Theorems for Random Lower Semi-Continuous Mappings

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We prove a general principle in Random Fixed Point Theory by introducing a condition named $(\mathcal{P})$ which was inspired by some of Petryshyn's work, and then we apply our result to prove some random fixed points theorems, including generalizations of some Bharucha-Reid theorems.

1. Introduction

Let (X, d) be a metric space and S a closed and nonempty subset of X. Denote by 2^X (resp., $C(X)$) the family of all nonempty (resp., nonempty and closed) subsets of X. A mapping $T : S \to 2^X$ is said to satisfy *condition*$(\mathcal{P})$ if, for every closed ball B of S with radius $r \geq 0$ and any sequence $\{x_n\}$ in S for which $d(x_n, B) \to 0$ and $d(x_n, T(x_n)) \to 0$ as $n \to \infty$, there exists $x_0 \in B$ such that $x_0 \in T(x_0)$ where $d(x, B) = \inf\{d(x, y) : y \in B\}$. If Ω is any nonempty set, we say that the operator $T : \Omega \times S \to 2^X$ satisfies *condition*$(\mathcal{P})$ if for each $\omega \in \Omega$, the mapping $T(\omega, \cdot) : S \to 2^X$ satisfies *condition*$(\mathcal{P})$. We should observe that this latter condition is related to a condition that was originally introduced by Petryshyn [1] for single-valued operators, in order to prove existence of fixed points. However, in our case, the condition is used to prove the measurability of a certain operator. On the other hand, in the year 2001, Shahzad (cf. [2]) using an idea of Itoh (cf. [3]), see also ([4]), proved that under a somewhat more restrictive condition, named condition (A), the following result.

Theorem S. *Let S be a nonempty separable complete subset of a metric space X and $T : \Omega \times C \to C(X)$ a continuous random operator satisfying condition (A). Then T has a deterministic fixed point if and only if T has a random fixed point.*

We shall show that the above result is still valid if the operator T is only lower semi-continuous. In addition, the assumption that each value $T(x)$ is closed has been relaxed without an extra assumption. Furthermore we state a new condition which generalizes condition (A) and allow us to generalize several known results, such as, Bharucha-Reid [5, Theorem 7], Domínguez Benavides et al. [6, Theorem 3.1] and Shahzad [2, Theorem 2.1].

2. Preliminaries

Let $(\Omega, \mathcal{A})$ be a measurable space and let (X, d) be a metric space. A mapping $F : \Omega \to 2^X$, is said to be measurable if $F^{-1}(G) = \{\omega \in \Omega : F(\omega) \cap G \neq \phi\}$ is measurable for each open subset G of X. This type of measurability is usually called weakly (cf. [7]), but since this is the only type of measurability we use in this paper, we omit the term "weakly". Notice that if X is separable and if, for each closed subset C of X, the set $F^{-1}(C)$ is measurable, then F is measurable.

Let C be a nonempty subset of X and $F : C \to 2^X$, then we say that F is lower (upper) semi-continuous if $F^{-1}(A)$ is open (closed) for all open (closed) subsets A of X. We say that F is continuous if F is lower and upper semi-continuous.

A mapping $F : \Omega \times X \to Y$ is called a random operator if, for each $x \in X$, the mapping $F(\cdot, x) : \Omega \to Y$ is measurable. Similarly a multivalued mapping $F : \Omega \times X \to 2^Y$ is also called a random operator if, for each $x \in X$, $F(\cdot, x) : \Omega \to 2^Y$ is measurable. A measurable mapping $\xi : \Omega \to Y$ is called a measurable selection of the operator $F : \Omega \to 2^Y$ if $\xi(\omega) \in F(\omega)$ for each $\omega \in \Omega$. A measurable mapping $\xi : \Omega \to X$ is called a random fixed point of the random operator $F : \Omega \times X \to X$ (or $F : \Omega \times X \to 2^X$) if for every $\omega \in \Omega$, $\xi(\omega) = F(\omega, \xi(\omega))$ (or $\xi(\omega) \in F(\omega, \xi(\omega))$). For the sake of clarity, we mention that $F(\omega, \xi(\omega)) = F(\omega, \cdot)(\xi(\omega))$.

Let C be a closed subset of the Banach space X, and suppose that F is a mapping from C into the topological vector space Y. We say the F is *demiclosed* at $y_0 \in Y$ if, for any sequences $\{x_n\}$ in C and $\{y_n\}$ in Y with $y_n \in F(x_n)$, $\{x_n\}$ converges weakly to x_0 and $\{y_n\}$ converges strongly to y_0, then it is the case that $x_0 \in C$ and $y_0 \in F(x_0)$. On the other hand, we say that F is *hemicompact* if each sequence $\{x_n\}$ in C has a convergent subsequence, whenever $d(x_n, F(x_n)) \to 0$ as $n \to \infty$.

3. Main Results

Theorem 3.1. *Let C be a closed separable subset of a complete metric space X, and let $T : \Omega \times C \to 2^X$ be measurable in ω and enjoy condition$(\mathcal{D})$. Suppose, for each $\omega \in \Omega$, that $h(\omega, x) = d(x, T(\omega, x))$ is upper semi-continuous and the set*

$$F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \phi. \tag{3.1}$$

Then T has a random fixed point.

Proof. Let $Z = \{z_n\}$ be a countable dense subset of C. Define $F : \Omega \to 2^C$ by $F(\omega) = \{x \in C : x \in T(\omega, x)\}$. Firstly, we show that F is measurable. To this end, let B_0 be an arbitrary closed ball of C, and set

$$L(B_0) = \bigcap_{k=1}^{\infty} \bigcup_{z \in Z_k} \left\{\omega \in \Omega : d(z, T(\omega, z)) < \frac{1}{k}\right\}, \tag{3.2}$$

where $Z_k = B_k \cap Z$ and $B_k = \{x \in C : d(x, B_0) < 1/k\}$. We claim that $F^{-1}(B_0) = L(B_0)$. To see this, let $\omega \in F^{-1}(B_0)$. Then there exists $x \in B_0$ such that $x \in T(\omega, x)$. Since $h(\omega, \cdot)$ is upper semi-continuous, for each $k \in \mathbb{N}$, there exists $z_{n_k} \in Z_k$ such that $d(z_{n_k}, T(\omega, z_{n_k})) < 1/k$. Therefore $\omega \in L(B_0)$. On the other hand, if $\omega \in L(B_0)$, then there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$d(z_{n_k}, B_0) < \frac{1}{k}, \qquad d(z_{n_k}, T(\omega, z_{n_k})) < \frac{1}{k} \tag{3.3}$$

for all $k \in \mathbb{N}$. This means that $d(z_{n_k}, B_0) \to 0$ and $d(z_{n_k}, T(\omega, z_{n_k})) \to 0$ as $n \to \infty$. Consequently, by $condition(\mathcal{D})$, there exists $x_0 \in B_0$ such that $x_0 \in T(\omega, x_0)$. Hence $\omega \in F^{-1}(B_0)$. Then we conclude that $F^{-1}(B_0) = L(B_0)$, and thus $F^{-1}(B_0)$ is measurable. To complete the proof, let G be an arbitrary open subset of C. Then by the separability of C,

$$G = \bigcup_{n=1}^{\infty} B_n \quad \text{where each } B_n \text{ is a closed ball of } C. \tag{3.4}$$

Since $F^{-1}(G) = \bigcup_{n=1}^{\infty} F^{-1}(B_n)$, we conclude that F is measurable. Additionally, we show that $F(\omega)$ is closed for each $\omega \in \Omega$. To see this, let $x_n \in F(\omega)$ such that $x_n \to x \in C$. Then, let $B_0 = \{x\}$ be a degenerated ball centered at x and radius $r = 0$, and since $d(x_n, T(\omega, x_n)) = 0$, $condition(\mathcal{D})$ implies that $x \in T(\omega, x)$. Hence $x \in F(\omega)$ and thus by the Kuratowski and Ryll-Nardzewski Theorem [8], F has a measurable selection $\xi : \Omega \to C$ such that $\xi(\omega) \in T(\omega, \xi(\omega))$ for each $\omega \in \Omega$. □

As a consequence of Theorem 3.1, we derive a new result for a lower semi-continuous random operator.

Theorem 3.2. *Let C be a closed separable subset of a complete metric space X, and let $T : \Omega \times C \to 2^X$ be a lower semi-continuous random operator, which enjoys $condition(\mathcal{D})$. Suppose, for each $\omega \in \Omega$, that the set*

$$F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \phi. \tag{3.5}$$

Then T has a random fixed point.

Proof. Due to Theorem 3.1, it is enough to show that $h(\omega, \cdot)$ is upper semi-continuous. To see this, we will prove that $A = \{x \in C : d(x, T(\omega, x)) < \alpha\}$ is open in C for $\alpha > 0$. Let $a \in A$ and select $\epsilon = \alpha - d(a, T(\omega, a))$. Then there exists $y \in T(\omega, a)$ so that $d(a, y) < \epsilon/3 + d(a, T(\omega, a))$. Since $T(\omega, \cdot)$ is lower semi-continuous, there exists a positive number $r < \epsilon/3$ such that $T(\omega, u) \cap B(y; \epsilon/3) \neq \emptyset$ for all $u \in B(a; r)$. Hence, we may choose $z_u \in T(\omega, u) \cap B(y; \epsilon/3)$ for which,

$$d(u, z_u) \leq d(u, a) + d(a, y) + d(y, z_u) < \alpha, \tag{3.6}$$

and consequently, $d(u, T(\omega, u)) < \alpha$. Therefore, A is open, and proof is complete. □

We observe that if the mapping $h(x) = d(x, T(x))$ is upper semi-continuous, then not necessarily the mapping T is lower semi-continuous. Consider the following example.

Let $T : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$T(x) = \begin{cases} 1, & x \neq 0 \\ [2,3], & x = 0. \end{cases} \tag{3.7}$$

Then $h(x) = |x-1|$ for $x \neq 0$ while $h(0) = 2$, which is upper semi-continuous. On the other hand, T is not lower semi-continuous.

Now, we derive several consequences of Theorem 3.2. We first obtain an extension of one of the main results of [6].

Theorem 3.3. *Let C be a weakly compact separable subset of a Banach space X, and let $T : \Omega \times C \to 2^X$ be a lower semi-continuous random operator. Suppose, for each $\omega \in \Omega$, that $I - T(\omega, \cdot)$ is demiclosed at 0 and the set*

$$F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \phi. \tag{3.8}$$

Then T has a random fixed point.

Proof. In order to apply Theorem 3.2, we just need to prove that T enjoys *condition*$(\mathcal{P})$. To this end, let ω be fixed in Ω. Suppose that B_0 is a closed ball of C with radius $r \geq 0$ where $\{x_n\}$ is a sequence in C such that $d(x_n, B_0) \to 0$ and $d(x_n, T(\omega, x_n)) \to 0$ as $n \to \infty$. Since C is separable, the weak topology on C is metrizable, and thus there exists a weakly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, so that $x_{n_k} \to x$ weakly, while $d(x_{n_k}, T(\omega, x_{n_k})) \to 0$ as $k \to \infty$. Consequently, for each $k \in \mathbb{N}$, there exists $z_k \in T(\omega, x_{n_k})$ such that

$$\|x_{n_k} - z_k\| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.9}$$

Hence, the demiclosedness of $I - T(\omega, \cdot)$ implies that $x \in T(\omega, x)$, and thus $T(\omega, \cdot)$ enjoys *condition*$(\mathcal{P})$.

Before we give an extension of the main result of [4], we observe that *condition*$(\mathcal{P})$ is basically applied to those closed balls directly used to prove the measurability of the mapping F, as will be seen in the proof of the next result. □

Theorem 3.4. *Let C be a closed separable subset of a complete metric space X, and let $T : \Omega \times C \to C(X)$ be a continuous hemicompact random operator. If, for each $\omega \in \Omega$, the set*

$$F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \phi, \tag{3.10}$$

then T has a random fixed point.

Proof. Due to Theorem 3.2, it would be enough to show that $T(\omega, \cdot)$ enjoys *condition*$(\mathcal{P})$ for every $\omega \in \Omega$. To see this, let B_0 be a closed ball of C, and let $\{x_n\}$ be a sequence in C such that $d(x_n, B_0) \to 0$ and $d(x_n, T(\omega, x_n)) \to 0$ as $n \to \infty$. Then by the hemicompactness of T, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, so that $x_{n_k} \to x \in B_0$. Hence

$d(x_{n_k}, T(\omega, x_{n_k})) \rightarrow 0$ as $k \rightarrow \infty$. This means that, for each $k \in \mathbb{N}$, there exists $z_k \in T(\omega, x_{n_k})$ such that

$$d(x_{n_k}, z_k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.11}$$

Consequently, $z_k \rightarrow x$. On the other hand, since T is upper semi-continuous at x, for every $\epsilon > 0$ there exist $k_0 \in \mathbb{N}$ such that

$$T(\omega, x_{n_k}) \subset B(T(\omega, x); \epsilon) \quad \text{for all } k \geq k_0. \tag{3.12}$$

Hence, $x \in \overline{B}(T(\omega, x); \epsilon)$. Since ϵ is arbitrary and $T(\omega, x)$ is closed, we derive that $x \in T(\omega, x)$, and thus T satisfies *condition*$(\mathcal{D})$. □

Corollary 3.5. *Let C be a locally compact separable subset of a complete metric space X, and let $T : \Omega \times C \rightarrow \mathcal{C}(X)$ be a continuous random operator. Suppose, for each $\omega \in \Omega$, that the set*

$$F(\omega) := \{x \in C : x \in T(\omega, x)\} \neq \phi. \tag{3.13}$$

Then T has a random fixed point.

Proof. Let G be an arbitrary open subset of C, and let $x \in G$. Since C is locally compact, there exists a compact ball B centered at x such that $B \subset G$. Now, we prove that *condition*$(\mathcal{D})$ holds with respect to B. To see this, let $\omega \in \Omega$, and let $\{x_n\}$ be a sequence in X such that $d(x_n, B) \rightarrow 0$ and $d(x_n, T(\omega, x_n)) \rightarrow 0$ as $n \rightarrow \infty$. Then there exists a sequence $\{y_n\}$ in B so that $d(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$. Since B is compact, there exists a convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \rightarrow x$, and consequently $x_{n_k} \rightarrow x$ with $x \in B$ as well as $d(x_{n_k}, T(\omega, x_{n_k})) \rightarrow 0$ as $k \rightarrow \infty$. Since T is upper semi-continuous, we derive, as in the proof of Theorem 3.4, that $x \in T(x)$. In addition, since T is lower semi-continuous, we may follow the proof of Theorem 3.1, to conclude that $F^{-1}(B)$ is measurable. Hence, the separability of C implies that we can select countably many compact balls B_i centered at corresponding points $x_i \in G$ such that

$$F^{-1}(G) = \bigcup_{i \in \mathbb{N}} F^{-1}(B_i). \tag{3.14}$$

Therefore, F is measurable. □

Next, we get a stochastic version of Schauder's Theorem, which is also an extension of a Theorem of Bharucha-Reid (see [5, Theorem 10]). We also observe that our proof is much easier and quite short.

Corollary 3.6. *Let C be a compact and convex subset of a Fréchet space X, and let $T : \Omega \times C \rightarrow C$ be a continuous random operator. Then T has a random fixed point.*

Proof. As we know, every Fréchet space is a complete metric space, and since C is compact, C itself is a complete separable metric space. In addition, for each $\omega \in \Omega$, there exists $x \in C$ such that $T(\omega, x) = x$. This means that the set $F(\omega)$, defined in Theorem 3.1, is nonempty.

Since C is compact, any sequence in C contains a convergent subsequence, which means that T is trivially a hemicompact operator. Consequently, by Theorem 3.4, T has a random fixed point. □

Before obtaining an extension of Bharucha-Reid [5, Theorem 3.7], we define a contraction mapping for metric spaces. Let X be a metric space, and let Ω be a measurable space. A random operator $T : \Omega \times X \to X$ is said to be a *random contraction* if there exists a mapping $k : \Omega \to [0,1)$ such that

$$d(T(\omega, x), T(\omega, y)) \leq k(\omega) d(x, y) \quad \text{for all } x, y \in X. \tag{3.15}$$

Theorem 3.7. *Let X be a complete separable metric space, and let $T : \Omega \times X \to X$ be a continuous random operator such that T^2 is a contraction with constant $k(\omega)$ for each $\omega \in \Omega$. Then T has a unique random fixed point.*

Proof. For each $\omega \in \Omega$, the mapping T^2 has a unique fixed point, $\xi(\omega)$, which is also the unique fixed point of T. It remains to show that the mapping $\xi : \Omega \to X$ defined by $T(\omega, \xi(\omega)) = \xi(\omega)$ is measurable. To see this, let $f_0 : \Omega \to X$ be an arbitrary measurable function. Then, we claim that $T(\omega, f_0(\omega))$ is measurable. To this end, let $Z = \{z_n\}$ be a countable dense set of X. Let $\omega \in \Omega$ and let $k \in \mathbb{N}$. Define

$$h_k : \Omega \longrightarrow X \quad \text{by } h_k(\omega) = z_m, \tag{3.16}$$

where m is the smallest natural number for which $d(z_m, f_0(\omega)) < 1/k$. Since f_0 is measurable, so are the sets $E_m = \{\omega \in \Omega : d(z_m, f_0(\omega)) < 1/k\}$, which, as a matter of fact, conform a disjoint covering of Ω. Consequently, $\{h_k\}$ is a sequence of measurable functions that converges pointwise to f_0. On the other hand, the range of each h_k is a subset of Z, and hence constant on each set E_m. Since the mapping T is measurable in ω, then, for each $k \in \mathbb{N}$, $T(\omega, h_k(\omega))$ is also measurable. Therefore the continuity of T on the second variable implies that

$$T(\omega, h_k(\omega)) \longrightarrow T(\omega, f_0(\omega)) \quad \text{as } k \longrightarrow \infty, \tag{3.17}$$

for each $\omega \in \Omega$. Hence $T(\omega, f_0(\omega))$ is measurable. Define the sequence

$$f_n(\omega) = T(\omega, f_{n-1}(\omega)), \quad n \in \mathbb{N}. \tag{3.18}$$

Then $\{f_n\}$ is a sequence of measurable functions. Since $f_n(\omega) = T^n(\omega, f_0(\omega))$, the fact that T^2 is a contraction implies that $f_n(\omega) \to \xi(\omega)$. Therefore, the mapping ξ is measurable, which completes the proof.

As a direct consequence of Theorem 3.7, we derive the extension mentioned earlier where the space X is more general, and the randomness on the mapping k has been removed. □

Corollary 3.8. *Let X be a complete separable metric space, and let $T : \Omega \times X \to X$ be a random contraction operator with constant $k(\omega)$ for each $\omega \in \Omega$. Then T has a unique random fixed point.*

Next, one can derive a corollary of the proof of Theorem 3.7, which is a theorem of Hans [9].

Corollary 3.9. *Let X be a complete separable metric space, and let $T : \Omega \times X \to X$ be a continuous random operator. Suppose, for each $\omega \in \Omega$, that there exists $n \in \mathbb{N}$ such that T^n is a contraction with constant $k(\omega)$. Then T has a unique random fixed point.*

Proof. As in the proof of the theorem, the mapping T has a unique fixed point for each $\omega \in \Omega$. The rest of the proof follows the proof of the theorem with the appropriate changes of the second power of T by the nth power of T. □

Notice that Theorem 3.7 holds for single-valued operators. The following question is formulated for multivalued operators taking closed and bounded values in X.

Open Question

Suppose that X is a complete separable metric space, and let $T : \Omega \times X \to \mathcal{CB}(X)$ be a continuous random operator such that T^2 is a contraction with constant $k(\omega)$ for each $\omega \in \Omega$. Then does T have a unique random fixed point?

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