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Research Article

Generalized Levitin-Polyak Well-Posedness of Vector Equilibrium Problems

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We study generalized Levitin-Polyak well-posedness of vector equilibrium problems with functional constraints as well as an abstract set constraint. We will introduce several types of generalized Levitin-Polyak well-posedness of vector equilibrium problems and give various criteria and characterizations for these types of generalized Levitin-Polyak well-posedness.

1. Introduction

It is well known that the well-posedness is very important for both optimization theory and numerical methods of optimization problems, which guarantees that, for approximating solution sequences, there is a subsequence which converges to a solution. The study of well-posedness originates from Tykhonov [1] in dealing with unconstrained optimization problems. Levitin and Polyak [2] extended the notion to constrained (scalar) optimization, allowing minimizing sequences $\{x_n\}$ to be outside of the feasible set X_0 and requiring $d(x_n, X_0)$ (the distance from x_n to X_0) to tend to zero. The Levitin and Polyak well-posedness is generalized in [3, 4] for problems with explicit constraint $g(x) \in K$, where g is a continuous map between two metric spaces and K is a closed set. For minimizing sequences $\{x_n\}$, instead of $d(x_n, X_0)$, here the distance $d(g(x_n), K)$ is required to tend to zero. This generalization is appropriate for penalty-type methods (e.g., penalty function methods, augmented Lagrangian methods) with iteration processes terminating when $d(g(x_n), K)$ is small enough (but $d(x_n, X_0)$ may be large). Recently, the study of generalized Levitin-Polyak well-posedness was extended to nonconvex vector optimization problems with abstract and functional constraints (see [5]), variational inequality problems with abstract and functional constraints (see [6]), generalized variational inequality problems with abstract and functional constraints [7], generalized vector variational inequality problems with abstract and functional constraints [8], and equilibrium problems with abstract and functional constraints [9]. Most recently, S. J. Li and M. H. Li [10] introduced and researched two types of Levitin-Polyak well-posedness of vector equilibrium problems with variable domination structures. Huang et al. [11] introduced and researched the Levitin-Polyak well-posedness of vector quasiequilibrium problems. Li et al. [12] introduced and researched the Levitin-Polyak well-posedness for two types of generalized vector quasiequilibrium problems. However, there is no study on the generalized Levitin-Polyak well-posedness for vector equilibrium problems and vector quasiequilibrium problems with explicit constraint $g(x) \in K$.

Motivated and inspired by the above works, in this paper, we introduce two types of generalized Levitin-Polyak well-posedness of vector equilibrium problems with functional constraints as well as an abstract set constraint and investigate criteria and characterizations for these two types of generalized Levitin-Polyak well-posedness. The results in this paper generalize and extend some known results in literature.

2. Preliminaries

Let (X, d_X), (Z, d_Z), and Y be locally convex Hausdorff topological vector spaces, where $d_X(d_Z)$ is the metric which compatible with the topology of $X(Z)$. Throughout this paper, we suppose that $K \subset Z$ and $X_1 \subset X$ are nonempty and closed sets, $C : X \to 2^Y$ is a set-valued mapping such that for any $x \in X$, $C(x)$ is a pointed, closed, and convex cone in Z with nonempty interior $\operatorname{int} C(x)$, $e : X \to Y$ is a continuous vector-valued mapping and satisfies that for any $x \in X$, $e(x) \in \operatorname{int} C(x)$, $f : X \times X_1 \to Y$ and $g : X_1 \to Z$ are two vector-valued mappings, and $X_0 = \{x \in X_1 : g(x) \in K\}$. We consider the following vector equilibrium problem with variable domination structures, functional constraints, as well as an abstract set constraint: finding a point $x^* \in X_0$, such that

$$f(x^*, y) \notin -\operatorname{int} C(x^*), \quad \forall y \in X_0. \tag{VEP}$$

We always assume that $X_0 \neq$ and g is continuous on X_1 and the solution set of (VEP) is denoted by Ω.

Let (P, d) be a metric space, $P_1 \subseteq P$, and $x \in P$. We denote by $d(x, P_1) = \inf\{d(x, p) : p \in P_1\}$ the distance function from the point $x \in P$ to the set P_1.

Definition 2.1. (i) A sequence $\{x_n\} \subset X_1$ is called a type I Levitin-Polyak (in short LP) approximating solution sequence for (VEP) if there exists $\{\epsilon_n\} \subset \mathbf{R}^1_+$ with $\epsilon_n \to 0$ such that

$$d(x_n, X_0) \leq \epsilon_n, \tag{2.1}$$

$$f(x_n, y) + \epsilon_n e(x_n) \notin -\operatorname{int} C(x_n), \quad \forall y \in X_0. \tag{2.2}$$

(ii) $\{x_n\} \subset X_1$ is called type II approximating solution sequence for (VEP) if there exists $\{\epsilon_n\} \subset \mathbf{R}^1_+$ with $\epsilon_n \to 0$ and $\{y_n\} \subset X_0$ satisfying (2.1), (2.2), and

$$f(x_n, y_n) - \epsilon_n e(x_n) \in -C(x_n). \tag{2.3}$$

(iii) $\{x_n\} \subset X_1$ is called a generalized type I approximating solution sequence for (VEP) if there exists $\{\epsilon_n\} \subset \mathbf{R}^1_+$ with $\epsilon_n \to 0$ satisfying

$$d\left(g(x_n), K\right) \leq \epsilon_n \tag{2.4}$$

and (2.2).

(iv) $\{x_n\} \subset X_1$ is called a generalized type II approximating solution sequence for (VEP) if there exists $\{\epsilon_n\} \subset \mathbf{R}^1_+$ with $\epsilon_n \to 0$ and $\{y_n\} \subset X_0$ satisfying (2.2), (2.3), and (2.4).

Definition 2.2. The vector equilibrium problem (VEP) is said to be type I (resp., type II, generalized type I, generalized type II) LP well-posed if $\Omega \neq \emptyset$ and for any type I (resp., type II, generalized type I, generalized type II) LP approximating solution sequence $\{x_n\}$ of (VEP), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\overline{x} \in \Omega$ such that $x_{n_j} \to \overline{x}$.

Remark 2.3. (i) If $Y = \mathbf{R}$ and $C(x) = \mathbf{R}^1_+ = \{r \in \mathbf{R} : r \geq 0\}$ for all $x \in X$, then the type I (resp., type II, generalized type I, generalized type II) LP well-posedness of (VEP) defined in Definition 2.2 reduces to the type I (resp., type II, generalized type I, generalized type II) LP well-posedness of the scalar equilibrium problem with abstract and functional constraints introduced by Long et al. [9]. Moreover, if X^* is the topological dual space of X, $F : X_1 \to X^*$ is a mapping, $\langle F(x), z\rangle$ denotes the value of the functional $F(x)$ at z, and $f(x,y) = \langle F(x), y - x\rangle$ for all $x, y \in X_1$, then the type I (resp., type II, generalized type I, generalized type II) LP well-posedness of (VEP) defined in Definition 2.2 reduces to the type I (resp., type II, generalized type I, generalized type II) LP well-posedness for the variational inequality with abstract and functional constraints introduced by Huang et al. [6]. If $K = Z$, then $X_1 = X_0$ and the type I (resp., type II) LP well-posedness of (VEP) defined in Definition 2.2 reduces to the type I (resp., type II) LP well-posedness of the vector equilibrium problem introduced by S. J. Li and M. H. Li [10].

(ii) It is clear that any (generalized) type II LP approximating solution sequence of (VEP) is a (generalized) type I LP approximating solution sequence of (VEP). Thus the (generalized) type I LP well-posedness of (VEP) implies the (generalized) type II LP well-posedness of (VEP).

(iii) Each type of LP well-posedness of (VEP) implies that the solution set Ω is nonempty and compact.

(iv) Let g be a uniformly continuous functions on the set

$$S(\delta_0) = \left\{x \in X_1 : d\left(g(x), K\right) \leq \delta_0\right\} \tag{2.5}$$

for some $\delta_0 > 0$. Then generalized type I (resp., type II) LP well-posedness implies type I (resp., type II) LP well-posedness.

3. Criteria and Characterizations for Generalized LP Well-Posedness of (VEP)

In this section, we present necessary and/or sufficient conditions for the various types of (generalized) LP well-posedness of (VEP) defined in Section 2.

3.1. Criteria and Characterizations without Using Gap Functions

In this subsection, we give some criteria and characterizations for the (generalized) LP well-posedness of (VEP) without using any gap functions of (VEP).

Now we introduce the Kuratowski measure of noncompactness for a nonempty subset A of X (see [13]) defined by

$$\alpha(A) = \inf\left\{\epsilon > 0 : A \subset \bigcup_{i=1}^{n} A_i,\ \text{for every } A_i,\ \operatorname{diam} A_i < \epsilon\right\}, \tag{3.1}$$

where $\operatorname{diam} A_i$ is the diameter of A_i defined by

$$\operatorname{diam} A_i = \sup\{d(x_1, x_2) : x_1, x_2 \in A_i\}. \tag{3.2}$$

Given two nonempty subsets A and B of X, the excess of set A to set B is defined by

$$e(A, B) = \sup\{d(a, B) : a \in A\}, \tag{3.3}$$

and the Hausdorff distance between A and B is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\}. \tag{3.4}$$

For any $\epsilon > 0$, four types of approximating solution sets for (VEP) are defined, respectively, by

$T_1(\epsilon) := \{x \in X_1 : d(g(x), K) \leq \epsilon$ and $f(x, y) + \epsilon e(x) \notin -\operatorname{int} C(x),$ for all $y \in X_0\}$,

$T_2(\epsilon) := \{x \in X_1 : d(x, X_0) \leq \epsilon$ and $f(x, y) + \epsilon e(x) \notin -\operatorname{int} C(x),$ for all $y \in X_0\}$,

$T_3(\epsilon) := \{x \in X_1 : d(g(x), K) \leq \epsilon$ and $f(x, y) + \epsilon e(x) \notin -\operatorname{int} C(x),$ for all $y \in X_0$ and $f(x, y) - \epsilon e(x) \in -C(x)$, for some $y \in X_0\}$,

$T_4(\epsilon) := \{x \in X_1 : d(x, X_0) \leq \epsilon$ and $f(x, y) + \epsilon e(x) \notin -\operatorname{int} C(x),$ for all $y \in X_0$ and $f(x, y) - \epsilon e(x) \in -C(x)$, for some $y \in X_0\}$.

Theorem 3.1. *Let X be complete.*

(i) (VEP) *is generalized type I LP well-posed if and only if the solution set Ω is nonempty and compact and*

$$e(T_1(\epsilon), \Omega) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{3.5}$$

(ii) (VEP) *is type I LP well-posed if and only if the solution set Ω is nonempty and compact and*

$$e(T_2(\epsilon), \Omega) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{3.6}$$

(iii) (VEP) *is generalized type II LP well-posed if and only if the solution set Ω is nonempty and compact and*

$$e(T_3(\epsilon), \Omega) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{3.7}$$

(iv) (VEP) *is type II LP well-posed if and only if the solution set* Ω *is nonempty and compact and*

$$e(T_4(\epsilon), \Omega) \longrightarrow 0 \quad \text{as } \epsilon \longrightarrow 0. \tag{3.8}$$

Proof. The proofs of (ii), (iii), and (iv) are similar with that of (i) and they are omitted here. Let (VEP) be generalized type I LP well-posed. Then Ω is nonempty and compact. Now we show that (3.5) holds. Suppose to the contrary that there exist $l > 0$, $\epsilon_n > 0$ with $\epsilon_n \to 0$ and $z_n \in T_1(\epsilon_n)$ such that

$$d(z_n, \Omega) \geq l. \tag{3.9}$$

Since $\{z_n\} \subset T_1(\epsilon_n)$ we know that $\{z_n\}$ is generalized type I LP approximating solution for (VEP). By the generalized type I LP well-posedness of (VEP), there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ converging to some element of Ω. This contradicts (3.9). Hence (3.5) holds.

Conversely, suppose that Ω is nonempty and compact and (3.5) holds. Let $\{x_n\}$ be a generalized type I LP approximating solution for (VEP). Then there exists a sequence $\{\epsilon_n\}$ with $\{\epsilon_n\} \subseteq \mathbf{R}^1_+$ and $\epsilon_n \to 0$ such that

$$\begin{gathered} d(g(x_n), K) \leq \epsilon_n, \\ f(x_n, y) + \epsilon_n e(x_n) \notin -\operatorname{int} C(x_n), \quad \forall y \in X_0. \end{gathered} \tag{3.10}$$

Thus, $\{x_n\} \subset T_1(\epsilon)$. It follows from (3.5) that there exists a sequence $\{z_n\} \subseteq \Omega$ such that

$$d(x_n, z_n) = d(x_n, \Omega) \leq e(T_1(\epsilon), \Omega) \longrightarrow 0. \tag{3.11}$$

Since Ω is compact, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging to $x_0 \in \Omega$. And so the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to x_0. Therefore (VEP) is generalized type I LP well-posed. This completes the proof. □

Theorem 3.2. *Let* X *be complete. Assume that*

(i) *for any* $y \in X_1$, *the vector-valued function* $x \mapsto f(x, y)$ *is continuous;*

(ii) *the mapping* $W : X \to 2^Y$ *defined by* $W(x) = Y \setminus -\operatorname{int} C(x)$ *is closed.*

Then (VEP) *is generalized type I LP well-posed if and only if*

$$T_1(\epsilon) \neq, \quad \forall \epsilon > 0, \quad \lim_{\epsilon \to 0} \alpha(T_1(\epsilon)) = 0. \tag{3.12}$$

Proof. First we show that for every $\epsilon > 0$, $T_1(\epsilon)$ is closed. In fact, let $\{x_n\} \subset T_1(\epsilon)$ and $x_n \to \overline{x}$. Then

$$\begin{gathered} d(g(x_n), K) \leq \epsilon, \\ f(x_n, y) + \epsilon e(x_n) \notin -\operatorname{int} C(x_n), \quad \forall y \in X_0. \end{gathered} \tag{3.13}$$

From (3.13), we get

$$d(g(\overline{x}), K) \leq \epsilon,$$
$$f(x_n, y) + \epsilon e(x_n) \in W(x_n), \quad \forall y \in X_0. \tag{3.14}$$

By assumptions (i), (ii), we have $f(\overline{x}, y) + \epsilon e(\overline{x}) \notin -\operatorname{int} C(\overline{x})$, for all $y \in X_0$. Hence $\overline{x} \in T_1(\epsilon)$.

Second, we show that

$$\Omega = \bigcap\nolimits_{\epsilon>0} T_1(\epsilon). \tag{3.15}$$

It is obvious that

$$\Omega \subset \bigcap\nolimits_{\epsilon>0} T_1(\epsilon). \tag{3.16}$$

Now suppose that $\epsilon_n > 0$ with $\epsilon_n \to 0$ and $x^* \in \bigcap_{n=1}^{\infty} T_1(\epsilon_n)$. Then

$$d(g(x^*), K) \leq \epsilon_n, \quad \forall n \in \mathbf{N}, \tag{3.17}$$

$$f(x^*, y) + \epsilon_n e(x^*) \notin -\operatorname{int} C(x^*), \quad \forall y \in X_0. \tag{3.18}$$

Since K is closed, g is continuous, and (3.17) holds, we have $x^* \in X_0$. By (3.18) and closedness of $W(x^*)$, we get $f(x^*, y) \in W(x^*)$, for all $y \in X_0$, that is, $x^* \in \Omega$. Hence (3.15) holds.

Now we assume that (3.12) holds. Clearly, $T_1(\cdot)$ is increasing with $\epsilon > 0$. By the Kuratowski theorem (see [14]), we have

$$H(T_1(\epsilon), \Omega) \longrightarrow 0, \quad \text{as } \epsilon \longrightarrow 0. \tag{3.19}$$

Let $\{x_n\}$ be any generalized type I LP approximating solution sequence for (VEP). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that (3.13) holds. Thus, $x_n \in T_1(\epsilon_n)$. It follows from (3.19) that $d(x_n, \Omega) \to 0$. So there exsist $u_n \in \Omega$, such that

$$d(x_n, u_n) \longrightarrow 0. \tag{3.20}$$

Since Ω is compact, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ and a solution $x^* \in \Omega$ satisfying

$$u_{n_j} \longrightarrow x^*. \tag{3.21}$$

From (3.20) and (3.21), we get $d(x_{n_j}, x^*) \to 0$.

Conversely, let (VEP) be generalized type I LP well-posed. Observe that for every $\epsilon > 0$,

$$H(T_1(\epsilon), \Omega) = \max\{e(T_1(\epsilon), \Omega), e(\Omega, T_1(\epsilon))\} = e(T_1(\epsilon), \Omega). \tag{3.22}$$

Hence,

$$\alpha(T_1(\epsilon)) \leq 2H(T_1(\epsilon), \Omega) + \alpha(\Omega) = 2e(T_1(\epsilon), \Omega), \tag{3.23}$$

where $\alpha(\Omega) = 0$ since Ω is compact. From Theorem 3.1(i), we know that $e(T_1(\epsilon), \Omega) \to 0$ as $\epsilon \to 0$. It follows from (3.23) that (3.12) holds. This completes the proof. □

Similar to Theorem 3.2, we can prove the following result.

Theorem 3.3. *Let X be complete. Assume that*

(i) *for any $y \in X_1$, the vector-valued function $x \mapsto f(x, y)$ is continuous;*

(ii) *the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -\operatorname{int} C(x)$ is closed;*

(iii) *the set-valued mapping $C : X_1 \to 2^Y$ is closed;*

(iv) *for any $x^* \in \Omega$, $f(x^*, y) \in -\partial C$, for some $y \in X_0$. Then* (VEP) *is generalized type II LP well-posed if and only if*

$$T_3(\epsilon) \neq , \quad \forall \epsilon > 0, \quad \lim_{\epsilon \to 0} \alpha(T_3(\epsilon)) = 0. \tag{3.24}$$

Definition 3.4. (VEP) is said to be generalized type I (resp., generalized type II) well-set if $\Omega \neq \emptyset$ and for any generalized type I (resp., generalized type II) LP approximating solution sequence $\{x_n\}$ for (VEP), we have

$$d(x_n, \Omega) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

From the definitions of the generalized LP well-posedness for (VEP) and those of the generalized well-set for (VEP), we can easily obtain the following proposition.

Proposition 3.5. *The relations between generalized LP well-posedness and generalized well set are*

(i) (VEP) *is generalized type I LP well-posed if and only if* (VEP) *is generalized type I well-set and Ω is compact.*

(ii) (VEP) *is generalized type II LP well-posed if and only if* (VEP) *is generalized type II well-set and Ω is compact.*

By combining the proof of Theorem 3.3 in [10] and that of Theorem 3.1, we can prove that the following results show that the relations between the generalized LP well-posedness for (VEP) and the solution set Ω of (VEP).

Theorem 3.6. *Let X be finite dimensional. Assume that*

(i) *for any $y \in X_1$, the vector-valued function $x \mapsto f(x, y)$ is continuous;*

(ii) *the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -\operatorname{int} C(x)$ is closed;*

(iii) *there exists $\epsilon_0 > 0$ such that $T_1(\epsilon_0)$ (resp., $T_3(\epsilon_0)$) is bounded.*

If Ω is nonempty, then (VEP) is generalized type I (resp., generalized type II) LP well-posed.

Corollary 3.7. *Suppose* $\Omega \neq$. *And assume that*

(i) *for any* $y \in X_1$ *the vector-valued function* $x \mapsto f(x,y)$ *is continuous;*

(ii) *the mapping* $W : X \to 2^Y$ *defined by* $W(x) = Y \setminus -\operatorname{int} C(x)$ *is closed;*

(iii) *there exists* $\epsilon_0 > 0$ *such that* $T_1(\epsilon_0)$ *(resp.,* $T_3(\epsilon_0)$*) is compact.*

If Ω is nonempty, then (VEP) is generalized type I (resp., generalized type II) LP well-posed.

3.2. Criteria and Characterizations Using Gap Functions

In this subsection, we give some criteria and characterizations for the (generalized) LP well-posedness of (VEP) using the gap functions of (VEP) introduced by S. J. Li and M. H. Li [10].

Chen et al. [15] introduced a nonlinear scalarization function $\xi_e : X \times Z \to \mathbf{R}$ defined by

$$\xi_e(x,y) = \inf\{\lambda \in \mathbf{R} : y \in \lambda e(x) - C(x)\}. \tag{3.26}$$

Definition 3.8 ([10]). A mapping $g : X \to \mathbf{R}$ is said to be a gap function on X_0 for (VEP) if

(i) $g(x) \geq 0$, for all $x \in X_0$;

(ii) $g(x^*) = 0$ and $x^* \in X_0$ if and only if $x^* \in \Omega$.

S. J. Li and M. H. Li [10] introduced a mapping $\phi : X \to \mathbf{R}$ defined as follows:

$$\phi(x) = \sup_{y \in X_0}\{-\xi_e(x, f(x,y))\}. \tag{3.27}$$

Lemma 3.9 (see [10]). *If for any* $x \in X_0$, $f(x,x) \in -\partial C(x)$, *where* $\partial C(x)$ *is the topological boundary of* $C(x)$, *then the mapping* ϕ *defined by* (3.27) *is a gap function on* X_0 *for* (VEP).

Now we consider the following general constrained optimization problems introduced and researched by Huang and Yang [4]:

$$\begin{gathered} (P)\min \phi(x) \\ s.t.\ x \in X_1,\ \ g(x) \in K. \end{gathered} \tag{3.28}$$

We use argmin ϕ *and* v^* *denote the optimal set and value of (P), respectively.*

The following example illustrates that it is useful to consider sequences that satisfy $d(g(x_n),K) \to 0$ instead of $d(x_n,X_0) \to +\infty$ for (VEP).

Example 3.10. Let $\alpha > 0$, $X = R^1$, $Z = R^1$, $C(x) = R^2_+$, and $e(x) = (1,1)$ for each $x \in X$, $K = R^1_-$,

$$X_1 = R^1_+, g(x) = \begin{cases} x, & \text{if } x \in [0,1], \\ \dfrac{1}{x^2}, & \text{if } x \geq 1, \end{cases}$$

$$f(x,y) = \begin{cases} (x^\alpha - y^\alpha, -x^\alpha - y - 1), & \text{if } x \in [0,1],\ \forall y \in X_1, \\ \left(\dfrac{1}{x^\alpha} - \dfrac{1}{y^\alpha}, -\dfrac{1}{x^\alpha} - y - 1\right), & \text{if } x > 1,\ \forall y \in X_1, \\ (-1,-1), & \text{if } x < 0,\ \forall y \in X_1. \end{cases} \tag{3.29}$$

Then, it is easy to verify that $X_0 = \{x \in X_1 : g(x) \in K\}$ and (VEP) is equivalent to the optimization problem (P) with

$$\phi(x) = \begin{cases} -x^\alpha, & \text{if } x \in [0,1], \\ -\dfrac{1}{x^\alpha}, & \text{if } x \geq 1. \end{cases} \tag{3.30}$$

Huang and Yang [4] showed that $x_n = (2n)^{1/\alpha}$ is the unique solution to the following penalty problem $(PP_\alpha(n))$:

$$(PP_\alpha(n)) \min_{x \in X_1} \phi(x) + n\left[\max\{0, g(x)\}\right]^\alpha, \quad n \in \mathbf{N}, \tag{3.31}$$

and $d(g(x_n), K) \to 0$ and $d(x_n, X_0) \to +\infty$.

Now, we recall the definitions about generalized well-posedness for (P) introduced by Huang and Yang [4] (or [7]) as follows

Definition 3.11. A sequence $\{x_n\} \subset X_1$ is called a generalized type I (resp., generalized type II) LP approximating solution sequence for (P) if the following (3.32) and (3.33) (resp., (3.32) and (3.34)) hold:

$$d(g(x_n), K) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.32}$$

$$\limsup_{n \to \infty} \phi(x_n) \leq v^*, \tag{3.33}$$

$$\lim_{n \to \infty} \phi(x_n) = v^*. \tag{3.34}$$

Definition 3.12. (P) is said to be generalized type I (resp., generalized type II) LP well-posed if

(i) $\operatorname{argmin} \phi \neq$;

(ii) for every generalized type I (resp., generalized type II) LP approximating solution sequence $\{x_n\}$ for (P), there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging to some element of $\operatorname{argmin} \phi$.

The following result shows the equivalent relations between the generalized LP well-posedness of (VEP) and the generalized LP well-posedness of (P).

Theorem 3.13. *Suppose that $f(x,x) \in -\partial C(x)$, for all $x \in X_0$. Then*

(i) (VEP) *is generalized type I well-posed if and only if (P) is generalized type I well-posed;*

(ii) (VEP) *is generalized type II well-posed if and only if (P) is generalized type II well-posed.*

Proof. (i) By Lemma 3.9, we know that ϕ is a gap function on X_0, $\overline{x} \in \Omega$ if and only if $\overline{x} \in \operatorname{argmin} \phi$ with $v^* = \phi(\overline{x}) = 0$.

Assume that $\{x_n\}$ is any generalized type I LP approximating solution sequence for (VEP). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$d(g(x_n), K) \leq \epsilon_n, \tag{3.35}$$

$$f(x_n, y) + \epsilon_n e(x_n) \notin -\operatorname{int} C(x_n), \quad \forall y \in X_0. \tag{3.36}$$

It follows from (3.35) and (3.36) that

$$d(g(x_n), K) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{3.37}$$

$$\xi_e(x_n, f(x_n, y)) \geq -\epsilon_n, \quad \forall y \in X_0. \tag{3.38}$$

Hence, we obtain

$$\phi(x_n) = \sup_{y \in X_0} \{-\xi_e(x_n, f(x_n, y))\} \leq \epsilon_n. \tag{3.39}$$

Thus,

$$\limsup_{n \to \infty} \phi(x_n) \leq 0 \quad \text{since } \epsilon_n \longrightarrow 0. \tag{3.40}$$

The above formula and (3.37) imply that $\{x_n\}$ is a generalized type I LP approximating solution sequence for (P).

Conversely, assume that $\{x_n\}$ is any generalized type I LP approximating solution sequence for (P). Then $d(g(x_n), K) \to 0$ and $\limsup_{n \to \infty} \phi(x_n) \leq 0$.

Thus, there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ satisfying (3.35) and

$$\phi(x_n) = \sup_{y \in X_0} \{-\xi_e(x_n, f(x_n, y))\} \leq \epsilon_n. \tag{3.41}$$

From (3.41), we have

$$\xi_e(x_n, f(x_n, y)) \geq -\epsilon_n, \quad \forall y \in X_0. \tag{3.42}$$

Equivalently, (3.36) holds. Hence, $\{x_n\}$ is a generalized type I LP approximating solution sequence for (VEP).

(ii) The proof is similar to (i) and is omitted. This completes the proof. □

Now we consider a real-valued function $c = c(t,s)$ defined for $t,s \geq 0$ sufficiently small, such that

$$\begin{gathered} c(t,s) \geq 0, \quad \forall t,s, \quad c(0,0) = 0, \\ s_n \longrightarrow 0, \quad t_n \geq 0, \quad c(t_n,s_n) \longrightarrow 0, \quad \text{imply } t_n \longrightarrow 0. \end{gathered} \tag{3.43}$$

Lemma 3.14 (see [4, Theorem 2.2]). *Suppose that $f(x,x) \in -\partial C(x)$ for any $x \in X_0$.*

(i) *If (P) is generalized type II LP well-posed, then there exists a function c satisfying* (3.43) *such that*

$$|\phi(x) - v^*| \geq c(d(x, \operatorname{argmin}\phi), d(g(x),K)), \quad \forall x \in X_1. \tag{3.44}$$

(ii) *Assume that* argmin ϕ *is nonempty and compact, and* (3.44) *holds for some c satisfying* (3.43). *Then (P) is generalized type II LP well-posed.*

The following theorem follows immediately from Lemma 3.14 and Theorem 3.13 with $\phi(x)$ defined by (3.27) *and $v^* = 0$.*

Theorem 3.15. *Suppose that $f(x,x) \in -\partial C(x)$ for any $x \in X_0$.*

(i) *If* (VEP) *is generalized type II LP well-posed, then there exists a function c satisfying* (3.43) *such that*

$$|\phi(x)| \geq c(d(x,\Omega), d(g(x),K)), \quad \forall x \in X_1. \tag{3.45}$$

(ii) *Assume that Ω is nonempty and compact, and* (3.45) *holds for some c satisfying* (3.43). *Then* (VEP) *is generalized type II LP well-posed.*

Definition 3.16 (see [4, 7]). (i) Let Z be a topological space and let $Z_1 \subset Z$ be a nonempty subset. Suppose that $G : Z \to R \cup \{+\infty\}$ is an extend real-valued function. Then the function G is said to be level-compact on Z_1 if for any $s \in R^1$ the subset $\{z \in Z_1 : G(z) \leq s\}$ is compact.

(ii) Let Z be a finite dimensional normed space and $Z_1 \subset Z$ be nonempty. A function $h : Z \to R^1 \cup \{+\infty\}$ is said to be level-bounded on Z_1 if Z_1 is bounded or

$$\lim_{z \in Z_1, \|z\| \to +\infty} h(z) = +\infty. \tag{3.46}$$

Proposition 3.17. *Assume that for any $y \in X_1$, the vector-valued function $x \mapsto f(x,y)$ is continuous and the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -\operatorname{int} C(x)$ is closed, and Ω is nonempty. Then,* (VEP) *is generalized type I LP well-posed if one of the following conditions holds:*

(i) *there exists $\delta_1 > 0$ such that $S(\delta_1)$ is compact, where*

$$S(\delta_1) = \{x \in X_1 : d(g(x),K) \leq \delta_1\}; \tag{3.47}$$

(ii) *the function* ϕ *defined by* (3.27) *is level-compact on* X_1*;*
(iii) X *is a finite-dimensional normed space and*

$$\lim_{x\in X_1,\|x\|\to+\infty} \max\{\phi(x), d(g(x), K)\} = +\infty; \tag{3.48}$$

(iv) *there exists* $\delta_1 > 0$ *such that* ϕ *is level-compact on* $S(\delta_1)$ *defined by* (3.47).

Proof. Let $\{x_n\} \subseteq X_1$ be a generalized type I LP approximating solution sequence for (VEP). Then there exists a sequence $\{\epsilon_n\} \subseteq R^1_+$ with $\epsilon_n > 0$ such that (3.35) and (3.36) hold. From (3.20), without loss of generality, we assume that $\{x_n\} \subset S(\delta_1)$. Since $S(\delta_1)$ is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x_0 \in S(\delta_1)$ such that $x_{n_j} \to x_0$. This fact combined with (3.35) yields that $x_0 \in X_0$. Furthermore, it follows from (3.36) and the continuity of f with respect to the first argument and the closedness of W that we have $f(x_0, y) \notin -\operatorname{int} C(x_0)$, for all $y \in X_0$. So $x_0 \in \Omega$. This implies that (VEP) is generalized type I LP well-posed.

It is easy to see that condition (ii) implies condition (iv). Now we show that condition (iii) implies condition (iv). It follows from [10, Proposition 4.2] that the function ϕ defined by (3.27) is lower semicontinuous, and thus for any $t \in R^1$, the set $\{x \in S(\delta_1) : \phi(x) \leq t\}$ is closed. Since X is a finite dimensional space, we need only to show that for any $t \in R^1$, the set $\{x \in S(\delta_1) : \phi(x) \leq t\}$ is bounded. Suppose to the contrary that there exists $t \in R^1$ and $\{x'_n\} \subset S(\delta_1)$ and $\phi(x'_n) \leq t$ such that $\|x'_n\| \to +\infty$. It follows from $\{x'_n\} \subset S(\delta_1)$ that $d(g(x'_n), K) \leq \delta_1$ and so

$$\max\{\phi(x'_n), d(g(x'_n), K)\} \leq \max\{t, \delta_1\}. \tag{3.49}$$

Which contradicts with (3.48).

Therefore, we only need to prove that if condition (iv) holds, then (VEP) is generalized type I LP well-posed. Suppose that condition (iv) holds and $\{x_n\}$ is a generalized type I LP approximating solution sequence for (VEP). Then there exists $\{\epsilon_n\} \subset R^1_+$ with $\epsilon_n > 0$ such that (3.35) and (3.36) hold. By (3.35), we can assume without loss of generality that

$$\{x_n\} \subset S(\delta_1). \tag{3.50}$$

It follows from (3.36) that $\xi_e(x_n, f(x_n, y)) \geq -\epsilon_n$, for all $y \in X_0$. Thus,

$$\phi(x_n) \leq \epsilon_n, \quad \forall n. \tag{3.51}$$

From (3.51), without loss of generality, we assume that $\{x_n\} \subseteq \{x \in S(\delta_1) : \phi(x) \leq b\}$ for some $b > 0$. Since ϕ is level-compact on $S(\delta_1)$, the subset $\{x \in S(\delta_1) : \phi(x) \leq b\}$ is compact. It follows that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $\overline{x} \in S(\delta_1)$ such that $x_{n_j} \to \overline{x}$. This together with (3.35) yields $\overline{x} \in X_0$. Furthermore by the continuity of f with respect to the first argument, the closedness of W, and (3.36) we have $x_0 \in \Omega$. This completes the proof. □

Similarly, we can prove Proposition 3.18.

Proposition 3.18. *Assume that for any $y \in X_1$, the vector-valued function $x \mapsto f(x,y)$ is continuous and the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -\operatorname{int} C(x)$ is closed, and Ω is nonempty. Then,* (VEP) *is type I LP well-posed if one of the following conditions holds:*

(i) *there exists $\delta_1 > 0$ such that $S_1(\delta_1)$ is compact where*

$$S_1(\delta_1) = \{x \in X_1 : d(x, X_0) \leq \delta_1\}; \tag{3.52}$$

(ii) *the function ϕ defined by* (3.27) *is level-compact on X_1;*

(iii) *X is a finite-dimensional normed space and*

$$\lim_{x \in X_1, \|x\| \to +\infty} \max\{\phi(x), d(x, X_0)\} = +\infty; \tag{3.53}$$

(iv) *there exists $\delta_1 > 0$ such that ϕ is level-compact on $S_1(\delta_1)$ defined by* (3.52).

Proposition 3.19. *Assume that X is a finite dimensional space, for any $y \in X_1$, the vector-valued function $x \mapsto f(x,y)$ is continuous and the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -\operatorname{int} C(x)$ is closed, and Ω is nonempty. Suppose that there exists $\delta_1 > 0$ such that the function $\phi(x)$ defined by* (3.27) *is level-bounded on the set $S(\delta_1)$ defined by* (3.47). *Then* (VEP) *is generalized type I LP well-posed.*

Proof. Let $\{x_n\}$ be a generalized type I LP approximating solution sequence for (VEP). Then there exists $\{\epsilon_n\}$ with $\epsilon_n > 0$ such that (3.35) and (3.36) hold.

From (3.35), without loss of generality, we assume that $\{x_n\} \subset S(\delta_1)$. Let us show by contradiction that $\{x_n\}$ is bounded. Otherwise we assume without loss of generality that $\|x_n\| \to +\infty$. By the level-boundedness of ϕ, we have

$$\lim_{\|x\| \to +\infty} \phi(x) = +\infty. \tag{3.54}$$

It follows from (3.36) and the proof in Proposition 3.17 that (3.51) holds. which contradicts with (3.54).

Now we assume without loss of generality that $x_n \to \overline{x}$. Furthermore by the continuity of f with respect to the first argument, the closedness of W, and (3.36) we have $x_0 \in \Omega$. This completes the proof. □

Similarly, we can prove the following Proposition 3.20.

Proposition 3.20. *Assume that X is a finite dimensional space, for any $y \in X_1$, the vector-valued function $x \mapsto f(x,y)$ is continuous and the mapping $W : X \to 2^Y$ defined by $W(x) = Y \setminus -\operatorname{int} C(x)$ is closed, and Ω is nonempty. Suppose that there exists $\delta_1 > 0$ such that the function $\phi(x)$ defined by* (3.27) *is level-bounded on the set $S_1(\delta_1)$ defined by* (3.52). *Then* (VEP) *is type I LP well-posed.*

Remark 3.21. Theorem 3.1 generalizes and extends [9, Theorems 3.1–3.6] from scalar-valued case to vector-valued case. Propositions 3.17–3.20, respectively, generalize and extend [9, Propositions 4.3, 4.2, 4.5, and 4.4] from scalar-valued case to vector-valued case. Theorems 3.2, 3.3, 3.6, 3.13, and 3.15, Proposition 3.5 and Corollary 3.7, respectively, extend [10, Theorems 3.1–3.3, 4.1, and 4.2, Proposition 3.1 and Corollary 3.1] from the well-posedness of (VEP) to the generalized well-posedness of (VEP). It is easy to see that the results in this paper generalize and extende the main results in [6] in several aspects.

Remark 3.22. The generalized Levitin-Polyak well-posedness for vectorquasiequilibrium problems and generalized vector-quasiequilibrium problems with explicit constraint $g(x) \in K$ is still an open question and we will do the research in the near future.

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