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Research Article

Fixed Points and Stability of the Cauchy Functional Equation in C^*-Algebras

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Using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras and Lie C^*-algebras and of derivations on C^*-algebras and Lie C^*-algebras for the Cauchy functional equation.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [6–19]).

J. M. Rassias [20, 21] following the spirit of the innovative approach of Th. M. Rassias [4] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [22] for a number of other new results).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Theorem 1.1 (see [23, 24]). *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.1}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \ \forall n \geq n_0$;

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ for all $y \in Y$.*

This paper is organized as follows. In Sections 2 and 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras and of derivations on C^*-algebras for the Cauchy functional equation.

In Sections 4 and 5, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Lie C^*-algebras and of derivations on Lie C^*-algebras for the Cauchy functional equation.

2. Stability of Homomorphisms in C^*-Algebras

Throughout this section, assume that A is a C^*-algebra with norm $\|\cdot\|_A$ and that B is a C^*-algebra with norm $\|\cdot\|_B$.

For a given mapping $f : A \to B$, we define

$$D_\mu f(x,y) := \mu f(x+y) - f(\mu x) - f(\mu y) \tag{2.1}$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} \mid |\nu| = 1\}$ and all $x, y \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* in C^*-algebras if H satisfies $H(xy) = H(x)H(y)$ and $H(x^*) = H(x)^*$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in C^*-algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ such that*

$$\|D_\mu f(x,y)\|_B \leq \varphi(x,y), \tag{2.2}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \varphi(x,y), \tag{2.3}$$

$$\|f(x^*) - f(x)^*\|_B \leq \varphi(x,x) \tag{2.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x, y) \le 2L\varphi(x/2, y/2)$ for all $x, y \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{1}{2 - 2L}\varphi(x, x) \tag{2.5}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \longrightarrow B\}, \tag{2.6}$$

and introduce the *generalized metric* on X:

$$d(g, h) = \inf\left\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \le C\varphi(x, x),\ \forall x \in A\right\}. \tag{2.7}$$

It is easy to show that (X, d) is complete.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x) \tag{2.8}$$

for all $x \in A$.

By [23, Theorem 3.1],

$$d(Jg, Jh) \le Ld(g, h) \tag{2.9}$$

for all $g, h \in X$.

Letting $\mu = 1$ and $y = x$ in (2.2), we get

$$\|f(2x) - 2f(x)\|_B \le \varphi(x, x) \tag{2.10}$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_B \le \frac{1}{2}\varphi(x, x) \tag{2.11}$$

for all $x \in A$. Hence $d(f, Jf) \le 1/2$.

By Theorem 1.1, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.12}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}. \tag{2.13}$$

This implies that H is a unique mapping satisfying (2.12) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x, x) \tag{2.14}$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x) \tag{2.15}$$

for all $x \in A$.

(3) $d(f, H) \leq (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{2 - 2L}. \tag{2.16}$$

This implies that the inequality (2.5) holds.

It follows from (2.2) and (2.15) that

$$\begin{aligned} \|H(x+y) - H(x) - H(y)\|_B &= \lim_{n\to\infty} \frac{1}{2^n} \left\| f(2^n(x+y)) - f(2^n x) - f(2^n y) \right\|_B \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0 \end{aligned} \tag{2.17}$$

for all $x, y \in A$. So

$$H(x+y) = H(x) + H(y) \tag{2.18}$$

for all $x, y \in A$.

Letting $y = x$ in (2.2), we get

$$\mu f(2x) = f(\mu 2x) \tag{2.19}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. By a similar method to above, we get

$$\mu H(2x) = H(2\mu x) \tag{2.20}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.3) that

$$\begin{aligned} \|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \left\| f(4^n xy) - f(2^n x) f(2^n y) \right\|_B \\ &\le \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \\ &\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) \\ &= 0 \end{aligned} \tag{2.21}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y) \tag{2.22}$$

for all $x, y \in A$.

It follows from (2.4) that

$$\begin{aligned} \left\|H(x^*) - H(x)^*\right\|_B &= \lim_{n\to\infty} \frac{1}{2^n} \left\| f(2^n x^*) - f(2^n x)^* \right\|_B \\ &\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n x) = 0 \end{aligned} \tag{2.23}$$

for all $x \in A$. So

$$H(x^*) = H(x)^* \tag{2.24}$$

for all $x \in A$.

Thus $H : A \to B$ is a C^*-algebra homomorphism satisfying (2.5), as desired. □

Corollary 2.2. *Let $0 < r < 1/2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\left\|D_\mu f(x,y)\right\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{2.25}$$

$$\|f(xy) - f(x)f(y)\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{2.26}$$

$$\left\|f(x^*) - f(x)^*\right\|_B \le \theta \|x\|_A^{2r} \tag{2.27}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{\theta}{2 - 4^r} \|x\|_A^{2r} \tag{2.28}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{2.29}$$

for all $x, y \in A$. Then $L = 2^{2r-1}$ and we get the desired result. □

Theorem 2.3. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (2.2), (2.3), *and* (2.4). *If there exists an $L < 1$ such that $\varphi(x,y) \leq (1/2)L\varphi(2x,2y)$ for all $x, y \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{L}{2-2L}\varphi(x,x) \tag{2.30}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right) \tag{2.31}$$

for all $x \in A$.

It follows from (2.10) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2}\varphi(x,x) \tag{2.32}$$

for all $x \in A$. Hence, $d(f, Jf) \leq L/2$.

By Theorem 1.1, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, that is,

$$H(2x) = 2H(x) \tag{2.33}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}. \tag{2.34}$$

This implies that H is a unique mapping satisfying (2.33) such that there exists $C \in (0,\infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x,x) \tag{2.35}$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x) \tag{2.36}$$

for all $x \in A$.

(3) $d(f,H) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,H) \le \frac{L}{2-2L}, \tag{2.37}$$

which implies that the inequality (2.30) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.25), (2.26), *and* (2.27). *Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{\theta}{4^r - 2}\|x\|_A^{2r} \tag{2.38}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{2.39}$$

for all $x, y \in A$. Then $L = 2^{1-2r}$ and we get the desired result. □

3. Stability of Derivations on C^*-Algebras

Throughout this section, assume that A is a C^*-algebra with norm $\|\cdot\|_A$.

Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of derivations on C^*-algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 3.1. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ such that*

$$\left\|D_\mu f(x,y)\right\|_A \le \varphi(x,y), \tag{3.1}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \le \varphi(x,y) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x,y) \leq 2L\varphi(x/2, y/2)$ for all $x, y \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{1}{2-2L}\varphi(x,x) \tag{3.3}$$

for all $x \in A$.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{3.4}$$

for all $x \in A$.

It follows from (3.2) that

$$\begin{aligned}\|\delta(xy) - \delta(x)y - x\delta(y)\|_A &= \lim_{n\to\infty} \frac{1}{4^n}\left\|f(4^n xy) - f(2^n x)\cdot 2^n y - 2^n x f(2^n y)\right\|_A \\ &\leq \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y) \\ &\leq \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) \\ &= 0\end{aligned} \tag{3.5}$$

for all $x, y \in A$. So

$$\delta(xy) = \delta(x)y + x\delta(y) \tag{3.6}$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). □

Corollary 3.2. *Let $0 < r < 1/2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that*

$$\left\|D_\mu f(x,y)\right\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{3.7}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{3.8}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{\theta}{2-4^r}\|x\|_A^{2r} \tag{3.9}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{3.10}$$

for all $x, y \in A$. Then $L = 2^{2r-1}$ and we get the desired result. □

Theorem 3.3. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (3.1) *and* (3.2)*. If there exists an $L < 1$ such that $\varphi(x,y) \le (1/2)L\varphi(2x,2y)$ for all $x, y \in A$, then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{L}{2-2L}\varphi(x,x) \tag{3.11}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1. □

Corollary 3.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.7) *and* (3.8)*. Then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{\theta}{4^r - 2}\|x\|_A^{2r} \tag{3.12}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{3.13}$$

for all $x, y \in A$. Then $L = 2^{1-2r}$ and we get the desired result. □

4. Stability of Homomorphisms in Lie C^*-Algebras

A C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x,y] := (xy - yx)/2$ on $\mathcal{C}$, is called a *Lie C^*-algebra* (see [9–11]).

Definition 4.1. Let A and B be Lie C^*-algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *Lie C^*-algebra homomorphism* if $H([x,y]) = [H(x), H(y)]$ for all $x, y \in A$.

Throughout this section, assume that A is a Lie C^*-algebra with norm $\|\cdot\|_A$ and that B is a C^*-algebra with norm $\|\cdot\|_B$.

We prove the generalized Hyers-Ulam stability of homomorphisms in Lie C^*-algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 4.2. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (2.2) *such that*

$$\|f([x,y]) - [f(x), f(y)]\|_B \le \varphi(x,y) \tag{4.1}$$

for all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x,y) \leq 2L\varphi(x/2, y/2)$ for all $x, y \in A$, then there exists a unique Lie C^*-algebra homomorphism $H : A \to B$ satisfying (2.5).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (2.5). The mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{4.2}$$

for all $x \in A$.

It follows from (4.1) that

$$\begin{aligned} \|H([x,y]) - [H(x), H(y)]\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \left\| f(4^n[x,y]) - \left[f(2^n x), f(2^n y)\right] \right\|_B \\ &\leq \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \\ &\leq \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) \\ &= 0 \end{aligned} \tag{4.3}$$

for all $x, y \in A$. So

$$H([x,y]) = [H(x), H(y)] \tag{4.4}$$

for all $x, y \in A$.

Thus $H : A \to B$ is a Lie C^*-algebra homomorphism satisfying (2.5), as desired. □

Corollary 4.3. *Let $r < 1/2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.25) *such that*

$$\|f([x,y]) - [f(x), f(y)]\|_B \leq \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{4.5}$$

for all $x, y \in A$. Then there exists a unique Lie C^*-algebra homomorphism $H : A \to B$ satisfying (2.28).

Proof. The proof follows from Theorem 4.2 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{4.6}$$

for all $x, y \in A$. Then $L = 2^{2r-1}$ and we get the desired result. □

Theorem 4.4. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying* (2.2) *and* (4.1). *If there exists an $L < 1$ such that $\varphi(x,y) \leq (1/2)L\varphi(2x, 2y)$ for all $x, y \in A$, then there exists a unique Lie C^*-algebra homomorphism $H : A \to B$ satisfying* (2.30).

Proof. The proof is similar to the proofs of Theorems 2.3 and 4.2. □

Corollary 4.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.25) *and* (4.5). *Then there exists a unique Lie C^*-algebra homomorphism $H : A \to B$ satisfying* (2.38).

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x, y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{4.7}$$

for all $x, y \in A$. Then $L = 2^{1-2r}$ and we get the desired result. □

Definition 4.6. A C^*-algebra A, endowed with the Jordan product $x \circ y := (xy + yx)/2$ for all $x, y \in A$, is called a *Jordan C^*-algebra* (see [25]).

Definition 4.7. Let A and B be Jordan C^*-algebras.

(i) A $\mathbb{C}$-linear mapping $H : A \to B$ is called a *Jordan C^* -algebra homomorphism* if $H(x \circ y) = H(x) \circ H(y)$ for all $x, y \in A$.

(ii) A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *Jordan derivation* if $\delta(x \circ y) = x \circ \delta(y) + \delta(x) \circ y$ for all $x, y \in A$.

Remark 4.8. If the Lie products $[\cdot, \cdot]$ in the statements of the theorems in this section are replaced by the Jordan products $\cdot \circ \cdot$, then one obtains Jordan C^*-algebra homomorphisms instead of Lie C^*-algebra homomorphisms.

5. Stability of Lie Derivations on C^*-Algebras

Definition 5.1. Let A be a Lie C^*-algebra. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *Lie derivation* if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in A$.

Throughout this section, assume that A is a Lie C^*-algebra with norm $\|\cdot\|_A$.

We prove the generalized Hyers-Ulam stability of derivations on Lie C^*-algebras for the functional equation $D_\mu f(x, y) = 0$.

Theorem 5.2. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying* (3.1) *such that*

$$\|f([x, y]) - [f(x), y] - [x, f(y)]\|_A \le \varphi(x, y) \tag{5.1}$$

for all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x, y) \le 2L\varphi(x/2, y/2)$ for all $x, y \in A$. Then there exists a unique Lie derivation $\delta : A \to A$ satisfying (3.3).

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique involutive $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{5.2}$$

for all $x \in A$.

It follows from (5.1) that

$$\begin{aligned}
&\|\delta([x,y]) - [\delta(x),y] - [x,\delta(y)]\|_A \\
&= \lim_{n\to\infty} \frac{1}{4^n} \left\| f(4^n[x,y]) - [f(2^n x), 2^n y] - [2^n x, f(2^n y)] \right\|_A \\
&\le \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \\
&\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) \\
&= 0
\end{aligned} \tag{5.3}$$

for all $x, y \in A$. So

$$\delta([x,y]) = [\delta(x), y] + [x, \delta(y)] \tag{5.4}$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). □

Corollary 5.3. *Let $0 < r < 1/2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.7) *such that*

$$\|f([x,y]) - [f(x),y] - [x,f(y)]\|_A \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{5.5}$$

for all $x, y \in A$. Then there exists a unique Lie derivation $\delta : A \to A$ satisfying (3.9).

Proof. The proof follows from Theorem 5.2 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{5.6}$$

for all $x, y \in A$. Then $L = 2^{2r-1}$ and we get the desired result. □

Theorem 5.4. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (3.1) *and* (5.1). *If there exists an $L < 1$ such that $\varphi(x,y) \le (1/2)L\varphi(2x,2y)$ for all $x, y \in A$, then there exists a unique Lie derivation $\delta : A \to A$ satisfying* (3.11).

Proof. The proof is similar to the proofs of Theorems 2.3 and 5.2. □

Corollary 5.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.7) *and* (5.5). *Then there exists a unique Lie derivation $\delta : A \to A$ satisfying* (3.12).

Proof. The proof follows from Theorem 5.4 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{5.7}$$

for all $x, y \in A$. Then $L= 2^{1-2r}$ and we get the desired result. □

Remark 5.6. If the Lie products $[\cdot,\cdot]$ in the statements of the theorems in this section are replaced by the Jordan products $\cdot \circ \cdot$, then one obtains Jordan derivations instead of Lie derivations.

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