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Research Article

A Fixed Point Approach to the Stability of an Additive-Quadratic-Cubic-Quartic Functional Equation

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Using the fixed point method, we prove the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation $f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$ in Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations is originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [6] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [8] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [9–19]).

In [20], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x), \tag{1.2}$$

which is called a *cubic functional equation*, and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [21], Lee et al. considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y), \tag{1.3}$$

which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*. Quartic functional equations have been investigated in [22, 23].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1 (see [24, 25]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.4}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all $n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Th. M. Rassias [26] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [27–32]).

This paper is organized as follows. In Section 2, we prove the generalized Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y) \tag{1.5}$$

in Banach spaces for an odd case. In Section 3, we prove the generalized Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation (1.5) in Banach spaces for an even case.

Throughout this paper, assume that X is a vector space and that Y is a Banach space.

2. Generalized Hyers-Ulam Stability of the Functional Equation (1.5): An Odd Case

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) &:= f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x) \\ &\quad - f(2y)-f(-2y)+4f(y)+4f(-y) \end{aligned} \tag{2.1}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y)=0$ in Banach spaces: an odd case.

Note that the fundamental ideas in the proofs of the main results in Sections 2 and 3 are contained in [24, 27, 28].

Theorem 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y) \le \frac{L}{8}\varphi(2x,2y) \tag{2.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.3}$$

for all $x, y \in X$. Then there is a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x)-2f(x)-C(x)\| \le \frac{L}{8-8L}(4\varphi(x,x)+\varphi(2x,x)) \tag{2.4}$$

for all $x \in X$.

Proof. Letting $x=y$ in (2.3), we get

$$\|f(3y)-4f(2y)+5f(y)\| \le \varphi(y,y) \tag{2.5}$$

for all $y \in X$.

Replacing x by $2y$ in (2.3), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \leq \varphi(2y, y) \tag{2.6}$$

for all $y \in X$.

By (2.5) and (2.6),

$$\begin{aligned}\|f(4y) - 10f(2y) + 16f(y)\| &\leq \|4(f(3y) - 4f(2y) + 5f(y))\| \\ &\quad + \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \\ &\leq 4\varphi(y, y) + \varphi(2y, y)\end{aligned} \tag{2.7}$$

for all $y \in X$. Letting $y := x/2$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{2.8}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \longrightarrow Y\}, \tag{2.9}$$

and introduce the generalized metric on S:

$$d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu(4\varphi(x, x) + \varphi(2x, x)),\ \forall x \in X\}, \tag{2.10}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see the proof of Lemma 2.1 of [33]).

Now we consider the linear mapping $J : S \rightarrow S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right) \tag{2.11}$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq 4\varphi(x, x) + \varphi(2x, x) \tag{2.12}$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right)\right\| \leq L(4\varphi(x, x) + \varphi(2x, x)) \tag{2.13}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h) \tag{2.14}$$

for all $g, h \in S$.

It follows from (2.8) that

$$\left\| g(x) - 8g\left(\frac{x}{2}\right) \right\| \le \frac{L}{8}(4\varphi(x,x) + \varphi(2x,x)) \tag{2.15}$$

for all $x \in X$. So $d(g, Jg) \le L/8$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following.

(1) C is a fixed point of J, that is,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.16}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}. \tag{2.17}$$

This implies that C is a unique mapping satisfying (2.16) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|g(x) - C(x)\| \le \mu(4\varphi(x,x) + \varphi(2x,x)) \tag{2.18}$$

for all $x \in X$.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x) \tag{2.19}$$

for all $x \in X$.

(3) $d(g,C) \le (1/(1-L))d(g,Jg)$, which implies the inequality

$$d(g,C) \le \frac{L}{8-8L}. \tag{2.20}$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\left\| 8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \le 8^n \left(\varphi\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) + 2\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right) \tag{2.21}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. So

$$\left\| 8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \leq L^n (\varphi(2x, 2y) + 2\varphi(x, y)) \tag{2.22}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. So

$$\|DC(x, y)\| = 0 \tag{2.23}$$

for all $x, y \in X$. Thus the mapping $C : X \to Y$ is cubic, as desired. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.24}$$

for all $x, y \in X$. Then there is a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{2^p + 9}{2^p - 8}\theta\|x\|^p \tag{2.25}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \tag{2.26}$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. □

Theorem 2.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.27}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.3)*. Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{8 - 8L}(4\varphi(x, x) + \varphi(2x, x)) \tag{2.28}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8}g(2x) \tag{2.29}$$

for all $x \in X$.

It follows from (2.8) that

$$\left\| g(x) - \frac{1}{8}g(2x) \right\| \le \frac{1}{8}(4\varphi(x,x) + \varphi(2x,x)) \tag{2.30}$$

for all $x \in X$. So $d(g, Jg) \le 1/8$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (2.24)*. Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{9 + 2^p}{8 - 2^p}\theta\|x\|^p \tag{2.31}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p) \tag{2.32}$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

Theorem 2.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{2}\varphi(2x,2y) \tag{2.33}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.3)*. Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \le \frac{L}{2 - 2L}(4\varphi(x,x) + \varphi(2x,x)) \tag{2.34}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Letting $y := x/2$ and $h(x) := f(2x) - 8f(x)$ for all $x \in X$ in (2.7), we get

$$\left\| h(x) - 2h\left(\frac{x}{2}\right) \right\| \le 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{2.35}$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 2h\left(\frac{x}{2}\right) \tag{2.36}$$

for all $x \in X$.

It follows from (2.35) that

$$\left\|h(x) - 2h\left(\frac{x}{2}\right)\right\| \le 2L\varphi(x,x) + \frac{L}{2}\varphi(2x,x) \tag{2.37}$$

for all $x \in X$. So $d(h, Jh) \le L/2$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let $\theta \ge 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (2.24). *Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \le \frac{2^p + 9}{2^p - 2}\theta\|x\|^p \tag{2.38}$$

for all $x \in X$.

Theorem 2.7. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{2.39}$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.3). *Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \le \frac{1}{2-2L}\left(4\varphi(x,x) + \varphi(2x,x)\right) \tag{2.40}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{2}h(2x) \tag{2.41}$$

for all $x \in X$.

It follows from (2.35) that

$$\left\|h(x) - \frac{1}{2}h(2x)\right\| \le 2\varphi(x,x) + \frac{1}{2}\varphi(2x,x) \tag{2.42}$$

for all $x \in X$. So $d(h, Jh) \le 1/2$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.8. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying* (2.24). *Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{9 + 2^p}{2 - 2^p}\theta\|x\|^p \tag{2.43}$$

for all $x \in X$.

3. Generalized Hyers-Ulam Stability of the Functional Equation (1.5): An Even Case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in Banach spaces: an even case.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{16}\varphi(2x, 2y) \tag{3.1}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3). *Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{L}{16 - 16L}(4\varphi(x, x) + \varphi(2x, x)) \tag{3.2}$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.3), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \leq \varphi(y, y) \tag{3.3}$$

for all $y \in X$.

Replacing x by $2y$ in (2.3), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \leq \varphi(2y, y) \tag{3.4}$$

for all $y \in X$.

By (3.4) and (3.5),

$$\begin{aligned} &\|f(4x) - 20f(2x) + 64f(x)\| \\ &\quad \leq \|4(f(3x) - 6f(2x) + 15f(x))\| + \|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\| \\ &\quad \leq 4\varphi(x, x) + \varphi(2x, x) \end{aligned} \tag{3.5}$$

for all $x \in X$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{3.6}$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

It follows from (3.16) that

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq \frac{L}{16}(4\varphi(x, x) + \varphi(2x, x)) \tag{3.7}$$

for all $x \in X$. So $d(g, Jg) \leq L/16$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.2. *Let $\theta \geq 0$ and let p be a real number with $p > 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.24)*. Then there is unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{2^p + 9}{2^p - 16}\theta\|x\|^p \tag{3.8}$$

for all $x \in X$.

Theorem 3.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 16L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.9}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3)*. Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{1}{16 - 16L}(4\varphi(x, x) + \varphi(2x, x)) \tag{3.10}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{16}g(2x) \tag{3.11}$$

for all $x \in X$.

It follows from (3.16) that

$$\left\| g(x) - \frac{1}{16}g(2x) \right\| \le \frac{1}{16}\left(4\varphi(x,x) + \varphi(2x,x)\right) \tag{3.12}$$

for all $x \in X$. So $d(g, Jg) \le 1/16$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.4. *Let $\theta \ge 0$ and let p be a real number with $0 < p < 4$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.24). Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \le \frac{9 + 2^p}{16 - 2^p}\theta\|x\|^p \tag{3.13}$$

for all $x \in X$.

Theorem 3.5. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{4}\varphi(2x,2y) \tag{3.14}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3). Then there is a unique quadratic mapping $T : X \to Y$ such that

$$\|f(2x) - 16f(x) - T(x)\| \le \frac{L}{4 - 4L}\left(4\varphi(x,x) + \varphi(2x,x)\right) \tag{3.15}$$

for all $x \in X$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 2.1.

Letting $h(x) := f(2x) - 16f(x)$ for all $x \in X$ in (3.6), we get

$$\left\| h(x) - 4h\left(\frac{x}{2}\right) \right\| \le 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{3.16}$$

for all $x \in X$.

Now we consider the linear mapping $J : S \to S$ such that

$$Jh(x) := 4h\left(\frac{x}{2}\right) \tag{3.17}$$

for all $x \in X$.

It follows from (3.16)that

$$\left\|h(x) - 4h\left(\frac{x}{2}\right)\right\| \leq L\varphi(x,x) + \frac{L}{4}\varphi(2x,x) \tag{3.18}$$

for all $x \in X$. So $d(h, Jh) \leq L/4$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.6. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.24)*. Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{2^p + 9}{2^p - 4}\theta\|x\|^p \tag{3.19}$$

for all $x \in X$.

Theorem 3.7. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{3.20}$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.3)*. Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{1}{4 - 4L}(4\varphi(x,x) + \varphi(2x,x)) \tag{3.21}$$

for all $x \in X$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jh(x) := \frac{1}{4}h(2x) \tag{3.22}$$

for all $x \in X$.

It follows from (3.16) that

$$\left\|h(x) - \frac{1}{4}h(2x)\right\| \leq \varphi(x,x) + \frac{1}{4}\varphi(2x,x) \tag{3.23}$$

for all $x \in X$. So $d(h, Jh) \leq 1/4$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 3.8. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.24). Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{9+2^p}{4-2^p}\theta\|x\|^p \tag{3.24}$$

for all $x \in X$.

4. Generalized Hyers-Ulam Stability of the Functional Equation (1.5)

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.5) if and only if the odd mapping $f : X \to Y$ is an additive-cubic mapping, that is,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x). \tag{4.1}$$

It was shown in of [34, Lemma 2.2] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = (1/6)g(x) - (1/6)h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.5) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, that is,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + 2f(2y) - 8f(y). \tag{4.2}$$

It was shown in of [35, Lemma 2.1] that $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and that $f(x) = (1/12)g(x) - (1/12)h(x)$. Functional equations of mixed type have been investigated in [36, 37].

Let $f_o(x) := (f(x) - f(-x))/2$ and $f_e(x) := (f(x) + f(-x))/2$. Then f_o is odd and f_e is even. f_o and f_e satisfy the functional equation (1.5). Let $g_o(x) := f_o(2x) - 2f_o(x)$ and $h_o(x) := f_o(2x) - 8f_o(x)$. Then $f_o(x) = (1/6)g_o(x) - (1/6)h_o(x)$. Let $g_e(x) := f_e(2x) - 4f_e(x)$ and $h_e(x) := f_e(2x) - 16f_e(x)$. Then $f_e(x) = (1/12)g_e(x) - (1/12)h_e(x)$. Thus

$$f(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x) + \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x). \tag{4.3}$$

So we obtain the following results.

Theorem 4.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{16}\varphi(2x,2y) \tag{4.4}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3)*. Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\ &\quad \leq \left(\frac{L}{12 - 12L} + \frac{L}{48 - 48L} + \frac{L}{48 - 48L} + \frac{L}{192 - 192L} \right)(4\varphi(x, x) + \varphi(2x, x))\end{aligned} \tag{4.5}$$

for all $x \in X$.

Proof. Since $\varphi(x, y) \leq (L/16)\varphi(2x, 2y)$, $\varphi(x, y) \leq (L/8)\varphi(2x, 2y)$, $\varphi(x, y) \leq (L/4)\varphi(2x, 2y)$ and $\varphi(x, y) \leq (L/2)\varphi(2x, 2y)$. The result follows from Theorems 2.1, 2.5, 3.1, and 3.5. □

Corollary 4.2. *Let $\theta \geq 0$ and let p be a real number with $p > 4$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and* (2.24)*. Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\ &\quad \leq \left(\frac{2^p + 9}{6(2^p - 2)} + \frac{2^p + 9}{12(2^p - 4)} + \frac{2^p + 9}{6(2^p - 8)} + \frac{2^p + 9}{12(2^p - 16)} \right)\theta\|x\|^p\end{aligned} \tag{4.6}$$

for all $x \in X$.

Theorem 4.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \tag{4.7}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3)*. Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$, and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\ &\quad \leq \left(\frac{1}{12 - 12L} + \frac{1}{48 - 48L} + \frac{1}{48 - 48L} + \frac{1}{192 - 192L} \right)(4\varphi(x, x) + \varphi(2x, x))\end{aligned} \tag{4.8}$$

for all $x \in X$.

Proof. Since $\varphi(x, y) \leq 2L\varphi(x/2, y/2)$, $\varphi(x, y) \leq 4L\varphi(x/2, y/2)$, $\varphi(x, y) \leq 8L\varphi(x/2, y/2)$ and $\varphi(x, y) \leq 16L\varphi(x/2, y/2)$. The result follows from Theorems 2.3, 2.7, 3.3, and 3.7. □

Corollary 4.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.24). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$, and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned} &\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\ &\quad \leq \left(\frac{2^p + 9}{6(2 - 2^p)} + \frac{2^p + 9}{12(4 - 2^p)} + \frac{2^p + 9}{6(8 - 2^p)} + \frac{2^p + 9}{12(16 - 2^p)} \right) \theta \|x\|^p \end{aligned} \tag{4.9}$$

for all $x \in X$.

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