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Research Article

Generalized IFSs on Noncompact Spaces

Alexandru Mihail and Radu Miculescu

Faculty of Mathematics and Computer Science, University of Bucharest, Academiei Street 14, 010014 Bucharest, Romania

Correspondence should be addressed to Radu Miculescu, miculesc@yahoo.com

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The aim of this paper is to continue the research work that we have done in a previous paper published in this journal (see Mihail and Miculescu, 2008). We introduce the notion of GIFS, which is a family of functions $f_1, \ldots, f_n : X^m \to X$, where (X, d) is a complete metric space (in the above mentioned paper the case when (X, d) is a compact metric space was studied) and $m, n \in \mathbb{N}$. In case that the functions f_k are Lipschitz contractions, we prove the existence of the attractor of such a GIFS and explore its properties (among them we give an upper bound for the Hausdorff-Pompeiu distance between the attractors of two such GIFSs, an upper bound for the Hausdorff-Pompeiu distance between the attractor of such a GIFS, and an arbitrary compact set of X and we prove its continuous dependence in the f_k's). Finally we present some examples of attractors of GIFSs. The last example shows that the notion of GIFS is a natural generalization of the notion of IFS.

1. Introduction

1.1. The Organization of the Paper

The paper is organized as follows. Section 2 contains a short presentation of the notion of an iterated function system (IFS), one of the most common and most general ways to generate fractals. This will serve as a framework for our generalization of an iterated function system.

Then, we introduce the notion of a GIFS, which is a finite family of Lipschitz contractions $f_k : X^m \to X$, where (X, d) is a complete metric space and $m \in \mathbb{N}$.

In Section 3 we prove the existence of the attractor of such a GIFS and explore its properties (among them we give an upper bound for the Hausdorff-Pompeiu distance between the attractors of two such GIFSs, an upper bound for the Hausdorff-Pompeiu distance between the attractor of such a GIFS, and an arbitrary compact set of X and we prove its continuous dependence in the f_k's).

Section 4, the last one, contains some examples and remarks. The last example shows that the notion of GIFS is a natural generalization of the notion of IFS.

1.2. Some Generalizations of the Notion of IFS

IFSs were introduced in their present form by John Hutchinson and popularized by Barnsley (see [1]). There is a current effort to extend Hutchinson's classical framework for fractals to more general spaces and infinite IFSs. Some papers containing results on this direction are [2–7].

1.3. Some Physical Applications of IFSs

In the last period IFSs have attracted much attention being used by researchers who work on autoregressive time series, engineer sciences, physics, and so forth. For applications of IFSs in image processing theory, in the theory of stochastic growth models, and in the theory of random dynamical systems one can consult [8–10]. Concerning the physical applications of iterated function systems we should mention the seminal paper [11] of El Naschie which draws attention to an informal but instructive analogy between iterated function systems and the two-slit experiment which is quite valuable in illuminating the role played by the possibly DNA-like Cantorian nature of microspacetime and clarifies the way in which probability enters into this subject. We also mention the paper [12] of Słomczyński where a new definition of quantum entropy is introduced and one method (using the theory of iterated function systems) of calculating coherent states entropy is presented. The coherent states entropy is computed as the integral of the Boltzmann-Shannon entropy over a fractal set.

In [13], Bahar described bifurcation from a fixed-point generated by an iterated function system (IFS) as well as the generation of "chaotic" orbits by an IFS, and in [14] unusual and quite interesting patterns of bifurcation from a fixed-point in an IFS system, as well as the routes to chaos taken by IFS-generated orbits, are discussed. Moreover, in [15] it is shown that random selection of transformation in the IFS is essential for the generation of a chaotic attractor. In [16, Section 6.4], one can find a lengthy but elementary explanation which features of randomness play the main role.

2. Preliminaries

Notations. Let (X, d_X) and (Y, d_Y) be two metric spaces.

As usual, $C(X,Y)$ denotes the set of continuous functions from X to Y, and $\overline{d} : C(X,Y) \times C(X,Y) \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ defined by

$$\overline{d}(f,g) = \sup_{x\in X} d_Y(f(x), g(x)) \tag{2.1}$$

is the generalized metric on $C(X,Y)$.

For a sequence $(f_n)_n$ of elements of $C(X,Y)$ and $f \in C(X,Y)$, $f_n \xrightarrow{s} f$ denotes the punctual convergence, $f_n \xrightarrow{u.c} f$ denotes the uniform convergence on compact sets, and $f_n \xrightarrow{u} f$ denotes the uniform convergence, that is, the convergence in the generalized metric $\overline{d}$.

Definition 2.1. Let (X, d) be a complete metric space and $m \in \mathbb{N}$.

For a function $f : X^m = \times_{k=1}^m X \to X$, the number

$$\inf\{c : d(f(x_1, \dots, x_m), f(y_1, \dots, y_m)) \leq c \max\{d(x_1, y_1), \dots, d(x_m, y_m)\},\\ \forall x_1, \dots, x_m, y_1, \dots, y_m \in X\}, \tag{2.2}$$

which is the same with

$$\sup_{x_1,\dots,x_m,y_1,\dots,y_m \in X; \max\{d(x_1,y_1),\dots,d(x_m,y_m)\}>0} \frac{d(f(x_1, \dots, x_m), f(y_1, \dots, y_m))}{\max\{d(x_1, y_1), \dots, d(x_m, y_m)\}}, \tag{2.3}$$

is denoted by $\mathbf{Lip}(f)$ and it is called the Lipschitz constant of f.

A function $f : X^m \to X$ is called a Lipschitz function if $\mathbf{Lip}(f) < \infty$ and a Lipschitz contraction if $\mathbf{Lip}(f) < 1$.

We will use the notation $LCon_m(X)$ for the set $\{f : X^m \to X : \mathbf{Lip}(f) < 1\}$.

Notations. $P(X)$ denotes the subsets of a given set X and $P^*(X)$ denotes the set $P(X) - \{\emptyset\}$.

For a subset A of $P(X)$, by A^* we mean $A - \{\emptyset\}$.

Given a metric space (X, d), $K(X)$ denotes the set of compact subsets of X and $B(X)$ denotes the set of closed bounded subsets of X.

Remark 2.2. It is obvious that $K(X) \subseteq B(X) \subseteq P(X)$.

Definition 2.3. For a metric space (X, d), we consider on $P^*(X)$ the generalized Hausdorff-Pompeiu pseudometric $h : P^*(X) \times P^*(X) \to [0, +\infty]$ defined by $h(A, B) = \max(d(A, B), d(B, A)) = \inf\{r \in [0, \infty] : A \subseteq B(B, r) \text{ and } B \subseteq B(A, r)\}$, where $B(A, r) = \{x \in X : d(x, A) < r\}$ and $d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A}(\inf_{y \in B} d(x, y))$.

Remark 2.4. The Hausdorff-Pompeiu pseudometric is a metric on $B^*(X)$ and, in particular, on $K^*(X)$.

Remark 2.5. The metric spaces $(B^*(X), h)$ and $(K^*(X), h)$ are complete, provided that (X, d) is a complete metric space (see [1, 7, 17]).

The following proposition contains the important properties of the Hausdorff-Pompeiu semimetric (see [1, 17] or [18]).

Proposition 2.6. *Let (X, d_X) and (Y, d_Y) be two metric spaces. Then one has the following:*

(i) *if H and K are two nonempty subsets of X, then*

$$h(H, K) = h\left(\overline{H}, \overline{K}\right); \tag{2.4}$$

(ii) *if* $(H_i)_{i\in I}$ *and* $(K_i)_{i\in I}$ *are two families of nonempty subsets of* X*, then*

$$h\left(\underset{i\in I}{\cup}H_i, \underset{i\in I}{\cup}K_i\right) = h\left(\overline{\underset{i\in I}{\cup}H_i}, \overline{\underset{i\in I}{\cup}K_i}\right) \leq \sup_{i\in I} h(H_i, K_i); \tag{2.5}$$

(iii) *if* H *and* K *are two nonempty subsets of* X *and* $f : X \to X$ *is a Lipschitz function, then*

$$h(f(K), f(H)) \leq \mathbf{Lip}(f) h(K, H). \tag{2.6}$$

Definition 2.7. An iterated function system on X consists of a finite family of Lipschitz contractions $(f_k)_{k=\overline{1,n}}$ on X and is denoted $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

Theorem 2.8. *Let* (X, d) *be a complete metric space, let* $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ *be an IFS. Then there exists a unique* $A(\mathcal{S}) \in K^*(X)$ *such that*

$$F_{\mathcal{S}}(A(\mathcal{S})) \overset{\text{def}}{=} f_1(A(\mathcal{S})) \bigcup \cdots \bigcup f_n(A(\mathcal{S})) = A(\mathcal{S}). \tag{2.7}$$

The set $A(\mathcal{S})$ *is called the attractor of the IFS* $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$.

Given a metric space (X, d), the idea of our generalization of the notion of an IFS is to consider contractions from $X^m = \times_{k=1}^m X$ to X, rather than contractions from X to itself.

Definition 2.9. Let (X, d) be a complete metric space and $m \in \mathbb{N}$. A generalized iterated function system on X of order m (for short a GIFS or a GmIFS), denoted $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, consists of a finite family of functions $(f_k)_{k=\overline{1,n}}$, $f_k : X^m \to X$ such that $f_1, \ldots, f_n \in LCon_m(X)$.

Earlier several authors tried to coin the name generalized IFS. One should note the paper [19] in which notion tightly corresponds to contractive multivalued IFS from [2].

3. The Existence of the Attractor of a GIFS for Lipschitz Contractions

In this section m is a fixed natural number, (X, d) will be a fixed complete metric space, and all the GIFSs are of order m and have the form $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$, where n is a natural number.

We prove the existence of the attractor of $\mathcal{S}$ (Theorem 3.9) and study its properties (among them we give an upper bound for the Hausdorff-Pompeiu distance between the attractors of two such GIFSs (Theorem 3.12), an upper bound for the Hausdorff-Pompeiu distance between the attractor of such a GIFS, and an arbitrary compact set of X (Theorem 3.17) and we prove its continuous dependence in the f_k's (Theorem 3.15)).

Definition 3.1. Let $f : X^m \to X$ be a function. The function $F_f : K^*(X)^m \to K^*(X)$ defined by

$$F_f(K_1, K_2, \ldots, K_m) = f(K_1, K_2, \ldots, K_m) = \{f(x_1, x_2, \ldots, x_m) : x_j \in K_j \quad \forall j \in \{1, \ldots, m\}\}, \tag{3.1}$$

for all $K_1, K_2, \ldots, K_m \in K^*(X)$, is called the set function associated to the function f.

The function $F_{\mathcal{S}} : K^*(X)^m \to K^*(X)$ defined by

$$F_{\mathcal{S}}(K_1, K_2, \ldots, K_m) = \bigcup_{k=1}^{n} F_{f_k}(K_1, K_2, \ldots, K_m), \tag{3.2}$$

for all $K_1, K_2, \ldots, K_m \in K^*(X)$, is called the set function associated to the GIFS $\mathcal{S}$.

Lemma 3.2. *For a sequence $(f_n)_n$ of elements of $C(X^m, X)$ and $f \in C(X^m, X)$ such that $f_n \overset{u}{\to} f$, one has*

$$f_n(K_1, K_2, \ldots, K_m) \longrightarrow f(K_1, K_2, \ldots, K_m), \tag{3.3}$$

in $(\mathcal{K}^*(X), h)$, for all $K_1, K_2, \ldots, K_m \in K^*(X)$.

Proposition 3.3. *Let (X, d_X) and (Y, d_Y) be two complete metric spaces and let $f_n, f \in C(X, Y)$ be such that $\sup_{n\geq 1}\mathbf{Lip}(f_n) < +\infty$ and $f_n \overset{s}{\to} f$ on a dense set in X. Then*

$$\begin{gathered} \mathbf{Lip}(f) \leq \sup_{n\geq 1}\mathbf{Lip}(f_n), \\ f_n \overset{u.c}{\to} f. \end{gathered} \tag{3.4}$$

Proof. In this proof, by M we mean $\sup_{n\geq 1}\mathbf{Lip}(f_n)$.

Let us consider $A = \{x \in X \mid f_n(x) \to f(x)\}$, which is a dense set in X, let K be a compact set in X, and let $\varepsilon > 0$.

Since f is uniformly continuous on K, there exists $\delta \in (0, \varepsilon/3(M+1))$ such that if $x, y \in K$ and $d_X(x, y) < \delta$, then

$$d_Y(f(x), f(y)) < \frac{\varepsilon}{3}. \tag{3.5}$$

Since K is compact, there exist $x_1, x_2, \ldots, x_p \in K$ such that

$$K \subseteq \bigcup_{i=1}^{p} B\left(x_i, \frac{\delta}{2}\right). \tag{3.6}$$

Taking into account the fact that A is dense in X, we can choose $y_1, y_2, \ldots, y_p \in A$ such that

$$y_1 \in B\left(x_1, \frac{\delta}{2}\right), \ldots, y_p \in B\left(x_p, \frac{\delta}{2}\right). \tag{3.7}$$

Since, for all $i \in \{1, \ldots, p\}$, $\lim_{n\to\infty} f_n(y_i) = f(y_i)$, there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, we have

$$d_Y(f_m(y_i), f(y_i)) < \frac{\varepsilon}{3}, \tag{3.8}$$

for every $i \in \{1, \ldots, p\}$.

For $x \in K$, there exists $i \in \{1, \dots, p\}$, such that $x \in B(x_i, \delta/2)$ and therefore

$$d_X(x, y_i) \leq d_X(x, x_i) + d_X(x_i, y_i) < \frac{\delta}{2} + \frac{\delta}{2} < \delta < \frac{\varepsilon}{3(M+1)}, \tag{3.9}$$

and so

$$d_Y(f(y_i), f(x)) < \frac{\varepsilon}{3}. \tag{3.10}$$

Hence, for $n \geq n_\varepsilon$, we have

$$\begin{aligned} d_Y(f_n(x), f(x)) &\leq d_Y(f_n(x), f_n(y_i)) + d_Y(f_n(y_i), f(y_i)) + d_Y(f(y_i), f(x)) \\ &\leq M d_X(x, y_i) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq M \frac{\varepsilon}{3(M+1)} + \frac{2\varepsilon}{3} < \varepsilon. \end{aligned} \tag{3.11}$$

Consequently, as x was arbitrary chosen in K, we infer that $f_n \stackrel{u}{\to} f$ on K, and so

$$f_n \stackrel{u.c}{\to} f. \tag{3.12}$$

The inequality

$$\mathbf{Lip}(f) \leq \sup_{n \geq 1} \mathbf{Lip}(f_n) \tag{3.13}$$

is obvious. □

Lemma 3.4. *Let $A_1, A_2, \dots, A_m$ be subsets of $\mathbb{R}$.*
Then

(1) $\inf_{a_1 \in A_1, \dots, a_m \in A_m} \max\{a_1, \dots, a_m\} = \max\{\inf A_1, \dots, \inf A_m\}$;

(2) $\sup_{a_1 \in A_1, \dots, a_m \in A_m} \max\{a_1, \dots, a_m\} = \max\{\sup A_1, \dots, \sup A_m\}$.

Lemma 3.5. *If $f : X^m \to X$ is a Lipschitz function, then*

$$\mathbf{Lip}(F_f) = \mathbf{Lip}(f). \tag{3.14}$$

Lemma 3.6. *In the framework of this section, one has*

$$\mathbf{Lip}(F_{\mathcal{S}}) \leq \max\{\mathbf{Lip}(f_1), \dots, \mathbf{Lip}(f_n)\}. \tag{3.15}$$

The proofs of the above lemmas are almost obvious.

Theorem 3.7 (Banach contraction principle for $LCon_m(X)$). *For every $f \in LCon_m(X)$, there exists a unique $\alpha \in X$, such that*

$$f(\alpha, \alpha, \dots, \alpha) = \alpha. \tag{3.16}$$

For every $x_0, x_1, \dots, x_{m-1} \in X$, the sequence $(x_n)_{n\geq 1}$, defined by $x_{k+m} = f(x_{k+m-1}, x_{k+m-2}, \dots, x_k)$, for all $k \in \mathbb{N}$, has the property that

$$\lim_{n\to\infty} x_n = \alpha. \tag{3.17}$$

Concerning the speed of the convergence, one has the following estimation:

$$d(x_n, \alpha) \leq \frac{m(\mathbf{Lip}(f))^{[n/m]} \max\{d(x_0, x_1), d(x_1, x_2), \dots, d(x_{n-1}, x_n)\}}{1 - \mathbf{Lip}(f)} \tag{3.18}$$

for every $n \in \mathbb{N}$.

Proof. See [20, Remark 5.1]. □

Remark 3.8. The point α from the above theorem is called the fixed point of f.

From Theorem 3.7 and Lemma 3.6 we have the following.

Theorem 3.9. *In the framework of this section, there exists a unique $A(\mathcal{S}) \in K^*(X)$ such that*

$$F_{\mathcal{S}}(A(\mathcal{S}), A(\mathcal{S}), \dots, A(\mathcal{S})) = A(\mathcal{S}). \tag{3.19}$$

Moreover, for any $H_0, H_1, \dots, H_{m-1} \in K^*(X)$, the sequence $(H_n)_{n\geq 1}$ defined by $H_{k+m} = F_{\mathcal{S}}(H_{k+m-1}, H_{k+m-2}, \dots, H_k)$, for all $k \in \mathbb{N}$, has the property that

$$\lim_{n\to\infty} H_n = A(\mathcal{S}). \tag{3.20}$$

Concerning the speed of the convergence, one has the following estimation:

$$h(H_n, A(\mathcal{S})) \leq \frac{m(\max\{\mathbf{Lip}(f_1), \dots, \mathbf{Lip}(f_n)\})^{[n/m]} \max\{h(H_0, H_1), \dots, h(H_{m-1}, H_m)\}}{1 - \max\{\mathbf{Lip}(f_1), \dots, \mathbf{Lip}(f_n)\}} \tag{3.21}$$

for all $n \in \mathbb{N}$.

Definition 3.10. The unique set $A(\mathcal{S})$ given by the previous theorem is called the attractor of the GIFS $\mathcal{S}$.

Theorem 3.11. *If $f, g \in LCon_m(\mathrm{X})$ have the fixed points α and β, then*

$$\begin{aligned} d(\alpha,\beta) &\le \min\left\{\frac{1}{1-\mathbf{Lip}(f)}d(f(\beta,\beta,\dots,\beta),\beta), \frac{1}{1-\mathbf{Lip}(g)}d(\alpha,g(\alpha,\alpha,\dots,\alpha))\right\} \\ &\le \frac{1}{1-\min\{\mathbf{Lip}(f),\mathbf{Lip}(g)\}}\overline{d}(f,g). \end{aligned} \tag{3.22}$$

Proof. We have

$$\begin{aligned} d(\alpha,\beta) &= d(f(\alpha,\dots,\alpha), g(\beta,\dots,\beta)) \\ &\le d(f(\alpha,\dots,\alpha), f(\beta,\dots,\beta)) + d(f(\beta,\dots,\beta), g(\beta,\dots,\beta)) \\ &= d(f(\alpha,\dots,\alpha), f(\beta,\dots,\beta)) + d(f(\beta,\dots,\beta),\beta) \\ &\le \mathbf{Lip}(f)d(\alpha,\beta) + d(f(\beta,\dots,\beta),\beta), \end{aligned} \tag{3.23}$$

so

$$d(\alpha,\beta) \le \frac{1}{1-\mathbf{Lip}(f)}d(f(\beta,\beta,\dots,\beta),\beta), \tag{3.24}$$

and in a similar manner we get

$$d(\alpha,\beta) \le \frac{1}{1-\mathbf{Lip}(g)}d(\alpha,g(\alpha,\alpha,\dots,\alpha)). \tag{3.25}$$

Therefore

$$\begin{aligned} d(\alpha,\beta) &\le \min\left\{\frac{1}{1-\mathbf{Lip}(f)}d(f(\beta,\beta,\dots,\beta),\beta), \frac{1}{1-\mathbf{Lip}(g)}d(\alpha,g(\alpha,\alpha,\dots,\alpha))\right\} \\ &= \min\left\{\frac{1}{1-\mathbf{Lip}(f)}d(f(\beta,\dots,\beta),g(\beta,\dots,\beta)), \frac{1}{1-\mathbf{Lip}(g)}d(f(\alpha,\dots,\alpha),g(\alpha,\dots,\alpha))\right\} \\ &\le \min\left\{\frac{1}{1-\mathbf{Lip}(f)}\overline{d}(f,g), \frac{1}{1-\mathbf{Lip}(g)}\overline{d}(f,g)\right\} = \frac{1}{1-\min\{\mathbf{Lip}(f),\mathbf{Lip}(g)\}}\overline{d}(f,g). \end{aligned} \tag{3.26}$$

□

From Theorem 3.11 and Lemma 3.6, we have the following.

Theorem 3.12. *In the framework of this section, if* $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ *and* $\mathcal{S}' = (X, (g_k)_{k=\overline{1,n}})$ *are two* m *dimensional GIFSs, then*

$$h(A(\mathcal{S}), A(\mathcal{S}')) \le \frac{1}{1-\mu} \max\left\{\overline{d}(f_1, g_1), \dots, \overline{d}(f_n, g_n)\right\}, \tag{3.27}$$

where $\mu = \min(\max\{\mathbf{Lip}(f_1), \dots, \mathbf{Lip}(f_n)\}, \max\{\mathbf{Lip}(g_1), \dots, \mathbf{Lip}(g_n)\})$.

Theorem 3.13. *Let* $f_n, f \in LCon_m(X)$ *with fixed points* α_n *and* α*, respectively, such that*

$$\begin{gathered} \sup_{n\ge1} \mathbf{Lip}(f_n) < 1, \\ f_n \xrightarrow{s} f \end{gathered} \tag{3.28}$$

on a dense set in X^m.

Then

$$\alpha_n \longrightarrow \alpha. \tag{3.29}$$

Proof. From the fact that $\sup_{n\ge1}\mathbf{Lip}(f_n) < 1$ and $f_n \xrightarrow{s} f$ on a dense set in X^m, it follows, using Proposition 3.3, that

$$f_n \xrightarrow{u.c.} f \tag{3.30}$$

on X^m and

$$\mathbf{Lip}(f) \le \sup_{n\ge1} \mathbf{Lip}(f_n). \tag{3.31}$$

From Theorem 3.11, we have

$$d(\alpha, \alpha_n) \le \frac{1}{1 - \mathbf{Lip}(f_n)} d(\alpha, f_n(\alpha, \alpha, \dots, \alpha)), \tag{3.32}$$

and hence

$$d(\alpha, \alpha_n) \le \frac{1}{1 - \sup_{n\ge1}\mathbf{Lip}(f_n)} d(\alpha, f_n(\alpha, \alpha, \dots, \alpha)) \tag{3.33}$$

for all $n \in \mathbb{N}$.

Since $f_n \xrightarrow{u.c} f$ on X^m, we obtain that

$$\lim_{n\to\infty} f_n(\alpha, \alpha, \dots, \alpha) = f(\alpha, \alpha, \dots, \alpha), \tag{3.34}$$

and consequently, using the above inequality, we obtain that

$$\lim_{n\to\infty} \alpha_n = \alpha. \tag{3.35}$$

□

Proposition 3.14. *Let $\mathcal{S}_j = (X, (f_k^j)_{k=\overline{1,n}})$, where $j \in \mathbb{N}^*$, and let $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be m-dimensional generalized iterated function systems such that*

$$\begin{gathered} \sup_{j\geq 1} \max\left\{\mathbf{Lip}\left(f_1^j\right), \ldots, \mathbf{Lip}\left(f_n^j\right)\right\} < 1, \\ f_k^j \xrightarrow{s} f_k \end{gathered} \tag{3.36}$$

on a dense set in X^m, for every $k \in \{1, \ldots, n\}$.

Then

$$F_{\mathcal{S}_j} \xrightarrow{u.c} F_{\mathcal{S}}. \tag{3.37}$$

Proof. Using Proposition 3.3, we obtain that

$$f_k^j \xrightarrow{u.c} f_k \tag{3.38}$$

on X^m and

$$\max\{\mathbf{Lip}(f_1), \ldots, \mathbf{Lip}(f_n)\} \leq \sup_{j\geq 1} \max\left\{\mathbf{Lip}\left(f_1^j\right), \ldots, \mathbf{Lip}\left(f_n^j\right)\right\}. \tag{3.39}$$

Then, using Lemma 3.2 and Proposition 2.6(ii), we get

$$F_{\mathcal{S}_j} \xrightarrow{s} F_{\mathcal{S}}. \tag{3.40}$$

Since, according to Lemma 3.6, we have

$$\mathbf{Lip}\left(F_{\mathcal{S}_j}\right) \leq \max\left\{\mathbf{Lip}\left(f_1^j\right), \ldots, \mathbf{Lip}\left(f_n^j\right)\right\} \leq \sup_{j\geq 1} \max\left\{\mathbf{Lip}\left(f_1^j\right), \ldots, \mathbf{Lip}\left(f_n^j\right)\right\} < 1 \tag{3.41}$$

for all $j \in \mathbb{N}$, we obtain, using again the arguments from Proposition 3.3, that

$$F_{\mathcal{S}_j} \xrightarrow{u.c} F_{\mathcal{S}}. \tag{3.42}$$

□

From Theorem 3.13, Proposition 3.14, and Lemma 3.6, we have the following.

Theorem 3.15. *Let $\mathcal{S}_j = (X, (f_k^j)_{k=\overline{1,n}})$, where $j \in \mathbb{N}^*$, and let $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ be m-dimensional generalized iterated function systems having the property that*

$$\sup_{j\geq 1} \max\left\{\mathbf{Lip}\left(f_1^j\right), \ldots, \mathbf{Lip}\left(f_n^j\right)\right\} < 1, \qquad f_k^j \xrightarrow{s} f_k \tag{3.43}$$

on a dense set in X^m, for every $k \in \{1, \ldots, n\}$.

Then

$$A(\mathcal{S}_j) \longrightarrow A(\mathcal{S}). \tag{3.44}$$

Theorem 3.16. *For $f \in LCon_m(X)$ having the unique fixed point α and for every $x \in X$, one has*

$$d_X(x, \alpha) \leq \frac{d(f(x, x, \ldots, x), x)}{1 - \mathbf{Lip}(f)}. \tag{3.45}$$

Proof. We can use the Banach contraction principle for $g \in LCon_1(X)$, where

$$g(x) = f(x, x, \ldots, x) \tag{3.46}$$

for all $x \in X$. □

Theorem 3.17. *For a generalized iterated function system $\mathcal{S} = (X, (f_k)_{k=\overline{1,n}})$ and $H \in K^*(X)$, the following inequality is valid:*

$$h(A(\mathcal{S}), H) \leq \frac{h(f(H, H, \ldots, H), H)}{1 - \max\{\mathbf{Lip}(f_1), \ldots, \mathbf{Lip}(f_n)\}}. \tag{3.47}$$

Proof. The function $G_{\mathcal{S}} : K^*(X) \to K^*(X)$, defined by

$$G_{\mathcal{S}}(K) = F_{\mathcal{S}}(K, K, \ldots, K) = \bigcup_{k=1}^{n} f_k(K, K, \ldots, K), \tag{3.48}$$

for all $K \in K^*(X)$, is a contraction and

$$\mathbf{Lip}(G_{\mathcal{S}}) \leq \mathbf{Lip}(F_{\mathcal{S}}) \leq \max\{\mathbf{Lip}(f_1), \ldots, \mathbf{Lip}(f_n)\}. \tag{3.49}$$

□

4. Examples

In this section we present some examples of attractors of GIFSs. Example 4.3 shows that the notion of GIFS is a natural generalization of the notion of IFS.

Example 4.1. Let $A_1, A_2, \dots, A_m \in B(\mathbb{X})$ and $\alpha \in \mathbb{X}$, where $\mathbb{X}$ is a Banach space and $B(\mathbb{X})$ is the set of linear and continuous operators from $\mathbb{X}$ to $\mathbb{X}$.

Let us consider the function $f : \mathbb{X}^m \to \mathbb{X}$, given by

$$f(x_1, x_2, \dots, x_m) = A_1 x_1 + A_2 x_2 + \cdots + A_m x_m + \alpha, \tag{4.1}$$

for every $x_1, x_2, \dots, x_m \in \mathbb{X}$.

Then

$$\begin{aligned} &\| f(x_1, x_2, \dots, x_m) - f(y_1, y_2, \dots, y_m) \| \\ &\quad = \| A_1(x_1 - y_1) + A_2(x_2 - y_2) + \cdots + A_m(x_m - y_m) \| \\ &\quad \le \sum_{k=1}^{m} \|A_k\| \|x_k - y_k\| \le \left(\sum_{k=1}^{m} \|A_k\| \right) \max\{ \|x_1 - y_1\|, \dots, \|x_m - y_m\| \}, \end{aligned} \tag{4.2}$$

for every $x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m \in \mathbb{X}$, and so

$$\mathbf{Lip}(f) \le \sum_{k=1}^{m} \|A_k\|. \tag{4.3}$$

In particular, if $\mathbb{X} = \mathbb{R}$ and $A_k = a_k I_{\mathbb{R}}$, for every $k \in \{1, \dots, m\}$, then

$$\mathbf{Lip}(f) \le \sum_{k=1}^{m} |a_k|. \tag{4.4}$$

Let us consider $f_0^m, f_1^m : \mathbb{R}^m \to \mathbb{R}$ given by

$$\begin{aligned} f_0(x_1, x_2, \dots, x_m) &= \sum_{k=1}^{m} \frac{1}{8} x_k, \\ f_1(x_1, x_2, \dots, x_m) &= \frac{8 - m}{8} + \sum_{k=1}^{m} \frac{1}{8} x_k \end{aligned} \tag{4.5}$$

for every $x_1, x_2, \dots, x_m \in \mathbb{R}$.

Then

$$\mathbf{Lip}(f_0) = \mathbf{Lip}(f_1) \le \frac{m}{8}. \tag{4.6}$$

If $m < 8$, then f_0, f_1 are contractions.

We consider the GIFS $\mathcal{S}^m = (\mathbb{R}, (f_0^m, f_1^m))$, where $m \in \{1, 2, \dots, 7\}$.

If $m \ge 4$, then

$$A(\mathcal{S}^m) = [0, 1]. \tag{4.7}$$

Indeed, $f_0^m([0,1],[0,1],\dots,[0,1]) = [0,m/8]$, $f_1^m([0,1],[0,1],\dots,[0,1]) = [1-m/8,1]$ and so $[0,1] = f_0^m([0,1],[0,1],\dots,[0,1]) \cup f_1^m([0,1],[0,1],\dots,[0,1])$, that is, $[0,1] = F_{\mathcal{S}^m}([0,1],\dots,[0,1])$. This proves that $A(\mathcal{S}^m) = [0,1]$.

If $m = 3$, *then*

$$A\left(\mathcal{S}^3\right) = \left[0,\frac{3}{8}\right] \cup \left[\frac{5}{8},1\right]. \tag{4.8}$$

Indeed, if $A = [0,3/8] \cup [5/8,1]$, then

$$\begin{aligned} f_0^3(A,A,A) &= f_0^3\left(\left[0,\frac{3}{8}\right],\left[0,\frac{3}{8}\right],\left[0,\frac{3}{8}\right]\right) \cup f_0^3\left(\left[0,\frac{3}{8}\right],\left[0,\frac{3}{8}\right],\left[\frac{5}{8},1\right]\right) \\ &\quad \cup f_0^3\left(\left[0,\frac{3}{8}\right],\left[\frac{5}{8},1\right],\left[\frac{5}{8},1\right]\right) \cup f_0^3\left(\left[\frac{5}{8},1\right],\left[\frac{5}{8},1\right],\left[\frac{5}{8},1\right]\right) \\ &= \left[0,\frac{9}{64}\right] \cup \left[\frac{5}{64},\frac{14}{64}\right] \cup \left[\frac{10}{64},\frac{21}{64}\right] \cup \left[\frac{15}{64},\frac{3}{8}\right] = \left[0,\frac{3}{8}\right] \end{aligned} \tag{4.9}$$

and $f_1^3(A,A,A) = [5/8,1]$. Hence $A = f_0^3(A,A,A) \cup f_1^3(A,A,A)$. This proves that $A(\mathcal{S}^3) = A = [0,3/8] \cup [5/8,1]$.

If $m = 2$, *then*

$$A\left(\mathcal{S}^2\right) = \left[0,\frac{2}{32}\right] \cup \left[\frac{3}{32},\frac{5}{32}\right] \cup \left[\frac{6}{32},\frac{8}{32}\right] \cup \left[\frac{24}{32},\frac{26}{32}\right] \cup \left[\frac{27}{32},\frac{29}{32}\right] \cup \left[\frac{30}{32},1\right]. \tag{4.10}$$

If $m = 1$, *then* $A(\mathcal{S}^1)$ *is a Cantor type set (more precisely* $A(\mathcal{S}^1)$ *consists of those elements of* $[0,1]$ *for which one can use the digits* 0 *and* 7 *in order to write them in base* 8*).*

Remark 4.2. Finally let us note that

$$A\left(\mathcal{S}^1\right) \subseteq A\left(\mathcal{S}^2\right) \subseteq A\left(\mathcal{S}^3\right) \subseteq A\left(\mathcal{S}^4\right) = A\left(\mathcal{S}^5\right) = A\left(\mathcal{S}^6\right) = A\left(\mathcal{S}^7\right). \tag{4.11}$$

Example 4.3. Let $\mathbb{X}$ be one of the spaces l_p, l_∞, or c_0, where $p \geq 1$.

Let $j : \mathbb{X} \to \mathbb{X}$, $i_m : \mathbb{R}^m \to \mathbb{X}$ and $\pi_1 : \mathbb{X} \to \mathbb{R}$ be given by

$$\begin{gathered} j\left((x_n)_{n\geq 1}\right) = (0, x_1, x_2, \dots, x_m, \dots), \\ i_m\left((x_n)_{n\geq 1}\right) = (x_1, x_2, \dots, x_m, 0, 0, 0, \dots), \\ \pi_1\left((x_n)_{n\geq 1}\right) = x_1 \end{gathered} \tag{4.12}$$

for all $(x_n)_{n\geq 1} \in \mathbb{X}$.

We consider the GIFS $\mathcal{S} = (\mathbb{X}, (f_0, f_1))$, *where* $f_0 : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ *and* $f_1 : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ *are given by*

$$f_0(x, y) = i_1\left(\frac{\pi_1(x)}{2}\right) + \frac{j(y)}{2},$$
$$f_1(x, y) = i_1\left(\frac{\pi_1(x)}{2} + \frac{1}{2}\right) + \frac{j(y)}{2} \tag{4.13}$$

for all $x, y \in \mathbb{X}$.

Then

$$A(\mathcal{S}) = \mathop{\times}_{k=0}^{\infty} \left[0, \frac{1}{2^k}\right]. \tag{4.14}$$

Indeed, if $A = \times_{k=0}^{\infty}[0, 1/2^k]$, then $j(A) = \{0\} \times (\times_{k=0}^{\infty}[0, 1/2^k])$ and $\pi_1(A) = [0, 1]$. Hence

$$f_0(A, A) = i_1^m\left(\frac{\pi_1(A)}{2}\right) + \frac{j(A)}{2} = \left[0, \frac{1}{2}\right] \times \{(0, 0, 0, \ldots)\} + \{0\} \times \left(\mathop{\times}_{k=0}^{\infty} \left[0, \frac{1}{2^{k+1}}\right]\right)$$
$$= \left[0, \frac{1}{2}\right] \times \left(\mathop{\times}_{k=0}^{\infty} \left[0, \frac{1}{2^{k+1}}\right]\right),$$
$$f_0(A, A) = i_1^m\left(\frac{\pi_1(A)}{2} + \frac{1}{2}\right) + \frac{j(A)}{2} = \left[\frac{1}{2}, 1\right] \times \{(0, 0, 0, \ldots)\} + \{0\} \times \left(\mathop{\times}_{k=0}^{\infty} \left[0, \frac{1}{2^{k+1}}\right]\right)$$
$$= \left[\frac{1}{2}, 1\right] \times \left(\mathop{\times}_{k=0}^{\infty} \left[0, \frac{1}{2^{k+1}}\right]\right), \tag{4.15}$$

and therefore $A = f_0(A, A) \cup f_1(A, A)$. This, together with the fact that A is compact, proves that $A(\mathcal{S}) = A = \times_{k=0}^{\infty}[0, 1/2^k]$.

On one hand it is obvious that $A(\mathcal{S})$ has infinite Hausdorff dimension. On the other hand, for every finite IFS $\mathcal{S}$, with contraction constant less then 1, we have $\dim_H(A(\mathcal{S})) < \infty$. Indeed, the proof of the above claim is similar with the one of Proposition 9.6, page 135, from [18].

Therefore there exists no finite IFS consisting of Lipschitz contractions having as attractor the set $A(\mathcal{S}) = \times_{k=0}^{\infty}[0, 1/2^k]$.

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References

[1] M. F. Barnsley, *Fractals Everywhere*, Academic Press, Boston, Mass, USA, 2nd edition, 1993.

[2] J. Andres, J. Fišer, G. Gabor, and K. Leśniak, "Multivalued fractals," *Chaos, Solitons and Fractals*, vol. 24, no. 3, pp. 665–700, 2005.

[3] G. Gwóźdź-Łukawska and J. Jachymski, "The Hutchinson-Barnsley theory for infinite iterated function systems," *Bulletin of the Australian Mathematical Society*, vol. 72, no. 3, pp. 441–454, 2005.
[4] A. Käenmäki, "On natural invariant measures on generalised iterated function systems," *Annales Academiæ Scientiarium Fennicæ. Mathematica*, vol. 29, no. 2, pp. 419–458, 2004.
[5] K. Leśniak, "Infinite iterated function systems: a multivalued approach," *Bulletin of the Polish Academy of Sciences. Mathematics*, vol. 52, no. 1, pp. 1–8, 2004.
[6] A. Łoziński, K. Życzkowski, and W. Słomczyński, "Quantum iterated function systems," *Physical Review E*, vol. 68, no. 4, Article ID 046110, 9 pages, 2003.
[7] R. Miculescu and A. Mihail, "Lipscomb's space ω^A is the attractor of an infinite IFS containing affine transformations of $l^2(A)$," *Proceedings of the American Mathematical Society*, vol. 136, no. 2, pp. 587–592, 2008.
[8] J. H. Elton and M. Piccioni, "Iterated function systems arising from recursive estimation problems," *Probability Theory and Related Fields*, vol. 91, no. 1, pp. 103–114, 1992.
[9] B. Forte and E. R. Vrscay, "Solving the inverse problem for function/image approximation using iterated function systems. I. Theoretical basis," *Fractals*, vol. 2, no. 3, pp. 325–334, 1994.
[10] L. Montrucchio and F. Privileggi, "Fractal steady states in stochastic optimal control models," *Annals of Operations Research*, vol. 88, pp. 183–197, 1999.
[11] M. S. El Naschie, "Iterated function systems and the two-slit experiment of quantum mechanics," *Chaos, Solitons and Fractals*, vol. 4, no. 10, pp. 1965–1968, 1994.
[12] W. Słomczyński, "From quantum entropy to iterated function systems," *Chaos, Solitons and Fractals*, vol. 8, no. 11, pp. 1861–1864, 1997.
[13] S. Bahar, "Chaotic orbits and bifurcation from a fixed point generated by an iterated function system," *Chaos, Solitons and Fractals*, vol. 5, no. 6, pp. 1001–1006, 1995.
[14] S. Bahar, "Further studies of bifurcations and chaotic orbits generated by iterated function systems," *Chaos, Solitons and Fractals*, vol. 7, no. 1, pp. 41–47, 1996.
[15] S. Bahar, "Chaotic attractors generated by iterated function systems: "harmonic decompositions" and the onset of chaos," *Chaos, Solitons and Fractals*, vol. 8, no. 3, pp. 303–312, 1997.
[16] H.-O. Peitgen, H. Jürgens, and D. Saupe, *Chaos and Fractals: New Frontiers of Science*, Springer, New York, NY, USA, 2nd edition, 2004.
[17] K. J. Falconer, *The Geometry of Fractal Sets*, vol. 85 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 1986.
[18] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, Chichester, UK, 1990.
[19] J. J. P. Veerman and L. B. Jonker, "Rigidity propertiesof locally scaling fractals," http://arxiv.org/abs/math.DS/9701216.
[20] M.-A. Şerban, "Fixed point theorems for operators on Cartesian product spaces and applications," *Seminar on Fixed Point Theory Cluj-Napoca*, vol. 3, pp. 163–172, 2002.