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Research Article

Hybrid Steepest-Descent Methods for Solving Variational Inequalities Governed by Boundedly Lipschitzian and Strongly Monotone Operators

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Let H be a real Hilbert space and let $F : H \to H$ be a boundedly Lipschitzian and strongly monotone operator. We design three hybrid steepest descent algorithms for solving variational inequality $\mathrm{VI}(C, F)$ of finding a point $x^* \in C$ such that $\langle Fx^*, x - x^*\rangle \geq 0$, for all $x \in C$, where C is the set of fixed points of a strict pseudocontraction, or the set of common fixed points of finite strict pseudocontractions. Strong convergence of the algorithms is proved.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot\rangle$ and the norm $\|\cdot\|$, let C be a nonempty closed convex subset of H, and let $F : C \to H$ be a nonlinear operator. We consider the problem of finding a point $x^* \in C$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

This is known as the variational inequality problem (i.e., $\mathrm{VI}(C, F)$), initially introduced and studied by Stampacchia [1] in 1964. In the recent years, variational inequality problems have been extended to study a large variety of problems arising in structural analysis, economics, optimization, operations research, and engineering sciences; see [1–6] and the references therein.

Yamada [7] proposed hybrid methods to solve $\mathrm{VI}(C, F)$, where C is composed of fixed points of a nonexpansive mapping; that is, C is of the form

$$C \equiv \mathrm{Fix}(T) := \{x \in H : Tx = x\}, \tag{1.2}$$

where $T : H \to H$ is a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$), $F : H \to H$ is Lipschitzian and strongly monotone.

He and Xu [8] proved that $\mathrm{VI}(C, F)$ has a unique solution and iterative algorithms can be devised to approximate this solution if F is a boundedly Lipschitzian and strongly monotone operator and C is a closed convex subset of H. In the case where C is the set of fixed points of a nonexpansive mapping, they invented a hybrid iterative algorithm to approximate the unique solution of $\mathrm{VI}(C, F)$ and this extended the Yamada's results.

The main purpose of this paper is to continue our research in [8]. We assume that F is a boundedly Lipschitzian and strongly monotone operator as in [8], but C is the set of fixed points of a strict pseudo-contraction $T : H \to H$, or the set of common fixed points of finite strict pseudo-contractions $T_i : H \to H$ $(i = 1, \ldots, N)$. For the two cases of C, we will design the hybrid iterative algorithms for solving $\mathrm{VI}(C, F)$ and prove their strong convergence, respectively. Relative definitions are stated as below.

Let C be a nonempty closed and convex subset of a real Hilbert space H, $F : C \to H$ and $T : C \to C$, then

(1) F is called Lipschitzian on C, if there there exists a positive constant L such that

$$\|Fx - Fy\| \le L\|x - y\|, \quad \forall x, y \in C; \tag{1.3}$$

(2) F is called boundedly Lipschitzian on C, if for each nonempty bounded subset B of C, there exists a positive constant κ_B depending only on the set B such that

$$\|Fx - Fy\| \le \kappa_B\|x - y\|, \quad \forall x, y \in B; \tag{1.4}$$

(3) F is said to be η-strongly monotone on C, if there exists a positive constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in C; \tag{1.5}$$

(4) T is said to be a κ-strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.6}$$

Obviously, the nonexpansive mapping class is a proper subclass of the strict pseudo-contraction class and the Lipschitzian operator class is a proper subclass of the boundedly Lipschitzian operator class, respectively.

We will use the following notations:

(i) $\rightharpoonup$ for weak convergence and $\to$ for strong convergence,

(ii) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$,

(iii) $S(u,r) = \{x : x \in H, \|x-u\| \le r\}$ denotes a closed ball with center u and radius r.

2. Preliminaries

We need some facts and tools which are listed as lemmas below.

Lemma 2.1. *Let H be a real Hilbert space. The following expressions hold:*

(i) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2$, *for all* $x, y \in H$, *for all* $t \in [0,1]$.

(ii) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$, *for all* $x, y \in H$.

Lemma 2.2 (see [9]). *Assume that $\{a_n\}$ is a sequence of nonnegtive real numbers satisfying the property*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\sigma_n, \quad n = 0,1,2\dots. \tag{2.1}$$

If $\{\gamma_n\}_{n=0}^{\infty} \subset (0,1)$ and $\{\sigma_n\}_{n=0}^{\infty}$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\gamma_n = 0$,

(ii) $\sum_{n=1}^{\infty}\gamma_n = \infty$,

(iii) $\limsup_{n\to\infty}\sigma_n \le 0$, *or* $\sum_{n=1}^{\infty}|\gamma_n\sigma_n| < \infty$,

then $\lim_{n\to\infty}a_n = 0$.

Lemma 2.3 (see [10]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ is a nonexpansive mapping. If a one has sequence $\{x_n\}$ in C such that $x_n \rightharpoonup z$ and $(I-T)x_n \to 0$, then $z = Tz$.*

Lemma 2.4 (see [11]). *Let C be a nonempty closed convex subset of a real Hilbert space H, if $T : C \to C$ is a κ-strict pseudo-contraction, then the mapping $I-T$ is demiclosed at 0. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-T)x_n \to 0$, then $(I-T)\tilde{x} = 0$.*

Lemma 2.5 (see [8]). *Assume that C is a nonempty closed convex subset of a real Hilbert space H, $F : C \to H$, if F is boundedly Lipschitzian and η-strongly monotone, then variational inequality* (1.1) *has a unique solution.*

Lemma 2.6. *Assume that $T : H \to H$ is a κ-strict pseudo-contraction, and the constant α satisfies $\kappa \le \alpha < 1$. Let*

$$T_\alpha = \alpha I + (1-\alpha)T, \tag{2.2}$$

then T_α is nonexpansive and $\mathrm{Fix}(T_\alpha) = \mathrm{Fix}(T)$.

Proof. Using Lemma 2.1(i) and the conception of κ-strict pseudo-contraction, we get

$$\begin{aligned}\left\|T_\alpha x - T_\alpha y\right\|^2 &= \left\|\alpha(x-y) + (1-\alpha)(Tx-Ty)\right\|^2 \\ &= \alpha\left\|x-y\right\|^2 + (1-\alpha)\left\|Tx-Ty\right\|^2 - \alpha(1-\alpha)\left\|(I-T)x-(I-T)y\right\|^2 \\ &\leq \alpha\left\|x-y\right\|^2 + (1-\alpha)\left[\left\|x-y\right\|^2 + \kappa\left\|(I-T)x-(I-T)y\right\|^2\right] \\ &\quad - \alpha(1-\alpha)\left\|(I-T)x-(I-T)y\right\|^2 \\ &= \left\|x-y\right\|^2 - (\alpha-\kappa)(1-\alpha)\left\|(I-T)x-(I-T)y\right\|^2 \\ &\leq \left\|x-y\right\|^2, \quad \forall x,y \in H,\end{aligned} \tag{2.3}$$

so T_α is nonexpansive. $\mathrm{Fix}(T_\alpha) = \mathrm{Fix}(T)$ is obvious. □

Lemma 2.7. *Assume that H is a real Hilbert space, $T : H \to H$ is a κ-strict pseudo-contraction such that $\mathrm{Fix}(T) \neq \emptyset$, and $F : H \to H$ is a boundedly Lipschitzian and η-strongly monotone operator. Take $x_0 \in \mathrm{Fix}(T)$ arbitrarily and set $\widehat{C} = S(x_0, 2\|Fx_0\|/\eta)$. Denote by $\widehat{L}$ the Lipschitz constant of F on $\widehat{C}$ and let*

$$T^{\alpha,\lambda} = (I - \mu\lambda F)T_\alpha, \tag{2.4}$$

where the constants μ and λ are such that $0 < \mu < \eta/\widehat{L}^2$ and $0 < \lambda < 1$, respectively, and T_α is defined as in Lemma 2.6 above. Then $T^{\alpha,\lambda}$ restricted to $\widehat{C}$ is a contraction.

Proof. If $x \in \widehat{C}$, that is, $\|x - x_0\| \leq 2\|Fx_0\|/\eta$, by Lemma 2.6, we have

$$\|T_\alpha x - x_0\| = \|T_\alpha x - T_\alpha x_0\| \leq \|x - x_0\| \leq \frac{2\|Fx_0\|}{\eta}. \tag{2.5}$$

It suggests that $T_\alpha x \in \widehat{C}$. Since F is Lipschitzian and η-strongly monotone on $\widehat{C}$, using Lemma 2.6, we obtain

$$\begin{aligned}\left\|T^{\alpha,\lambda}x - T^{\alpha,\lambda}y\right\|^2 &= \left\|(I-\mu\lambda F)T_\alpha x - (I-\mu\lambda F)T_\alpha y\right\|^2 \\ &= \left\|(T_\alpha x - T_\alpha y) - \mu\lambda(FT_\alpha x - FT_\alpha y)\right\|^2 \\ &= \left\|T_\alpha x - T_\alpha y\right\|^2 + \mu^2\lambda^2\left\|FT_\alpha x - FT_\alpha y\right\|^2 \\ &\quad - 2\mu\lambda\langle T_\alpha x - T_\alpha y, FT_\alpha x - FT_\alpha y\rangle\end{aligned}$$

$$\begin{aligned}
&\leq \left\|T_\alpha x - T_\alpha y\right\|^2 + \mu^2\lambda^2\widehat{L}^2\left\|T_\alpha x - T_\alpha y\right\|^2 \\
&\quad - 2\mu\lambda\eta\left\|T_\alpha x - T_\alpha y\right\|^2 \\
&= \left(1 + \mu^2\lambda^2\widehat{L}^2 - 2\mu\lambda\eta\right)\left\|T_\alpha x - T_\alpha y\right\|^2 \\
&\leq (1-\tau\lambda)^2\left\|x - y\right\|^2, \quad \forall x, y \in \widehat{C}.
\end{aligned} \tag{2.6}$$

Therefore, $T^{\alpha,\lambda}$ restricted to that $\widehat{C}$ is a contraction with coefficient $1-\tau\lambda$, where $\tau = 1/2\mu(2\eta - \mu\widehat{L}^2)$. □

Lemma 2.8 (see [11]). *Assume C is a closed convex subset of a Hilbert space H.*

(i) *Given an integer $N \geq 1$, assume that for each $1 \leq i \leq N$, $T_i : C \to C$ is a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$. Assume $\{\gamma_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \gamma_i = 1$. Then $T = \sum_{i=1}^N \gamma_i T_i$ is a κ-strict pseudo-contraction, with $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$.*

(ii) *Let $\{T_i\}_{i=1}^N$, $\{\gamma_i\}_{i=1}^N$, and T be given as in (i) above. Suppose that $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$, then*

$$\mathrm{Fix}(T) = \bigcap_{i=1}^N \mathrm{Fix}(T_i). \tag{2.7}$$

Lemma 2.9. *Assume that $T_i : H \to H$ is a κ_i-strict pseudo-contraction for some $0 \leq \kappa_i < 1$ $(1 \leq i \leq N)$, let $T_{\alpha_i} = \alpha_i I + (1-\alpha_i)T_i$, $\kappa_i < \alpha_i < 1$ $(1 \leq i \leq N)$, if $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$, then*

$$\mathrm{Fix}(T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N}) = \bigcap_{i=1}^N \mathrm{Fix}(T_{\alpha_i}). \tag{2.8}$$

Proof. We prove it by induction. For $N = 2$, set $T_{\alpha_1} = \alpha_1 I + (1-\alpha_1)T_1$, $T_{\alpha_2} = \alpha_2 I + (1-\alpha_2)T_2$, $\kappa_i < \alpha_i < 1$, $i = 1, 2$. Obviously

$$\mathrm{Fix}(T_{\alpha_1}) \bigcap \mathrm{Fix}(T_{\alpha_2}) \subset \mathrm{Fix}(T_{\alpha_1} T_{\alpha_2}). \tag{2.9}$$

Now we prove

$$\mathrm{Fix}(T_{\alpha_1} T_{\alpha_2}) \subset \mathrm{Fix}(T_{\alpha_1}) \bigcap \mathrm{Fix}(T_{\alpha_2}). \tag{2.10}$$

for all $q \in \operatorname{Fix}(T_{\alpha_1}T_{\alpha_2})$, $T_{\alpha_1}T_{\alpha_2}q = q$, if $T_{\alpha_2}q = q$, then $T_{\alpha_1}q = q$, the conclusion holds. In fact, we can claim that $T_{\alpha_2}q = q$. From Lemma 2.6, we know that T_{α_2} is nonexpansive and $\operatorname{Fix}(T_{\alpha_1}) \bigcap \operatorname{Fix}(T_{\alpha_2}) = \operatorname{Fix}(T_1) \bigcap \operatorname{Fix}(T_2) \neq \emptyset$. Take $p \in \operatorname{Fix}(T_{\alpha_1}) \bigcap \operatorname{Fix}(T_{\alpha_2})$, then

$$\begin{aligned}
\|p-q\|^2 &= \|p - T_{\alpha_1}T_{\alpha_2}q\|^2 = \|p - [\alpha_1(T_{\alpha_2}q) + (1-\alpha_1)T_1T_{\alpha_2}q]\|^2 \\
&= \|\alpha_1(p - T_{\alpha_2}q) + (1-\alpha_1)(p - T_1T_{\alpha_2}q)\|^2 \\
&= \alpha_1\|p - T_{\alpha_2}q\|^2 + (1-\alpha_1)\|p - T_1T_{\alpha_2}q\|^2 \\
&\quad - \alpha_1(1-\alpha_1)\|T_{\alpha_2}q - T_1T_{\alpha_2}q\|^2 \\
&\leq \alpha_1\|p - T_{\alpha_2}q\|^2 + (1-\alpha_1)\left[\|p - T_{\alpha_2}q\|^2 + \kappa_1\|T_{\alpha_2}q - T_1T_{\alpha_2}q\|^2\right] \\
&\quad - \alpha_1(1-\alpha_1)\|T_{\alpha_2}q - T_1T_{\alpha_2}q\|^2 \\
&\leq \|p - T_{\alpha_2}q\|^2 - (\alpha_1-\kappa_1)(1-\alpha_1)\|T_{\alpha_2}q - T_1T_{\alpha_2}q\|^2 \\
&\leq \|p - q\|^2 - (\alpha_1-\kappa_1)(1-\alpha_1)\|T_{\alpha_2}q - T_1T_{\alpha_2}q\|^2.
\end{aligned} \tag{2.11}$$

Since $\kappa_1 < \alpha_1 < 1$, we get

$$\|T_{\alpha_2}q - T_1T_{\alpha_2}q\|^2 \leq 0, \tag{2.12}$$

Namely, $T_{\alpha_2}q = T_1T_{\alpha_2}q$, that is,

$$T_{\alpha_2}q \in \operatorname{Fix}(T_1) = \operatorname{Fix}(T_{\alpha_1}), \quad T_{\alpha_2}q = T_{\alpha_1}T_{\alpha_2}q = q. \tag{2.13}$$

Suppose that the conclusion holds for $N = k$, we prove that

$$\operatorname{Fix}(T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}) = \bigcap_{i=1}^{k+1}\operatorname{Fix}(T_{\alpha_i}). \tag{2.14}$$

It suffices to verify

$$\operatorname{Fix}(T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}) \subset \bigcap_{i=1}^{k+1}\operatorname{Fix}(T_{\alpha_i}) \tag{2.15}$$

for all $q \in \operatorname{Fix}(T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}})$, $T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}q = q$. Using Lemma 2.6 again, take $p \in \bigcap_{i=1}^{k+1}\operatorname{Fix}(T_{\alpha_i})$,

$$\begin{aligned}
\|p-q\|^2 &= \left\|p - T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2 \\
&= \left\|p - [\alpha_1(T_{\alpha_2}\cdots T_{\alpha_{k+1}}q) + (1-\alpha_1)(T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q)]\right\|^2 \\
&= \left\|\alpha_1(p - T_{\alpha_2}\cdots T_{\alpha_{k+1}}q) + (1-\alpha_1)(p - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q)\right\|^2 \\
&= \alpha_1\left\|p - T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2 + (1-\alpha_1)\left\|p - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2 \\
&\quad - \alpha_1(1-\alpha_1)\left\|T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2 \\
&\le \alpha_1\left\|p - T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2 \\
&\quad + (1-\alpha_1)\left[\left\|p - T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2 + \kappa_1\left\|T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2\right] \\
&\quad - \alpha_1(1-\alpha_1)\left\|T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2 \\
&\le \|p-q\|^2 - (\alpha_1-\kappa_1)(1-\alpha_1)\left\|T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2.
\end{aligned} \tag{2.16}$$

Since $\kappa_1 < \alpha_1 < 1$, we have

$$\left\|T_{\alpha_2}\cdots T_{\alpha_{k+1}}q - T_1T_{\alpha_2}\cdots T_{\alpha_{k+1}}q\right\|^2 \le 0, \tag{2.17}$$

this implies that

$$T_{\alpha_2}\cdots T_{\alpha_{k+1}}q \in \operatorname{Fix}(T_1) = \operatorname{Fix}(T_{\alpha_1}), \tag{2.18}$$

Namely,

$$T_{\alpha_2}\cdots T_{\alpha_{k+1}}q = T_{\alpha_1}T_{\alpha_2}\cdots T_{\alpha_{k+1}}q = q. \tag{2.19}$$

From (2.19) and inductive assumption, we get

$$q \in \operatorname{Fix}(T_{\alpha_2}\cdots T_{\alpha_{k+1}}) = \bigcap_{i=2}^{k+1}\operatorname{Fix}(T_{\alpha_i}), \tag{2.20}$$

therefore

$$T_{\alpha_i}q = q, \quad i = 2,3,\ldots,k+1. \tag{2.21}$$

Substituting it into (2.19), we obtain $T_{\alpha_1}q = q$. Thus we assert that

$$q \in \bigcap_{i=1}^{k+1}\operatorname{Fix}(T_{\alpha_i}). \tag{2.22}$$

□

3. Further Extension of Hybrid Iterative Algorithm

Yamada got the following result.

Theorem 3.1 (see [7]). *Assume that H is a real Hilbert space, $T : H \to H$ is nonexpansive such that $\mathrm{Fix}(T) \neq \emptyset$, and $F : H \to H$ is η-strongly monotone and L-Lipschitzian. Fix a constant $\mu \in (0, 2\eta/L^2)$. Assume also that the sequence $\{\lambda_n\} \subset (0,1)$ satisfies the following conditions:*

(i) $\lambda_n \to 0,\ n \to \infty$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, *or* $\lim_{n\to\infty}(\lambda_n/\lambda_{n+1}) = 1$.

Take $x_0 \in \mathrm{Fix}(T)$ arbitrarily and define $\{x_n\}$ by

$$x_{n+1} = T^{\lambda_n} x_n = (I - \mu\lambda_n F)T x_n, \tag{3.1}$$

then $\{x_n\}$ converges strongly to the unique solution of $\mathrm{VI}(\mathrm{Fix}(T), F)$.

He and Xu [8] proved that $\mathrm{VI}(C, F)$ has a unique solution if F is a boundedly Lipschitzian and strongly monotone operator and C is a closed convex subset of H. Using this result, they were able to relax the global Lipschitz condition on F in Theorem 3.1 to the weaker bounded Lipschitz condition and invented a hybrid iterative algorithm to approximate the unique solution of $\mathrm{VI}(C, F)$. Their result extended the Yamada's above theorem.

In this section, we mainly focus on further extension of our hybrid algorithm in [8]. Consider $\mathrm{VI}(C, F)$, where C is composed of fixed points of a κ-strict pseudo-contraction $T : H \to H$ such that $\mathrm{Fix}(T) \neq \emptyset$ and $F : H \to H$ is still η-strongly monotone and boundedly Lipschitzian. Fix a point $x_0 \in \mathrm{Fix}(T)$ arbitrarily, set $\widehat{C} = S(x_0, 2\|Fx_0\|/\eta)$. Denote by $\widehat{L}$ the Lipschitz constant of F on $\widehat{C}$. Fix the constant μ satisfying $0 < \mu < \eta/\widehat{L}^2$. Assume also that the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy $\kappa \leq \alpha_n \leq \alpha < 1$ for a constant $\alpha \in (0,1)$ and $0 < \lambda_n < 1$ $(n \geq 0)$, respectively. Let $T_{\alpha_n} = \alpha_n I + (1 - \alpha_n)T$ and $T^{\alpha_n,\lambda_n} = (I - \mu\lambda_n F)T_{\alpha_n}$, define $\{x_n\}$ by the scheme:

$$x_{n+1} = T^{\alpha_n,\lambda_n} x_n = (I - \mu\lambda_n F)T_{\alpha_n} x_n, \quad (n \geq 0). \tag{3.2}$$

We have the following result.

Theorem 3.2. *If the sequences $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:*

(i) $\lambda_n \to 0 \quad (n \to \infty)$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, *or* $\lim_{n\to\infty}(\lambda_{n-1}/\lambda_n) = 1$, $\lim_{n\to\infty}(|\alpha_n - \alpha_{n-1}|/\lambda_n) = 0$,

then $\{x_n\}$ generated by (3.2) *converges strongly to the unique solution x^* of* $\mathrm{VI}(\mathrm{Fix}(T), F)$.

Proof. We prove that $x_n \in \widehat{C}$ for all $n \geq 0$ by induction. It is trivial that $x_0 \in \widehat{C}$. Suppose we have proved $x_n \in \widehat{C}$, that is,

$$\|x_n - x_0\| \leq \frac{2\|Fx_0\|}{\eta}. \tag{3.3}$$

Using Lemma 2.7, We then derive from (3.2) and (3.3) that

$$\begin{aligned} \|x_{n+1} - x_0\| &= \left\|T^{\alpha_n,\lambda_n}x_n - x_0\right\| \\ &\leq \left\|T^{\alpha_n,\lambda_n}x_n - T^{\alpha_n,\lambda_n}x_0\right\| + \left\|T^{\alpha_n,\lambda_n}x_0 - x_0\right\| \\ &\leq (1-\tau\lambda_n)\|x_n - x_0\| + \mu\lambda_n\|Fx_0\| \\ &= (1-\tau\lambda_n)\|x_n - x_0\| + \tau\lambda_n\frac{\mu}{\tau}\|Fx_0\| \\ &\leq \max\left\{\|x_n - x_0\|, \frac{\mu}{\tau}\|Fx_0\|\right\} \\ &\leq \max\left\{\frac{2}{\eta}, \frac{\mu}{\tau}\right\}\|Fx_0\|. \end{aligned} \tag{3.4}$$

However, since $0 < \mu < \eta/\widehat{L}^2$ and $\tau = (1/2)\mu(2\eta - \mu\widehat{L}^2)$, we get

$$\frac{\mu}{\tau} = \frac{\mu}{(1/2)\mu\left(2\eta - \mu\widehat{L}^2\right)} = \frac{2}{\eta + \left(\eta - \mu\widehat{L}^2\right)} \leq \frac{2}{\eta}. \tag{3.5}$$

This together with (3.4) implies that

$$\|x_{n+1} - x_0\| \leq \frac{2\|Fx_0\|}{\eta}. \tag{3.6}$$

It proves that $x_{n+1} \in \widehat{C}$. Therefore, $x_n \in \widehat{C}$ for all $n \geq 0$. Thus $\{x_n\}$ is bounded. It is not difficult to verify that the sequences $\{Tx_n\}$ and $\{FT_{\alpha_n}x_n\}$ are all bounded.

By (3.2) and Lemma 2.7, we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \left\|T^{\alpha_n,\lambda_n}x_n - T^{\alpha_{n-1},\lambda_{n-1}}x_{n-1}\right\| \\
&\le \left\|T^{\alpha_n,\lambda_n}x_n - T^{\alpha_n,\lambda_n}x_{n-1}\right\| + \left\|T^{\alpha_n,\lambda_n}x_{n-1} - T^{\alpha_{n-1},\lambda_n}x_{n-1}\right\| \\
&\quad + \left\|T^{\alpha_{n-1},\lambda_n}x_{n-1} - T^{\alpha_{n-1},\lambda_{n-1}}x_{n-1}\right\| \\
&\le (1-\tau\lambda_n)\|x_n-x_{n-1}\| + (1-\tau\lambda_n)\|T_{\alpha_n}x_{n-1} - T_{\alpha_{n-1}}x_{n-1}\| \\
&\quad + \mu|\lambda_n-\lambda_{n-1}|\|FT_{\alpha_{n-1}}x_{n-1}\| \\
&\le (1-\tau\lambda_n)\|x_n-x_{n-1}\| + (1-\tau\lambda_n)|\alpha_n-\alpha_{n-1}|\ [\|x_{n-1}\| + \|Tx_{n-1}\|] \\
&\quad + \mu|\lambda_n-\lambda_{n-1}|\|FT_{\alpha_{n-1}}x_{n-1}\| \\
&\le (1-\tau\lambda_n)\|x_n-x_{n-1}\| + M[|\alpha_n-\alpha_{n-1}| + |\lambda_n-\lambda_{n-1}|],
\end{aligned} \tag{3.7}$$

where $M = \sup_n[\|FT_{\alpha_n}x_n\|, \|x_n\|, \|Tx_n\|] < \infty$. By Lemma 2.2 and conditions (i)–(iii), we conclude that

$$\|x_{n+1}-x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.8}$$

Since $\lambda_n \to 0$, it is straitforward from (3.2) that

$$\|x_{n+1}-T_{\alpha_n}x_n\| = \mu\lambda_n\|FT_{\alpha_n}x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.9}$$

On the other hand

$$\begin{aligned}
\|x_{n+1}-T_{\alpha_n}x_n\| &= \|x_{n+1} - [\alpha_n x_n + (1-\alpha_n)Tx_n]\| \\
&= \|(x_{n+1}-x_n) + (1-\alpha_n)(x_n-Tx_n)\| \\
&\ge (1-\alpha_n)\|x_n-Tx_n\| - \|x_{n+1}-x_n\|.
\end{aligned} \tag{3.10}$$

By the condition $\alpha_n \le \alpha < 1$ and (3.8)–(3.10), we obtain

$$\begin{aligned}
\|x_n-Tx_n\| &\le \frac{1}{1-\alpha_n}[\|x_{n+1}-T_{\alpha_n}x_n\| + \|x_{n+1}-x_n\|] \\
&\le \frac{1}{1-\alpha}[\|x_{n+1}-T_{\alpha_n}x_n\| + \|x_{n+1}-x_n\|] \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{3.11}$$

By Lemma 2.4 and (3.11), we obtain

$$\omega_w(x_n) \subset \mathrm{Fix}(T). \tag{3.12}$$

Lemma 2.5 asserts that $\mathrm{VI}(\mathrm{Fix}(T), F)$ has a unique solution $x^* \in \mathrm{Fix}(T)$. Now we prove that $\|x_n - x^*\| \to 0$ $(n \to \infty)$. By Lemma 2.1(ii), (3.2), and Lemma 2.7, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|T^{\alpha_n,\lambda_n} x_n - x^*\right\|^2 \\
&= \left\|\left(T^{\alpha_n,\lambda_n} x_n - T^{\alpha_n,\lambda_n} x^*\right) + \left(T^{\alpha_n,\lambda_n} x^* - x^*\right)\right\|^2 \\
&\leq \left\|T^{\alpha_n,\lambda_n} x_n - T^{\alpha_n,\lambda_n} x^*\right\|^2 + 2\left\langle T^{\alpha_n,\lambda_n} x^* - x^*, x_{n+1} - x^*\right\rangle \\
&\leq (1 - \tau\lambda_n)\|x_n - x^*\|^2 + 2\mu\lambda_n\langle -Fx^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.13}$$

Let us show that

$$\limsup_{n\to\infty}\langle -Fx^*, x_n - x^*\rangle \leq 0. \tag{3.14}$$

In fact, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty}\langle -Fx^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle -Fx^*, x_{n_j} - x^*\rangle. \tag{3.15}$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \tilde{x} \in \mathrm{Fix}(T)$. Since x^* is the unique solution of $\mathrm{VI}(\mathrm{Fix}(T), F)$, we obtain

$$\limsup_{n\to\infty}\langle -Fx^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle -Fx^*, x_{n_j} - x^*\rangle = -\langle Fx^*, \tilde{x} - x^*\rangle \leq 0. \tag{3.16}$$

Finally conditions (i)–(iii) and (3.14) allow us to apply Lemma 2.2 to the relation (3.13) to conclude that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. □

4. Parallel Algorithm and Cyclic Algorithm

In this section, we discuss the parallel algorithm and the cyclic algorithm, respectively, for solving the variational inequality over the set of the common fixed points of finite strict pseudo-contractions.

Let H be a real Hilbert space and $F : H \to H$ a η-strongly monotone and boundedly Lipschitzian operator. Let N be a positive integer and $T_i : H \to H$ a κ_i-strict pseudo-contraction for some $\kappa_i \in (0,1)$ $(i = 1,\ldots,N)$ such that $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. We consider the problem of finding $x^* \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ such that

$$\langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^N \mathrm{Fix}(T_i). \tag{4.1}$$

Since $\bigcap_{i=1}^N \mathrm{Fix}(T_i)$ is a nonempty closed convex subset of H, VI(4.1) has a unique solution. Throughout this section, $x_0 \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ is an arbitrary fixed point, $\widehat{C} = S(x_0, 2\|Fx_0\|/\eta)$,

$\widehat{L}$ is the Lipschitz constant of F on $\widehat{C}$, the fixed constant μ satisfies $0 < \mu < \eta/\widehat{L}^2$, and the sequence $\{\lambda_n\}$ belongs to $(0,1)$.

Firstly we consider the parallel algorithm. Take a positive sequence $\{\gamma_i\}_{i=1}^N$ such that $\sum_{i=1}^N \gamma_i = 1$ and let

$$T = \sum_{i=1}^{N} \gamma_i T_i. \tag{4.2}$$

By using Lemma 2.8, we assert that T is a κ-strict pseudo-contraction with $\kappa = \max\{\kappa_i : i = 1,\ldots,N\}$ and $\mathrm{Fix}(T) = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ holds. Thus VI(4.1) is equivalent to $\mathrm{VI}(\mathrm{Fix}(T),F)$ and we can use scheme (3.2) to solve VI(4.1). In fact, taking $T = \sum_{i=1}^N \gamma_i T_i$ in the scheme (3.2), we get the so-called parallel algorithm

$$x_{n+1} = T^{\alpha_n,\lambda_n} x_n = \left(I - \mu\lambda_n F\right) T_{\alpha_n} x_n \quad (n \geq 0). \tag{4.3}$$

Using Lemma 2.8 and Thorem 3.2, the following conclusion can be deduced directly.

Theorem 4.1. *Suppose that $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the same conditions as in Theorem 3.2. Then the sequence $\{x_n\}$ generated by the parallel algorithm* (4.3) *converges strongly to the unique solution x^* of VI* (4.1).

For each $i = 1,\ldots,N$, let

$$T_{\alpha_i} = \alpha_i I + (1-\alpha_i) T_i, \tag{4.4}$$

where the constant α_i such that $\kappa_i < \alpha_i < 1$. Then we turn to defining the cyclic algorithm as follows:

$$\begin{aligned} x_1 &= T_{\alpha_1} x_0 - \mu\lambda_0 F(T_{\alpha_1} x_0), \\ x_2 &= T_{\alpha_2} x_1 - \mu\lambda_1 F(T_{\alpha_2} x_1), \\ &\cdots \\ x_N &= T_{\alpha_N} x_{N-1} - \mu\lambda_{N-1} F(T_{\alpha_N} x_{N-1}), \\ x_{N+1} &= T_{\alpha_1} x_N - \mu\lambda_N F(T_{\alpha_1} x_N), \\ &\cdots. \end{aligned} \tag{4.5}$$

Indeed, the algorithm above can be rewritten as

$$x_{n+1} = T_{\alpha_{[n+1]}} x_n - \mu\lambda_n F\left(T_{\alpha_{[n+1]}} x_n\right), \tag{4.6}$$

where $T_{\alpha_{[n]}} = \alpha_{[n]} I + (1-\alpha_{[n]}) T_{[n]}$, $T_{[n]} = T_{n \bmod N}$, namely, $T_{[n]}$ is one of $T_1, T_2, \ldots, T_N$ circularly. For convenience, we denote (4.6) as

$$x_{n+1} = T^{\alpha_{[n+1]},\lambda_n} x_n. \tag{4.7}$$

We get the following result

Theorem 4.2. *If* $\{\lambda_n\} \subset (0,1)$ *satisfies the following conditions:*

(i) $\lambda_n \to 0,\ n \to \infty$;

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$;

(iii) $\sum_{n=0}^{\infty} |\lambda_{n+N} - \lambda_n| < \infty$, *or* $\lim_{n\to\infty}(\lambda_n/\lambda_{n+N}) = 1$,

then the sequence $\{x_n\}$ *generated by* (4.6) *converges strongly to the unique solution* x^* *of* $VI(4.1)$.

Proof. We break the proof process into six steps.

(1) $x_n \in \widehat{C}$. We prove it by induction. Definitely $x_0 \in \widehat{C}$. Suppose $x_n \in \widehat{C}$, that is,

$$\|x_n - x_0\| \le \frac{2\|Fx_0\|}{\eta}. \tag{4.8}$$

We have from $x_0 \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$, (4.8), and Lemma 2.7 that

$$\begin{aligned}
\|x_{n+1} - x_0\| &= \left\|T^{\alpha_{[n+1]},\lambda_n} x_n - x_0\right\| \\
&\le \left\|T^{\alpha_{[n+1]},\lambda_n} x_n - T^{\alpha_{[n+1]},\lambda_n} x_0\right\| + \left\|T^{\alpha_{[n+1]},\lambda_n} x_0 - x_0\right\| \\
&\le (1 - \tau\lambda_n)\|x_n - x_0\| + \mu\lambda_n\|Fx_0\| \\
&= (1 - \tau\lambda_n)\|x_n - x_0\| + \tau\lambda_n\frac{\mu}{\tau}\|Fx_0\| \\
&\le \max\left\{\|x_n - x_0\|, \frac{\mu}{\tau}\|Fx_0\|\right\} \\
&\le \max\left\{\frac{2}{\eta}, \frac{\mu}{\tau}\right\}\|Fx_0\|,
\end{aligned} \tag{4.9}$$

where $\tau = (1/2)\mu(2\eta - \mu\widehat{L}^2)$. Observing $0 < \mu < \eta/\widehat{L}^2$, we get

$$\frac{\mu}{\tau} = \frac{\mu}{1/2\mu\left(2\eta - \mu\widehat{L}^2\right)} = \frac{2}{\eta + \left(\eta - \mu\widehat{L}^2\right)} \le \frac{2}{\eta}. \tag{4.10}$$

This together with (4.9) implies that

$$\|x_{n+1} - x_0\| \le \frac{2\|Fx_0\|}{\eta}. \tag{4.11}$$

It suggests that $x_{n+1} \in \widehat{C}$. Therefore, $x_n \in \widehat{C}$ for all $n \ge 0$. We can also prove that the sequences $\{x_n\}$, $\{T_{\alpha_{[n]}}x_n\}$,$\{FT_{\alpha_{[n]}}x_n\}$ are all bounded.

(2) $\|x_{n+N} - x_n\| \to 0$ $(n \to \infty)$. By (4.6) and Lemma 2.7, we have

$$\begin{aligned}
\|x_{n+N} - x_n\| &= \left\|T^{\alpha_{[n+N]},\lambda_{n+N-1}} x_{n+N-1} - T^{\alpha_{[n]},\lambda_{n-1}} x_{n-1}\right\| \\
&\le \left\|T^{\alpha_{[n+N]},\lambda_{n+N-1}} x_{n+N-1} - T^{\alpha_{[n+N]},\lambda_{n+N-1}} x_{n-1}\right\| \\
&\quad + \left\|T^{\alpha_{[n+N]},\lambda_{n+N-1}} x_{n-1} - T^{\alpha_{[n]},\lambda_{n-1}} x_{n-1}\right\| \\
&\le (1-\tau\lambda_{n+N-1})\|x_{n+N-1} - x_{n-1}\| \\
&\quad + \mu|\lambda_{n+N-1} - \lambda_{n-1}|\left\|FT_{\alpha_{[n]}} x_{n-1}\right\| \\
&\le (1-\tau\lambda_{n+N-1})\|x_{n+N-1} - x_{n-1}\| + M|\lambda_{n+N-1} - \lambda_{n-1}|,
\end{aligned} \tag{4.12}$$

where $M = \sup_n\|FT_{\alpha_{[n]}} x_{n-1}\| < \infty$. Since $\{\lambda_n\}$ satisfies (i)–(iii), using Lemma 2.2, we get

$$\|x_{n+N} - x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{4.13}$$

(3) $\|x_n - T_{\alpha_{[n+N]}} T_{\alpha_{[n+N-1]}} \cdots T_{\alpha_{[n+1]}} x_n\| \to 0$ $(n \to \infty)$. By (4.3) and $\lambda_n \to 0$, we have

$$\left\|x_{n+1} - T_{\alpha_{[n+1]}} x_n\right\| = \mu\lambda_n\left\|FT_{\alpha_{[n+1]}} x_n\right\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{4.14}$$

Recursively,

$$\begin{gathered}
x_{n+N} - T_{\alpha_{[n+N]}} x_{n+N-1} \longrightarrow 0 \quad (n \longrightarrow \infty), \\
x_{n+N-1} - T_{\alpha_{[n+N-1]}} x_{n+N-2} \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{gathered} \tag{4.15}$$

By Lemma 2.6, $T_{\alpha_{[n+N]}}$ is nonexpansive, we obtain

$$\begin{gathered}
T_{\alpha_{[n+N]}} x_{n+N-1} - T_{\alpha_{[n+N]}} T_{\alpha_{[n+N-1]}} x_{n+N-2} \longrightarrow 0 \quad (n \longrightarrow \infty), \\
T_{\alpha_{[n+N]}} T_{\alpha_{[n+N-1]}} x_{n+N-2} - T_{\alpha_{[n+N]}} T_{\alpha_{[n+N-1]}} T_{\alpha_{[n+N-2]}} x_{n+N-3} \longrightarrow 0 \quad (n \longrightarrow \infty), \\
\cdots \\
T_{\alpha_{[n+N]}} \cdots T_{\alpha_{[n+2]}} x_{n+1} - T_{\alpha_{[n+N]}} \cdots T_{\alpha_{[n+1]}} x_n \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{gathered} \tag{4.16}$$

Adding all the expressions above, we get

$$\left\|x_{n+N} - T_{\alpha_{[n+N]}} T_{\alpha_{[n+N-1]}} \cdots T_{\alpha_{[n+1]}} x_n\right\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{4.17}$$

Using this together with the conclusion of step (2), we obtain

$$\left\|x_n - T_{\alpha_{[n+N]}} T_{\alpha_{[n+N-1]}} \cdots T_{\alpha_{[n+1]}} x_n\right\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{4.18}$$

(4) $\omega_w(x_n) \subset \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Assume that $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup \widehat{x}$, we prove $\widehat{x} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. By the conclusion of step (3), we get

$$\left\| x_{n_j} - T_{\alpha_{[n_j+N]}} T_{\alpha_{[n_j+N-1]}} \cdots T_{\alpha_{[n_j+1]}} x_{n_j} \right\| \longrightarrow 0, \quad (j \longrightarrow \infty). \tag{4.19}$$

Observe that, for each n_j, $T_{\alpha_{[n_j+N]}} T_{\alpha_{[n_j+N-1]}} \cdots T_{\alpha_{[n_j+1]}}$ is some permutation of the mappings $T_{\alpha_1}, T_{\alpha_2}, \ldots, T_{\alpha_N}$, since $T_{\alpha_1}, T_{\alpha_2}, \ldots, T_{\alpha_N}$ are finite, all the full permutation are $N!$, there must be some permutation that appears infinite times. Without loss of generality, suppose that this permutation is $T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N}$, we can take a subsequence $\{x_{n_{jk}}\} \subset \{x_{n_j}\}$ such that

$$\left\| x_{n_{jk}} - T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N} x_{n_{jk}} \right\| \longrightarrow 0 \quad (k \longrightarrow \infty). \tag{4.20}$$

It is easy to prove that $T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N}$ is nonexpansive. By Lemma 2.3, we get

$$\widehat{x} = T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N} \widehat{x}. \tag{4.21}$$

Using Lemmas 2.6 and 2.9, we obtain

$$\widehat{x} \in \mathrm{Fix}(T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_N}) = \bigcap_{i=1}^N \mathrm{Fix}(T_{\alpha_i}) = \bigcap_{i=1}^N \mathrm{Fix}(T_i). \tag{4.22}$$

(5) $\limsup_{n\to\infty} \langle -Fx^*, x_n - x^* \rangle \le 0$. In fact, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that

$$\limsup_{n\to\infty} \langle -Fx^*, x_n - x^* \rangle = \lim_{j\to\infty} \langle -Fx^*, x_{n_j} - x^* \rangle. \tag{4.23}$$

Without loss of generality, we may further assume that $x_{n_j} \rightharpoonup \widetilde{x} \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$. Since x^* is the solution of VI(4.1), we obtain

$$\limsup_{n\to\infty} \langle -Fx^*, x_n - x^* \rangle = \lim_{j\to\infty} \langle -Fx^*, x_{n_j} - x^* \rangle = -\langle Fx^*, \widetilde{x} - x^* \rangle \le 0. \tag{4.24}$$

(6) $x_n \to x^*$. By (4.6), Lemmas 2.1(ii), and 2.7, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\| T^{\alpha_{[n+1]},\lambda_n} x_n - x^* \right\|^2 \\
&= \left\| \left(T^{\alpha_{[n+1]},\lambda_n} x_n - T^{\alpha_{[n+1]},\lambda_n} x^*\right) + \left(T^{\alpha_{[n+1]},\lambda_n} x^* - x^*\right) \right\|^2 \\
&\le \left\| T^{\alpha_{[n+1]},\lambda_n} x_n - T^{\alpha_{[n+1]},\lambda_n} x^* \right\|^2 + 2\langle T^{\alpha_{[n+1]},\lambda_n} x^* - x^*, x_{n+1} - x^* \rangle \\
&\le (1 - \tau\lambda_n)\|x_n - x^*\|^2 + 2\mu\lambda_n \langle -Fx^*, x_{n+1} - x^* \rangle.
\end{aligned} \tag{4.25}$$

From the conclusion of step (5) and Lemma 2.2, we get

$$x_n \to x^* \quad (n \to \infty). \tag{4.26}$$

□

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