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Research Article

Common Fixed Point of Multivalued Generalized φ-Weak Contractive Mappings

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Fixed point and coincidence results are presented for multivalued generalized φ-weak contractive mappings on complete metric spaces, where $\varphi : [0,+\infty) \longrightarrow [0,+\infty)$ is a lower semicontinuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Our results extend previous results by Zhang and Song (2009), as well as by Rhoades (2001), Nadler (1969), and Daffer and Kaneko (1995).

1. Introduction

Let (X, d) be a metric space. We denote the family of all nonempty closed and bounded subsets of X by $CB(X)$.

A mapping $T : X \to X$ is said to be φ-weak contractive if there exists a map $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.1}$$

for all $x, y \in X$.

Also two mappings $T, S : X \to X$ are called generalized φ-weak contractions if there exists a map $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Sy) \leq M(x, y) - \varphi(M(x, y)) \tag{1.2}$$

for all $x, y \in X$, where

$$M(x,y) := \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}. \tag{1.3}$$

A mapping $T : X \to CB(X)$ is said to be a weak contraction if there exists $0 \le \alpha < 1$ such that

$$H(Tx,Ty) \le \alpha N(x,y), \tag{1.4}$$

for all $x, y \in X$, where H denotes the Hausdorff metric on $CB(X)$ induced by d, that is,

$$H(A,B) := \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \tag{1.5}$$

for all $A, B \in CB(X)$, and where

$$N(x,y) := \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}. \tag{1.6}$$

A mapping $T : X \to CB(X)$ is said to be φ-weak contractive if there exists a map $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$H(Tx,Ty) \le d(x,y) - \varphi(d(x,y)), \tag{1.7}$$

for all $x, y \in X$.

The concepts of weak and φ-weak contractive mappings were defined by Daffer and Kaneko [1] in 1995.

Many authors have studied fixed points for multivalued mappings. Among many others, see, for example, [1–4], and the references therein.

In the following theorem, Nadler [3] extended the Banach Contraction Principle to multivalued mappings.

Theorem 1.1. *Let (X,d) be a complete metric space. Suppose $T : X \to CB(X)$ is a contraction mapping in the sense that for some $0 \le \alpha < 1$,*

$$H(Tx,Ty) \le \alpha d(x,y), \tag{1.8}$$

for all $x, y \in X$. Then there exists a point $x \in X$ such that $x \in Tx$ (i.e., x is a fixed point of T).

Daffer and Kaneko [1] proved the existence of a fixed point for a multivalued weak contraction mapping of a complete metric space X into $CB(X)$.

Theorem 1.2. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be such that*

$$H(Tx,Ty) \le \alpha N(x,y), \tag{1.9}$$

for some $0 \leq \alpha < 1$ *and for all* $x, y \in X$ *(i.e., weak contraction). If* $x \mapsto d(x, Tx)$ *is lower semicontinuous (l.s.c.), then there exists* $x_0 \in X$ *such that* $x_0 \in Tx_0$.

In Section 3 we extend Nadler and Daffer-Kaneko's theorems to multivalued generalized weak contraction mappings (see Definition 2.1).

Rhoades [5, Theorem 2] proved the following fixed point theorem for φ-weak contractive single valued mappings, giving another generalization of the Banach Contraction Principle.

Theorem 1.3. *Let* (X, d) *be a complete metric space, and let* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{1.10}$$

for every $x, y \in X$ *(i.e.,* φ*-weak contractive), where* $\varphi : [0, +\infty) \to [0, +\infty)$ *is a continuous and nondecreasing function with* $\varphi(0) = 0$ *and* $\varphi(t) > 0$ *for all* $t > 0$*. Then* T *has a unique fixed point.*

By choosing $\psi(t) = t - \varphi(t)$, φ-weak contractions become mappings of Boyd and Wong type [6], and by defining $k(t) = (1 - \varphi(t))/t$ for $t > 0$ and $k(0) = 0$, then φ-weak contractions become mappings of Reich type [7].

Recently Zhang and Song [8] proved the following theorem on the existence of a common fixed point for two single valued generalized φ-weak contraction mappings.

Theorem 1.4. *Let* (X, d) *be a complete metric space, and let* $T, S : X \to X$ *be two mappings such that for all* $x, y \in X$

$$d(Tx, Sy) \leq M(x, y) - \varphi(M(x, y)), \tag{1.11}$$

(i.e., generalized φ*-weak contractions), where* $\varphi : [0, +\infty) \to [0, +\infty)$ *is an l.s.c. function with* $\varphi(0) = 0$ *and* $\varphi(t) > 0$ *for all* $t > 0$*. Then there exists a unique point* $x \in X$ *such that* $x = Tx = Sx$.

In Section 4, we extend Theorem 1.3 by assuming φ to be only l.s.c., and extend Theorem 1.4 to multivalued mappings.

2. Preliminaries

In this paper, (X, d) denotes a complete metric space and H denotes the Hausdorff metric on $CB(X)$.

Definition 2.1. Two mappings $T, S : X \to CB(X)$ are called generalized weak contractions if there exists $0 \leq \alpha < 1$ such that

$$H(Tx, Sy) \leq \alpha M(x, y), \tag{2.1}$$

for all $x, y \in X$.

Definition 2.2. Two mappings $T, S : X \to CB(X)$ are called generalized φ-weak contractive if there exists a map $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$H(Tx, Sy) \leq M(x, y) - \varphi(M(x, y)) \tag{2.2}$$

for all $x, y \in X$.

In the proof of our main results, we will use the following well-known lemma, and refer to Nadler [3] or Assad and Kirk [9] for its proof.

Lemma 2.3. *If $A, B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$, there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon. \tag{2.3}$$

3. Extension of Nadler and Daffer-Kaneko's Theorems

The following theorem extends Nadler and Daffer-Kaneko's Theorems to a coincidence theorem, without assuming $x \mapsto d(x, Tx)$ to be l.s.c.

Theorem 3.1. *Let (X, d) be a complete metric space, and let $T, S : X \to CB(X)$ be two multivalued mappings such that for all $x, y \in X$,*

$$H(Tx, Sy) \leq \alpha M(x, y), \tag{3.1}$$

where $0 \leq \alpha < 1$ (i.e., multivalued generalized weak contractions). Then there exists a point $x \in X$ such that $x \in Tx$ and $x \in Sx$ (i.e., T and S have a common fixed point). Moreover, if either T or S is single valued, then this common fixed point is unique.

Proof. Obviously $M(x, y) = 0$ if and only if $x = y$ is a common fixed point of T and S.

Let $\varepsilon > 0$ be such that $\beta = \alpha + \varepsilon < 1$. Let $x_0 \in X$ and $x_1 \in Sx_0$. By Lemma 2.3, there exists $x_2 \in Tx_1$ such that $d(x_2, x_1) \leq H(Tx_1, Sx_0) + \varepsilon M(x_1, x_0)$. Again by using Lemma 2.3, there exists $x_3 \in Sx_2$ such that $d(x_3, x_2) \leq H(Sx_2, Tx_1) + \varepsilon M(x_2, x_1)$. By induction and using Lemma 2.3, we can find in this way a sequence $\{x_n\}$ in X such that $x_{2k+1} \in Sx_{2k}$ and

$$d(x_{2k+1}, x_{2k}) \leq H(Sx_{2k}, Tx_{2k-1}) + \varepsilon M(x_{2k}, x_{2k-1}) \tag{3.2}$$

and $x_{2k+2} \in Tx_{2k+1}$ and

$$d(x_{2k+2}, x_{2k+1}) \leq H(Tx_{2k+1}, Sx_{2k}) + \varepsilon M(x_{2k+1}, x_{2k}). \tag{3.3}$$

It follows that

$$\begin{aligned}
&d(x_{2n+1}, x_{2n}) \\
&\quad \le H(Tx_{2n-1}, Sx_{2n}) + \varepsilon M(x_{2n-1}, x_{2n}) \\
&\quad \le \beta M(x_{2n-1}, x_{2n}) \\
&\quad = \beta \max\left\{ d(x_{2n-1}, x_{2n}), d(x_{2n-1}, Tx_{2n-1}), d(x_{2n}, Sx_{2n}), \frac{d(x_{2n-1}, Sx_{2n}) + d(x_{2n}, Tx_{2n-1})}{2} \right\} \\
&\quad \le \beta \max\left\{ d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \frac{d(x_{2n-1}, x_{2n+1}) + 0}{2} \right\} \\
&\quad = \beta \max\{ d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}) \} \\
&\quad = \beta d(x_{2n-1}, x_{2n}),
\end{aligned} \tag{3.4}$$

since if otherwise $d(x_{2n}, x_{2n+1}) > d(x_{2n}, x_{2n-1})$, then $d(x_{2n}, x_{2n+1}) \le \beta d(x_{2n}, x_{2n+1})$ and so $d(x_{2n}, x_{2n+1}) = 0$. Hence $0 = d(x_{2n}, x_{2n+1}) > d(x_{2n}, x_{2n-1})$ and this is a contradiction.

Similarly,

$$d(x_{2n+2}, x_{2n+1}) \le \beta d(x_{2n+1}, x_{2n}). \tag{3.5}$$

From (3.4) and (3.5), we conclude that

$$d(x_{k+1}, x_k) \le \beta d(x_k, x_{k-1}), \tag{3.6}$$

for all $k \in \mathbb{N}$. Since $\beta < 1$ and (3.6) holds, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x \in X$ such that $\lim_{n \to \infty} x_n = x$.

We have

$$\begin{aligned}
d(x_{2n+2}, Sx) &\le H(Tx_{2n+1}, Sx) \\
&\le \alpha M(x_{2n+1}, x) \\
&= \alpha \max\left\{ d(x_{2n+1}, x), d(x_{2n+1}, Tx_{2n+1}), d(x, Sx), \frac{d(x_{2n+1}, Sx) + d(x, Tx_{2n+1})}{2} \right\} \\
&\le \alpha \max\left\{ d(x_{2n+1}, x), d(x_{2n+1}, x_{2n+2}), d(x, Sx), \frac{d(x_{2n+1}, Sx) + d(x, x_{2n+2})}{2} \right\}.
\end{aligned} \tag{3.7}$$

Letting $n \to \infty$ in the above inequality, we conclude that $d(x, Sx) \le \alpha d(x, Sx)$. So $d(x, Sx) = 0$. Since $Sx \in CB(X)$, we have $x \in Sx$.

Similarly, $x \in Tx$. Therefore, T and S have a common fixed point.

Furthermore, if T is single valued, then this common fixed point is unique. In fact, if x and y are two common fixed points for T and S, then

$$\begin{aligned} d(x,y) &\leq H(\{x\},Sy) \\ &= H(\{Tx\},Sy) \\ &\leq \alpha M(x,y) \\ &= \alpha \max\left\{ d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2} \right\} \\ &\leq \alpha \max\left\{ d(x,y), 0, 0, \frac{d(x,y)+d(y,x)}{2} \right\} \\ &= \alpha d(x,y). \end{aligned} \tag{3.8}$$

Since $0 \leq \alpha < 1$, $d(x,y) = 0$, and so $x = y$. □

Remark 3.2. The last part of the proof of Theorem 3.1 shows that if $S,T : X \to CB(X)$ are multivalued and x_0 is a common fixed point, and Tx_0 or Sx_0 is a singleton, then the common fixed point of T and S is unique.

By taking $T = S$ in Theorem 3.1, we get the following corollary that extends the Daffer and Kaneko theorem (Theorem 1.2).

Corollary 3.3. *Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be such that*

$$H(Tx,Ty) \leq \alpha N(x,y), \tag{3.9}$$

for some $0 \leq \alpha < 1$ and for all $x,y \in X$ (i.e., weak contraction). Then there exists $x_0 \in X$ such that $x_0 \in Tx_0$.

Example 3.4. Let $X = [0,1]$ be endowed with the Euclidean metric. Let $S,T : X \to CB(X)$ be defined by $Tx = [0, x/4]$ and $Sy = \{y/4\}$. Obviously,

$$\begin{aligned} H(Tx,Sy) &= \max\left\{ \left|\frac{y}{4} - \frac{x}{4}\right|, \frac{y}{4} \right\} \\ &\leq \frac{1}{2}\max\left\{ |y-x|, \left|y - \frac{y}{4}\right| \right\} \\ &= \frac{1}{2}\max\{d(x,y), d(y,Sy)\} \\ &\leq \frac{1}{2}M(x,y). \end{aligned} \tag{3.10}$$

So T and S have a common fixed point ($x = 0$), and since S is single valued, this fixed point is unique.

4. Extension of Rhoades and Zhang-Song's Theorems

First we extend Zhang and Song's theorem (Theorem 1.4) to the case where one of the mappings is multivalued.

Theorem 4.1. *Let (X, d) be a complete metric space and let $T : X \to X$ and $S : X \to CB(X)$ be two mappings such that for all $x, y \in X$,*

$$H(\{Tx\}, Sy) \leq M(x, y) - \varphi(M(x, y)), \tag{4.1}$$

(i.e., generalized φ-weak contractive) where $\varphi : [0, +\infty) \to [0, +\infty)$ is l.s.c. with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then there exists a unique point $x \in X$ such that $Tx = x \in Sx$.

Proof. Unicity of the common fixed point follows from (4.1).

Obviously $M(x, y) = 0$ if and only if $x = y$ is a common fixed point of T and S.

Let $x_0 \in X$ and $x_1 \in Sx_0$. Let $x_2 := Tx_1$. By Lemma 2.3, there exists $x_3 \in Sx_2$ such that

$$d(x_3, x_2) \leq H(Sx_2, \{Tx_1\}) + \frac{1}{2}\varphi(M(x_2, x_1)). \tag{4.2}$$

We let $x_4 := Tx_3$. Inductively, we let $x_{2n} := Tx_{2n-1}$, and by Lemma 2.3, we choose $x_{2n+1} \in Sx_{2n}$ such that

$$d(x_{2n+1}, x_{2n}) \leq H(Sx_{2n}, \{Tx_{2n-1}\}) + \frac{1}{2}\varphi(M(x_{2n}, x_{2n-1})). \tag{4.3}$$

We break the argument into four steps. □

Step 1. $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Proof. Using (4.1) and (4.3),

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &\leq H(\{Tx_{2n-1}\}, Sx_{2n}) + \frac{1}{2}\varphi(M(x_{2n-1}, x_{2n})) \\ &\leq M(x_{2n-1}, x_{2n}) - \frac{1}{2}\varphi(M(x_{2n-1}, x_{2n})), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
&d(x_{2n-1}, x_{2n}) \\
&\quad \le M(x_{2n-1}, x_{2n}) \\
&\quad = \max\left\{ d(x_{2n-1}, x_{2n}), d(x_{2n-1}, Tx_{2n-1}), d(x_{2n}, Sx_{2n}), \frac{d(x_{2n-1}, Sx_{2n}) + d(x_{2n}, Tx_{2n-1})}{2} \right\} \\
&\quad \le \max\left\{ d(x_{2n-1}, x_{2n}), d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \frac{d(x_{2n-1}, x_{2n+1}) + 0}{2} \right\} \\
&\quad = \max\{ d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}) \} \\
&\quad = d(x_{2n-1}, x_{2n}) \quad (\text{by } (4.4)).
\end{aligned} \tag{4.5}$$

So $M(x_{2n-1}, x_{2n}) = d(x_{2n-1}, x_{2n})$. Hence by (4.4),

$$d(x_{2n+1}, x_{2n}) \le d(x_{2n}, x_{2n-1}). \tag{4.6}$$

Also

$$\begin{aligned}
d(x_{2n+2}, x_{2n+1}) &= d(Tx_{2n+1}, x_{2n+1}) \\
&\le H(\{Tx_{2n+1}\}, Sx_{2n}) \\
&\le M(x_{2n+1}, x_{2n}) - \varphi(M(x_{2n+1}, x_{2n})),
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
&d(x_{2n+1}, x_{2n}) \\
&\quad \le M(x_{2n+1}, x_{2n}) \\
&\quad = \max\left\{ d(x_{2n+1}, x_{2n}), d(x_{2n+1}, Tx_{2n+1}), d(x_{2n}, Sx_{2n}), \frac{d(x_{2n+1}, Sx_{2n}) + d(x_{2n}, Tx_{2n+1})}{2} \right\} \\
&\quad \le \max\left\{ d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}), \frac{0 + d(x_{2n}, x_{2n+2})}{2} \right\} \\
&\quad = \max\{ d(x_{2n+1}, x_{2n}), d(x_{2n+1}, x_{2n+2}) \} \\
&\quad = d(x_{2n+1}, x_{2n}) \quad (\text{by } (4.7)).
\end{aligned} \tag{4.8}$$

So $M(x_{2n+1}, x_{2n}) = d(x_{2n+1}, x_{2n})$. Hence by (4.7),

$$d(x_{2n+2}, x_{2n+1}) \le d(x_{2n+1}, x_{2n}). \tag{4.9}$$

Therefore, by (4.6) and (4.9), we conclude that

$$d(x_{k+1}, x_k) \leq d(x_k, x_{k-1}), \tag{4.10}$$

for all $k \in \mathbb{N}$.

Therefore, the sequence $\{d(x_{k+1}, x_k)\}$ is monotone nonincreasing and bounded below. So there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} M(x_{n+1}, x_n) = r. \tag{4.11}$$

Since φ is l.s.c.,

$$\varphi(r) \leq \liminf_{n\to\infty} \varphi(M(x_n, x_{n-1})) \leq \liminf_{n\to\infty} \varphi(M(x_{2n-1}, x_{2n})). \tag{4.12}$$

By (4.4), we conclude that

$$r \leq r - \frac{1}{2}\varphi(r), \tag{4.13}$$

and so $\varphi(r) = 0$. Hence $r = 0$. □

Step 2. $\{x_n\}$ is a bounded sequence.

Proof. If $\{x_n\}$ were unbounded, then by Step 1, $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are unbounded. We choose the sequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(1) = 1$, $n(2) > n(1)$ is even and minimal in the sense that $d(x_{n(2)}, x_{n(1)}) > 1$, and $d(x_{n(2)-2}, x_{n(1)}) \leq 1$, and similarly $n(3) > n(2)$ is odd and minimal in the sense that $d(x_{n(3)}, x_{n(2)}) > 1$, and $d(x_{n(3)-2}, x_{n(2)}) \leq 1, \ldots, n(2k) > n(2k-1)$ is even and minimal in the sense that $d(x_{n(2k)}, x_{n(2k-1)}) > 1$ and $d(x_{n(2k)-2}, x_{n(2k-1)}) \leq 1$, and $n(2k+1) > n(2k)$ is odd and minimal in the sense that $d(x_{n(2k+1)}, x_{n(2k)}) > 1$ and $d(x_{n(2k+1)-2}, x_{n(2k)}) \leq 1$.

Obviously $n(k) \geq k$ for every $k \in \mathbb{N}$. By Step 1, there exists $N_0 \in \mathbb{N}$ such that for all $k \geq N_0$ we have $d(x_{k+1}, x_k) < 1/4$. So for every $k \geq N_0$, we have $n(k+1) - n(k) \geq 2$ and

$$\begin{aligned} 1 &< d\left(x_{n(k+1)}, x_{n(k)}\right) \\ &\leq d\left(x_{n(k+1)}, x_{n(k+1)-1}\right) + d\left(x_{n(k+1)-1}, x_{n(k+1)-2}\right) + d\left(x_{n(k+1)-2}, x_{n(k)}\right) \\ &\leq d\left(x_{n(k+1)}, x_{n(k+1)-1}\right) + d\left(x_{n(k+1)-1}, x_{n(k+1)-2}\right) + 1. \end{aligned} \tag{4.14}$$

Hence $\lim_{k\to\infty} d(x_{n(k+1)}, x_{n(k)}) = 1$. Also

$$\begin{aligned} 1 &< d\left(x_{n(k+1)}, x_{n(k)}\right) \\ &\leq d\left(x_{n(k+1)}, x_{n(k+1)+1}\right) + d\left(x_{n(k+1)+1}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) \\ &\leq d\left(x_{n(k+1)}, x_{n(k+1)+1}\right) + d\left(x_{n(k+1)+1}, x_{n(k+1)}\right) + d\left(x_{n(k+1)}, x_{n(k)}\right) \\ &\quad + d\left(x_{n(k)}, x_{n(k+1)}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) \\ &\leq 2d\left(x_{n(k+1)}, x_{n(k+1)+1}\right) + d\left(x_{n(k+1)}, x_{n(k)}\right) + 2d\left(x_{n(k)+1}, x_{n(k)}\right), \end{aligned} \tag{4.15}$$

and this shows that $\lim_{k\to\infty} d(x_{n(k+1)+1}, x_{n(k)+1}) = 1$.

So if $n(k+1)$ is odd, then

$$d\left(x_{n(k+1)+1}, x_{n(k)+1}\right) \leq M\left(x_{n(k+1)}, x_{n(k)}\right) - \varphi\left(M\left(x_{n(k+1)}, x_{n(k)}\right)\right), \tag{4.16}$$

where

$$\begin{aligned}
1 < d\left(x_{n(k+1)}, x_{n(k)}\right) &\leq M\left(x_{n(k+1)}, x_{n(k)}\right) \\
&= \max\left\{ d\left(x_{n(k+1)}, x_{n(k)}\right), d\left(x_{n(k+1)}, Tx_{n(k+1)}\right), d\left(x_{n(k)}, Sx_{n(k)}\right), \right. \\
&\qquad \left. \frac{d\left(x_{n(k+1)}, Sx_{n(k)}\right) + d\left(x_{n(k)}, Tx_{n(k+1)}\right)}{2} \right\} \\
&\leq \max\left\{ d\left(x_{n(k+1)}, x_{n(k)}\right), d\left(x_{n(k+1)}, x_{n(k+1)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right), \right. \\
&\qquad \left. \frac{d\left(x_{n(k+1)}, x_{n(k)+1}\right) + d\left(x_{n(k)}, x_{n(k+1)+1}\right)}{2} \right\} \\
&\leq \max\left\{ d\left(x_{n(k+1)}, x_{n(k)}\right), d\left(x_{n(k+1)}, x_{n(k+1)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right), \right. \\
&\qquad \left. \frac{2d\left(x_{n(k+1)}, x_{n(k)}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) + d\left(x_{n(k+1)+1}, x_{n(k+1)}\right)}{2} \right\},
\end{aligned} \tag{4.17}$$

and this shows that $\lim_{k \to \infty} M(x_{n(k+1)}, x_{n(k)}) = 1$. Since φ is l.s.c. and (4.16) holds, we have $1 \leq 1 - \varphi(1)$. So $\varphi(1) = 0$ and this is a contradiction. □

Step 3. $\{x_n\}$ is Cauchy.

Proof. Let $C_n = \sup\{d(x_i, x_j) : i, j \geq n\}$. Since $\{x_n\}$ is bounded, $C_n < +\infty$ for all $n \in \mathbb{N}$. Obviously $\{C_n\}$ is decreasing. So there exists $C \geq 0$ such that $\lim_{n \to \infty} C_n = C$. We need to show that $C = 0$.

For every $k \in \mathbb{N}$, there exists $n(k), m(k) \in \mathbb{N}$ such that $m(k) > n(k) \geq k$ and

$$C_k - \frac{1}{k} \leq d\left(x_{m(k)}, x_{n(k)}\right) \leq C_k. \tag{4.18}$$

By (4.18), we conclude that

$$\lim_{k \to \infty} d\left(x_{m(k)}, x_{n(k)}\right) = C. \tag{4.19}$$

From Step 1 and (4.19), we have

$$\begin{aligned}
\lim_{k \to \infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) &= \lim_{k \to \infty} d\left(x_{m(k)+1}, x_{n(k)}\right) \\
&= \lim_{k \to \infty} d\left(x_{m(k)}, x_{n(k)+1}\right) = \lim_{k \to \infty} d\left(x_{m(k)}, x_{n(k)}\right) = C.
\end{aligned} \tag{4.20}$$

So we may assume that for every $k \in \mathbb{N}$, $m(k)$ is odd and $n(k)$ is even. Hence

$$\begin{aligned} d\left(x_{m(k)+1}, x_{n(k)+1}\right) &= d\left(Tx_{m(k)}, x_{n(k)+1}\right) \\ &\le H\left(\left\{Tx_{m(k)}\right\}, Sx_{n(k)}\right) \\ &\le M\left(x_{m(k)}, x_{n(k)}\right) - \varphi\left(M\left(x_{m(k)}, x_{n(k)}\right)\right), \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} d\left(x_{m(k)}, x_{n(k)}\right) &\le M\left(x_{m(k)}, x_{n(k)}\right) \\ &= \max\Bigg\{ d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)}, Tx_{m(k)}\right), d\left(x_{n(k)}, Sx_{n(k)}\right), \\ &\qquad \frac{d\left(x_{m(k)}, Sx_{n(k)}\right) + d\left(x_{n(k)}, Tx_{m(k)}\right)}{2} \Bigg\} \\ &\le \max\Bigg\{ d\left(x_{m(k)}, x_{n(k)}\right), d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)}, x_{n(k)+1}\right), \\ &\qquad \frac{d\left(x_{m(k)}, x_{n(k)+1}\right) + d\left(x_{n(k)}, x_{m(k)+1}\right)}{2} \Bigg\}. \end{aligned} \tag{4.22}$$

This inequality shows that $\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = C$. Since φ is l.s.c. and (4.21) holds, we have $C \le C - \varphi(C)$. Hence $\varphi(C) = 0$ and so $C = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence. □

Step 4. T and S have a common fixed point.

Proof. Since (X, d) is complete and $\{x_n\}$ is Cauchy, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. For every $n \in \mathbb{N}$

$$\begin{aligned} d(x_{2n+2}, Sx) = d(Tx_{2n+1}, Sx) &\le H(\{Tx_{2n+1}\}, Sx) \\ &\le M(x_{2n+1}, x) - \varphi(M(x_{2n+1}, x)), \end{aligned} \tag{4.23}$$

where

$$\begin{aligned} &M(x_{2n+1}, x) \\ &\quad = \max\left\{ d(x_{2n+1}, x), d(x_{2n+1}, Tx_{2n+1}), d(x, Sx), \frac{d(x_{2n+1}, Sx) + d(x, Tx_{2n+1})}{2} \right\} \\ &\quad = \max\left\{ d(x_{2n+1}, x), d(x_{2n+1}, x_{2n+2}), d(x, Sx), \frac{d(x_{2n+1}, Sx) + d(x, x_{2n+2})}{2} \right\}, \end{aligned} \tag{4.24}$$

and this shows that $\lim_{n\to\infty} M(x_{2n+1}, x) = d(x, Sx)$.

Since φ is l.s.c. and (4.23) holds, letting $n \to \infty$ in (4.23) we get

$$d(x, Sx) \le d(x, Sx) - \varphi(d(x, Sx)). \tag{4.25}$$

So $\varphi(d(x, Sx)) = 0$ and hence $d(x, Sx) = 0$. Since $Sx \in CB(X)$, then $x \in Sx$.Also

$$d(Tx, x) \leq H(\{Tx\}, Sx) \leq M(x, x) - \varphi(M(x, x)), \tag{4.26}$$

where

$$M(x, x) = \max\left\{d(x, x), d(x, Tx), d(x, Sx), \frac{d(x, Sx) + d(x, Tx)}{2}\right\} = d(x, Tx). \tag{4.27}$$

So from (4.26), we have

$$d(Tx, x) \leq d(Tx, x) - \varphi(d(Tx, x)). \tag{4.28}$$

Thus $\varphi(d(Tx, x)) = 0$, and hence $d(Tx, x) = 0$. Therefore, $x = Tx$. □

Remark 4.2. In the proof of Theorem 2.1 in Zhang and Song [8], the boundedness of the sequence $\{C_n\}$ is used, but not proved. Also, for the proof that $\{x_n\}$ is a Cauchy sequence, the monotonicity of φ is used, without being explicitly mentioned.

In our proof of Theorem 4.1, which is different from [8, Theorem 2.1], φ is not assumed to be nondecreasing.

The following theorem extends Rhoades' theorem by assuming φ to be only l.s.c..

Theorem 4.3. *Let (X, d) be a complete metric space, and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{4.29}$$

for every $x, y \in X$ (i.e., φ-weak contractive), where $\varphi : [0, +\infty) \to [0, +\infty)$ is an l.s.c. function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then T has a unique fixed point.

Proof. The proof is similar to the proof of Theorem 4.1, by taking $S = T$, and replacing $M(x, y)$ with $d(x, y)$. □

Remark 4.4. With a similar proof as in Theorem 4.1, in Theorem 4.3 we can replace the inequality (4.29) by the following inequality (4.30) for two single valued mappings $T, S : X \to X$.

$$d(Tx, Sy) \leq M(x, y) - \varphi(d(x, y)). \tag{4.30}$$

5. Conclusion and Future Directions

We have extended Nadler and Daffer-Kaneko's theorems to a coincidence theorem without assuming the lower semicontinuity of the mapping $x \mapsto d(x, Tx)$.

We have also extended Rhoades' theorem by assuming φ to be only l.s.c., as well as Zhang and Song's theorem to the case where one of the mappings is multivalued. Future directions to be pursued in the context of this research include the investigation of the case where both mappings in Zhang and Song's theorem are multivalued.

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References

[1] P. Z. Daffer and H. Kaneko, "Fixed points of generalized contractive multi-valued mappings," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 2, pp. 655–666, 1995.
[2] C. Chifu and G. Petrusel, "Existence and data dependence of fixed points and strict fixed points for contractive-type multivalued operators," *Fixed Point Theory and Applications*, vol. 2007, Article ID 34248, 8 pages, 2007.
[3] S. B. Nadler, "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.
[4] N. Shahzad and A. Lone, "Fixed points of multimaps which are not necessarily nonexpansive," *Fixed Point Theory and Applications*, no. 2, pp. 169–176, 2005.
[5] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 4, pp. 2683–2693, 2001.
[6] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 458–464, 1969.
[7] S. Reich, "Some fixed point problems," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 57, no. 3-4, pp. 194–198, 1974.
[8] Q. Zhang and Y. Song, "Fixed point theory for generalized φ-weak contractions," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 75–78, 2009.
[9] N. A. Assad and W. A. Kirk, "Fixed point theorems for set-valued mappings of contractive type," *Pacific Journal of Mathematics*, vol. 43, pp. 553–562, 1972.