

Research Article

Hyers-Ulam Stability of Nonlinear Integral Equation

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We will apply the successive approximation method for proving the Hyers-Ulam stability of a nonlinear integral equation.

1. Introduction

We say a functional equation is stable if, for every approximate solution, there exists an exact solution near it. In 1940, Ulam posed the following problem concerning the stability of functional equations [1]: we are given a group G and a metric group G' with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \rightarrow G'$ satisfies

$$\rho(f(xy), f(x)f(y)) < \delta, \quad (1.1)$$

for all $x, y \in G$, then a homomorphism $h : G \rightarrow G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$? The problem for the case of the approximately additive mappings was solved by Hyers [2] when G and G' are Banach space. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [3–5]). Recently, Y. Li and L. Hua proved the stability of Banach's fixed point theorem [6]. The interested reader can also find further details in the book of Kuczma (see [7, chapter XVII]). Examples of some recent developments, discussions, and critiques of that idea of stability can be found, for example, in [8–12].

In this paper, we study the Hyers-Ulam stability for the nonlinear Volterra integral equation of second kind. Jung was the author who investigated the Hyers-Ulam stability of Volterra integral equation on any compact interval. In 2007, he proved the following [13].

Given $a \in \mathbb{R}$ and $r > 0$, let $I(a; r)$ denote a closed interval $\{x \in \mathbb{R} \mid a - r \leq x \leq a + r\}$ and let $f : I(a; r) \times \mathbb{C} \rightarrow \mathbb{C}$ be a continuous function which satisfies a Lipschitz condition $|f(x, y) - f(x, z)| \leq L|y - z|$ for all $x \in I(a; r)$ and $y, z \in \mathbb{C}$, where L is a constant with $0 < Lr < 1$. If a continuous function $y : I(a; r) \rightarrow \mathbb{C}$ satisfies

$$\left| y(x) - b - \int_a^x f(x, t, u(t)) dt \right| \leq \epsilon, \quad (1.2)$$

for all $x \in I(a; r)$ and for some $\epsilon \geq 0$, where b is a complex number, then there exists a unique continuous function $u : I(a; r) \rightarrow \mathbb{C}$ such that

$$y(x) = b + \int_a^x f(x, t, u(t)) dt, \quad |u(x) - y(x)| \leq \frac{\epsilon}{1 - Lr}, \quad (1.3)$$

for all $x \in I(a; r)$.

The purpose of this paper is to discuss the Hyers-Ulam stability of the following nonhomogeneous nonlinear Volterra integral equation:

$$u(x) = f(x) + \varphi \left(\int_a^x F(x, t, u(t)) dt \right) \equiv Tu, \quad (1.4)$$

where $x \in I = [a, b]$, $-\infty < a < b < \infty$. We will use the successive approximation method, to prove that (1.4) has the Hyers-Ulam stability under some appropriate conditions. The method of this paper is distinctive. This new technique is simpler and clearer than methods which are used in some papers, (cf. [13, 14]). On the other hand, Hyers-Ulam stability constant obtained in our paper is different to the other works, [13].

2. Basic Concepts

Consider the nonhomogeneous nonlinear Volterra integral equation (1.4). We assume that $f(x)$ is continuous on the interval $[a, b]$ and $F(x, t, u(t))$ is continuous with respect to the three variables x , t , and u on the domain $D = \{(x, t, u) : x \in [a, b], t \in [a, b], u(t) \in [c, d]\}$; and $F(x, t, u(t))$ is Lipschitz with respect to u . In this paper, we consider the complete metric space $(X := C[a, b], \|\cdot\|_\infty)$ and assume that φ is a bounded linear transformation on X .

Note that, the linear mapping $\varphi : X \rightarrow X$ is called bounded, if there exists $M > 0$ such that $\|\varphi x\| \leq M\|x\|$, for all $x \in X$. In this case, we define $\|\varphi\| = \sup\{(\|\varphi x\|/\|x\|); x \neq 0, x \in X\}$. Thus φ is bounded if and only if $\|\varphi\| < \infty$, [15].

Definition 2.1 (cf. [5, 13]). One says that (1.4) has the Hyers-Ulam stability if there exists a constant $K \geq 0$ with the following property: for every $\epsilon > 0$, $y \in X$, if

$$\left| y(x) - f(x) - \varphi \left(\int_a^x F(x, t, y(t)) dt \right) \right| \leq \epsilon, \quad (2.1)$$

then there exists some $u \in X$ satisfying $u(x) = f(x) + \varphi(\int_a^x F(x, t, u(t))dt)$ such that

$$|u(x) - y(x)| \leq K\varepsilon. \quad (2.2)$$

We call such K a Hyers-Ulam stability constant for (1.4).

3. Existence of the Solution of Nonlinear Integral Equations

Consider the iterative scheme

$$u_{n+1}(x) = f(x) + \varphi\left(\int_a^x F(x, t, u_n(t))dt\right) \equiv Tu_n, \quad n = 1, 2, \dots \quad (3.1)$$

Since $F(x, t, u(t))$ is assumed Lipschitz, we can write

$$\begin{aligned} |u_{n+1}(x) - u_n(x)| &= \left| \varphi\left(\int_a^x F(x, t, u_n(t))dt\right) - \varphi\left(\int_a^x F(x, t, u_{n-1}(t))dt\right) \right| \\ &= \left| \varphi\left(\int_a^x F(x, t, u_n(t))dt - \int_a^x F(x, t, u_{n-1}(t))dt\right) \right| \\ &\leq \|\varphi\| \int_a^x |F(x, t, u_n(t)) - F(x, t, u_{n-1}(t))| dt \\ &\leq \|\varphi\| L \int_a^x |u_n(t) - u_{n-1}(t)| dt. \end{aligned} \quad (3.2)$$

Hence,

$$\begin{aligned} |u_{n+1}(x) - u_n(x)| &\leq \|\varphi\| L \int_a^x |u_n(t_1) - u_{n-1}(t_1)| dt_1 \\ &\leq (\|\varphi\| L)^2 \int_a^x \int_a^{t_1} |u_{n-1}(t_2) - u_{n-2}(t_2)| dt_2 dt_1 \\ &\vdots \\ &\leq (\|\varphi\| L)^{n-1} \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-2}} |u_2(t_{n-1}) - u_1(t_{n-1})| dt_{n-1} \cdots dt_2 dt_1 \\ &\leq (\|\varphi\| L)^{n-1} d(Tu_1, u_1) \int_a^x \int_a^{t_1} \cdots \int_a^{t_{n-2}} dt_{n-1} \cdots dt_2 dt_1, \end{aligned} \quad (3.3)$$

in which $d(f, g) = \max_{x \in [a, b]} |f(x) - g(x)|$, for all $f, g \in C[a, b]$. So, we can write

$$|u_{n+1}(x) - u_n(x)| \leq (\|\varphi\| L)^{n-1} \frac{(x-a)^{n-1}}{(n-1)!} d(Tu_1, u_1). \quad (3.4)$$

Therefore, since x is complete metric space, if $u_1 \in X$, then

$$\sum_{n=1}^{\infty} [u_{n+1}(x) - u_n(x)] \quad (3.5)$$

is absolutely and uniformly convergent by Weirstrass's M-test theorem. On the other hand, $u_n(x)$ can be written as follows:

$$u_n(x) = u_1(x) + \sum_{k=1}^{n-1} [u_{k+1}(x) - u_k(x)]. \quad (3.6)$$

So there exists a unique solution $u \in X$ such that $\lim_{n \rightarrow \infty} u_n(x) = u$. Now by taking the limit of both sides of (3.1), we have

$$\begin{aligned} u &= \lim_{n \rightarrow \infty} u_{n+1}(x) = \lim_{n \rightarrow \infty} \left(f(x) + \varphi \left(\int_a^x F(x, t, u_n(t)) dt \right) \right) \\ &= f(x) + \varphi \left(\int_a^x F \left(x, t, \lim_{n \rightarrow \infty} u_n(t) \right) dt \right) \\ &= f(x) + \varphi \left(\int_a^x F(x, t, u(t)) dt \right). \end{aligned} \quad (3.7)$$

So, there exists a unique solution $u \in X$ such that $Tu = u$.

4. Main Results

In this section, we prove that the nonlinear integral equation in (1.4) has the Hyers-Ulam stability.

Theorem 4.1. *The equation $Tx = x$, where T is defined by (1.4), has the Hyers-Ulam stability; that is, for every $\xi \in X$ and $\epsilon > 0$ with*

$$d(T\xi, \xi) \leq \epsilon, \quad (4.1)$$

there exists a unique $u \in X$ such that

$$\begin{aligned} Tu &= u, \\ d(\xi, u) &\leq K\epsilon, \end{aligned} \quad (4.2)$$

for some $K \geq 0$.

Proof. Let $\xi \in X$, $\epsilon > 0$, and $d(T\xi, \xi) \leq \epsilon$. In the previous section we have proved that

$$u(t) \equiv \lim_{n \rightarrow \infty} T^n \xi(t) \quad (4.3)$$

is an exact solution of the equation $Tx = x$. Clearly there is n with $d(T^n \xi, u) \leq \epsilon$, because $T^n \xi$ is uniformly convergent to u as $n \rightarrow \infty$. Thus

$$\begin{aligned}
 d(\xi, u) &\leq d(\xi, T^n \xi) + d(T^n \xi, u) \\
 &\leq d(\xi, T\xi) + d(T\xi, T^2\xi) + d(T^2\xi, T^3\xi) + \cdots + d(T^{n-1}\xi, T^n\xi) + d(T^n\xi, u) \\
 &\leq d(\xi, T\xi) + \frac{k}{1!}d(\xi, T\xi) + \frac{k^2}{2!}d(\xi, T\xi) + \cdots + \frac{k^{n-1}}{(n-1)!}d(\xi, T\xi) + d(T^n\xi, u) \quad (4.4) \\
 &\leq d(\xi, T\xi) \left(1 + \frac{k}{1!} + \frac{k^2}{2!} + \cdots + \frac{k^{n-1}}{(n-1)!} \right) + \epsilon \\
 &\leq \epsilon(e^k) + \epsilon = (1 + e^k)\epsilon,
 \end{aligned}$$

where $k = \|\varphi\|L(b-a)$. This completes the proof. \square

Corollary 4.2. *For infinite interval, Theorem 4.1 is not true necessarily. For example, the exact solution of the integral equation $u(x) = 1 + \int_a^x u(t)dt \equiv T(u)$, $x \in [0, \infty)$, is $u(x) = e^x$. By choosing $\epsilon = 1$ and $\xi(x) = 0$, $T(\xi) = 1$ is obtained, so $d(T\xi, \xi) \leq \epsilon = 1$, $d(\xi, u) = \infty$. Hence, there exists no Hyers-Ulam stability constant $K \geq 0$ such that the relation $d(\xi, u) \leq K\epsilon$ is true.*

Corollary 4.3. *Theorem 4.1 holds for every finite interval $[a, b]$, $[a, b)$, $(a, b]$, and (a, b) , when $-\infty < a < b < \infty$.*

Corollary 4.4. *If one applies the successive approximation method for solving (1.4) and $u_i(x) = u_{i+1}(x)$ for some $i = 1, 2, \dots$, then $u(x) = u_i(x)$, such that $u(x)$ is the exact solution of (1.4).*

Example 4.5. If we put $F(x, t, u(t)) = K(x, t)u(t)$ and $\varphi(x) = \lambda x$ (λ is constant), (1.4) will be a linear Volterra integral equation of second kind in the following form:

$$u(x) = f(x) + \lambda \int_a^x k(x, t)u(t)dt. \quad (4.5)$$

In this example, if $|k(x, t)| < M$ on square $R = \{(x, y) : x \in [a, b], y \in [a, b]\}$, then $F(x, t, u(t)) = K(x, t)u(t)$ satisfies in the Lipschitz condition, where M is the Lipschitz constant. Also $\|\varphi\| = |\lambda|$; therefore, if $|\lambda| < \infty$, the Volterra equation (4.5) has the Hyers-Ulam stability.

5. Conclusions

Let $I = [a, b]$ be a finite interval, and let $X = C[a, b]$ and $y = Ty$ be integral equations in which $T : X \rightarrow X$ is a nonlinear integral map. In this paper, we showed that T has the Hyers-Ulam stability; that is, if y° is an approximate solution of the integral equation and $d(y^\circ, Ty^\circ) \leq \epsilon$ for all $t \in I$ and $\epsilon \geq 0$, then $d(y^*, y^\circ) \leq K\epsilon$, in which y^* is the exact solution and K is positive constant.

6. Ideas

In this paper, we proved that (1.4) has the Hyers-Ulam stability. In (1.4), φ is a linear transformation. It is an open problem that raises the following question: “What can we say about the Hyers-Ulam stability of the general nonlinear Volterra integral equation (1.4) when φ is not necessarily linear?”

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