Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 859032, 15 pages
doi:10.1155/2011/859032

Research Article

Strong Convergence Theorems for an Infinite Family of Equilibrium Problems and Fixed Point Problems for an Infinite Family of Asymptotically Strict Pseudocontractions

Shenghua Wang,[1] Shin Min Kang,[2] and Young Chel Kwun[3]

[1] *School of Applied Mathematics and Physics, North China Electric Power University, Baoding 071003, China*
[2] *Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea*
[3] *Department of Mathematics, Dong-A University, Pusan 614-714, Republic of Korea*

Correspondence should be addressed to Young Chel Kwun, yckwun@dau.ac.kr

Received 12 October 2010; Accepted 29 January 2011

Academic Editor: Jong Kim

We prove a strong convergence theorem for an infinite family of asymptotically strict pseudo-contractions and an infinite family of equilibrium problems in a Hilbert space. Our proof is simple and different from those of others, and the main results extend and improve those of many others.

1. Introduction

Let C be a closed convex subset of a Hilbert space H. Let $S : C \to H$ be a mapping and if there exists an element $x \in C$ such that $x = Sx$, then x is called a *fixed point* of S. The set of fixed points of S is denoted by $F(S)$. Recall that

(1) S is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C, \tag{1.1}$$

(2) S is called *asymptotically nonexpansive* [1] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\|S^n x - S^n y\| \le k_n \|x - y\|, \quad \forall x, y \in C, \ n \ge 1, \tag{1.2}$$

(3) S is called to be a *κ-strict pseudo-contraction* [2] if there exists a constant κ with $0 \le \kappa < 1$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa\|(x - y) - (Sx - Sy)\|^2, \quad \forall x, y \in C, \tag{1.3}$$

(4) S is called an *asymptotically κ-strict pseudo-contraction* [3, 4] if there exists a constant κ with $0 \le \kappa < 1$ and a sequence $\{\gamma_n\} \subset [0, \infty)$ with $\lim_{n\to\infty}\gamma_n = 0$ such that

$$\|S^n x - S^n y\|^2 \le (1 + \gamma_n)\|x - y\|^2 + \kappa\|(x - y) - (S^n x - S^n y)\|^2, \quad \forall x, y \in C, \ n \ge 1. \tag{1.4}$$

It is clear that every asymptotically nonexpansive mapping is an asymptotically 0-strict pseudo-contraction and every κ-strict pseudo-contraction is an asymptotically κ-strict pseudo-contraction with $\gamma_n = 0$ for all $n \ge 1$. Moreover, every asymptotically κ-strict pseudo-contraction with sequence $\{\gamma_n\}$ is uniformly L-Lispchitzian, where $L = \sup\{(\kappa + \sqrt{1 + \gamma_n(1 - \kappa)})/(1 - \kappa) : n \ge 1\}$ and the fixed point set of asymptotically κ-strict pseudo-contraction is closed and convex; see [3, Proposition 2.6].

Let Φ be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\Phi : C \times C \to \mathbb{R}$ is to find $x \in C$ such that $\Phi(x, y) \ge 0$ for all $y \in C$. The set of such solutions is denoted by $\mathrm{EP}(\Phi)$.

In 2007, S. Takahashi and W. Takahashi [5] first introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of the equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space H and proved a strong convergence theorem which is connected with Combettes and Hirstoaga's result [6] and Wittmann's result [7]. More precisely, they gave the following theorem.

Theorem 1.1 (see [5]). *Let C be a nonempty closed convex subset of H. Let Φ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the following assumptions:*

(A1) $\Phi(x, x) = 0$ *for all* $x \in C$;

(A2) Φ *is monotone, that is,* $\Phi(x, y) + \Phi(y, x) \le 0$ *for all* $x, y \in C$;

(A3) *for all* $x, y, z \in C$,

$$\lim_{t\downarrow 0} \Phi(tz + (1 - t)x, y) \le \Phi(x, y); \tag{1.5}$$

(A4) *for all* $x \in C$, $y \mapsto \Phi(x, y)$ *is convex and lower semicontinuous.*

Let $S : C \to H$ be a nonexpansive mapping such that $F(S) \cap \mathrm{EP}(\Phi) \neq \emptyset$, $f : H \to H$ be a contraction and $\{x_n\}$, $\{u_n\}$ be the sequences generated by

$$\begin{gathered} x_1 \in H, \\ \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S u_n, \quad \forall n \geq 1, \end{gathered} \tag{1.6}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:

$$\begin{gathered} \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \liminf_{n\to\infty} r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{gathered} \tag{1.7}$$

Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap \mathrm{EP}(\Phi)$, where $z = P_{F(S)\cap \mathrm{EP}(\Phi)} f(z)$.

In [8], Tada and Takahashi proposed a hybrid algorithm to find a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem and proved the following strong convergence theorem.

Theorem 1.2 (see [8]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let Φ be a bifunction from $C \times C \to \mathbb{R}$ satisfying (A1)–(A4) and let S be a nonexpansive mapping of C into H such that $F(S) \cap \mathrm{EP}(\Phi) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and*

$$\begin{gathered} u_n \in C \text{ such that } \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ w_n = (1-\alpha_n)x_n + \alpha_n S u_n, \\ C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x, \quad \forall n \geq 1, \end{gathered} \tag{1.8}$$

where $\{\alpha_n\} \subset [a,1]$ for some $a \in (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{F(S)\cap \mathrm{EP}(\Phi)} x$.

Many methods have been proposed to solve the equilibrium problems and fixed point problems; see [9–13].

Recently, Kim and Xu [3] proposed a hybrid algorithm for finding a fixed point of an asymptotically κ-strict pseudo-contraction and proved a strong convergence theorem in a Hilbert space.

Theorem 1.3 (see [3]). *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be an asymptotically κ-strict pseudo-contraction for some $0 \leq \kappa < 1$. Assume that $F(T)$ is nonempty and bounded. Let $\{x_n\}$ be the sequence generated by the following algorithm:*

$$
\begin{aligned}
&x_0 \in C \text{ chosen arbitrarily}, \\
&y_n = \alpha_n x_n + (1-\alpha_n) T^n x_n, \\
C_n &= \left\{ z \in H : \|y_n - z\| \leq \|x_n - z\|^2 + [\kappa - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n \right\}, \\
D_n &= \{ z \in H : \langle x_n - z, x_0 - x_n \rangle \geq 0 \}, \\
&x_{n+1} = P_{C_n \cap D_n} x_0, \quad \forall n \geq 1,
\end{aligned} \tag{1.9}
$$

where

$$
\theta_n = \Delta_n^2 (1-\alpha_n)\gamma_n \longrightarrow 0 \quad (n \longrightarrow \infty), \qquad \Delta_n = \sup\{\|x_n - z\| : z \in F(T)\} < \infty. \tag{1.10}
$$

Assume that the control sequence $\{\alpha_n\}$ is chosen such that $\limsup_{n\to\infty} \alpha_n < 1-\kappa$. Then $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.

In this paper, motivated by [3, 8], we propose a new algorithm for finding a common element of the set of fixed points of an infinite family of asymptotically strict pseudo-contractions and the set of solutions of an infinite family of equilibrium problems and prove a strong convergence theorem. Our proof is simple and different from those of others, and the main results extend and improve those Kim and Xu [3], Tada and Takahashi [8], and many others.

2. Preliminaries

Let H be a Hilbert space, and let C be a nonempty closed convex subset of H. It is well known that, for all $x, y \in C$ and $t \in [0,1]$,

$$
\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|, \tag{2.1}
$$

and hence

$$
\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2, \tag{2.2}
$$

which implies that

$$
\left\| \sum_{i=1}^{n} t_i x_i \right\|^2 \leq \sum_{i=1}^{n} t_i \|x_i\|^2 \tag{2.3}
$$

for all $\{x_i\} \subset H$ and $\{t_i\} \subset [0,1]$ with $\sum_{i=1}^{n} t_i = 1$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$z = P_C x \Longleftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

Let I denote the identity operator of H, and let $\{x_n\}$ be a sequence in a Hilbert space H and $x \in H$. Throughout the rest of the paper, $x_n \rightarrow x$ denotes the strong convergence of $\{x_n\}$ to x.

We need the following lemmas for our main results in this paper.

Lemma 2.1 (see [14]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let Φ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$\Phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall y \in C. \tag{2.5}$$

Lemma 2.2 (see [6]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let Φ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *For any $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r x = \left\{ z \in C : \Phi(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}, \quad \forall x \in H. \tag{2.6}$$

Then the following hold:

(1) *T_r is single-valued,*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r x - T_r y \right\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \tag{2.7}$$

(3) $F(T_r) = \mathrm{EP}(\Phi)$*, and*

(4) $\mathrm{EP}(\Phi)$ *is closed and convex.*

3. Main Results

Now, we are ready to give our main results.

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T : C \rightarrow C$ be an asymptotically κ-strict pseudo-contraction with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $F(T) \neq \emptyset$. Assume that $\{\beta_n\} \subset [\kappa, 1]$ and define a mapping $S_n = \beta_n I + (1 - \beta_n) T^n$ for each $n \geq 1$. Then the following hold:*

$$\begin{gathered} \left\| S_n x - S_n y \right\|^2 \leq (1 + \gamma_n) \left\| x - y \right\|^2, \quad \forall x, y \in C, \\ \| S_n x - x \|^2 \leq \gamma_n \| x - x^* \|^2 + 2\langle x - S_n x, x - x^* \rangle, \quad \forall x \in C, \ x^* \in F(T). \end{gathered} \tag{3.1}$$

Proof. For all $x, y \in C$, we have

$$\begin{aligned}
\left\|S_n x - S_n y\right\|^2 &= \|\beta_n(x-y) + (1-\beta_n)(T^n x - T^n y)\|^2 \\
&= \beta_n\|x-y\|^2 + (1-\beta_n)\|T^n x - T^n y\|^2 - \beta_n(1-\beta_n)\|(I-T^n)x - (I-T^n)y\|^2 \\
&\le \beta_n\|x-y\|^2 + (1-\beta_n)\left[(1+\gamma_n)\|x-y\|^2 + \kappa\|(I-T^n)x-(I-T^n)y\|^2\right] \\
&\quad - \beta_n(1-\beta_n)\|(I-T^n)x-(I-T^n)y\|^2 \\
&= \beta_n\|x-y\|^2 + (1-\beta_n)(1+\gamma_n)\|x-y\|^2 \\
&\quad + (1-\beta_n)(\kappa-\beta_n)\|(I-T^n)x-(I-T^n)y\|^2 \\
&\le \beta_n\|x-y\|^2 + (1-\beta_n)(1+\gamma_n)\|x-y\|^2 \\
&\le (1+\gamma_n)\|x-y\|^2.
\end{aligned} \tag{3.2}$$

By this result, for all $x \in C$ and $x^* \in F(T)$, we have

$$\begin{aligned}
(1+\gamma_n)\|x-x^*\|^2 \ge \|S_n x - S_n x^*\|^2 &= \|S_n x - x + x - x^*\|^2 \\
&= \|S_n x - x\|^2 + \|x-x^*\|^2 + 2\langle S_n x - x, x - x^*\rangle,
\end{aligned} \tag{3.3}$$

and hence

$$\|S_n x - x\|^2 \le \gamma_n\|x-x^*\|^2 + 2\langle x - S_n x, x - x^*\rangle. \tag{3.4}$$

This completes the proof. □

Lemma 3.2. *Let C be a nonempty closed subset of a Hilbert space H. Let $T : C \to C$ be an asymptotically κ-strict pseudo-contraction with sequence $\{\gamma_n\} \subset [0,\infty)$ satisfying $\gamma_n \to 0$ as $n \to \infty$. Let $\{z_n\}$ be a sequence in C such that $\|z_n - z_{n+1}\| \to 0$ and $\|z_n - T^n z_n\| \to 0$ as $n \to \infty$. Then $\|z_n - Tz_n\| \to 0$ as $n \to \infty$.*

Proof. The proof method of this lemma is mainly from [15, Lemma 2.7]. Since T is an asymptotically κ-strict pseudo-contraction, we obtain from [3, Proposition 2.6] that

$$\left\|T^{n+1}z_n - T^{n+1}z_{n+1}\right\| \le L\|z_n - z_{n+1}\|, \tag{3.5}$$

where $L = \sup\{(\kappa + \sqrt{1+\gamma_n(1-\kappa)})/(1-\kappa) : n \ge 1\}$. Note that $\|z_n - z_{n+1}\| \to 0$, which implies that $\|T^{n+1}z_n - T^{n+1}z_{n+1}\| \to 0$, and observe that

$$\begin{aligned}
\|z_n - Tz_n\| &\le \|z_n - z_{n+1}\| + \left\|z_{n+1} - T^{n+1}z_{n+1}\right\| + \left\|T^{n+1}z_{n+1} - T^{n+1}z_n\right\| + \left\|T^{n+1}z_n - Tz_n\right\| \\
&\le (1+L)\|z_n - z_{n+1}\| + \left\|z_{n+1} - T^{n+1}z_{n+1}\right\| + \left\|T^{n+1}z_n - Tz_n\right\|.
\end{aligned} \tag{3.6}$$

Since T is uniformly Lipschitzian, T is uniformly continuous. So we have

$$\left\|T^{n+1}z_n - Tz_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.7}$$

It follows from $\|z_n - z_{n+1}\| \to 0$ and $\|z_n - T^n z_n\| \to 0$ as $n \to \infty$ that $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$. This completes the proof. □

Let H be a Hilbert space, and, let C be a nonempty closed and convex subset of H. Let $\{\Phi_n\}$ be a countable family of bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $\{r_n\}$ be a real number sequence in (r, ∞) with $r > 0$. Define

$$T_{r_i}x = \left\{z \in C : \Phi_i(z, y) + \frac{1}{r_i}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}, \quad \forall x \in H. \tag{3.8}$$

Lemma 2.2 shows that every T_{r_i} $(i \geq 1)$ is a firmly nonexpansive mapping and hence nonexpansive and $F(T_{r_i}) = \mathrm{EP}(\Phi_i)$.

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_i\} : C \to C$ be an infinite family of asymptotically κ_i-strict pseudocontractions with the sequence $\{\gamma_{i,n}\} \subset [0, \infty)$ satisfying $\gamma_{i,n} \to 0$ as $n \to \infty$ for each $i \geq 1$ and $\gamma_{1,n} \geq \gamma_{i,n}$ for each $i \geq 1$ and $n \geq 1$. Let $\{\Phi_n\}$ be a countable family of bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Assume that $\Omega = \bigcap_{i=1}^{\infty}(F(T_i) \cap \mathrm{EP}(\Phi_i))$ is nonempty and bounded. Set $\alpha_0 = 1$ and $\theta_0 = 1$. Assume that $\{\alpha_i\}$ is a strictly decreasing sequence in $[0, a]$ for some $0 < a < 1$, $\{\theta_n\}$ is a strictly decreasing sequence in $(0, 1)$, $\{\beta_{i,n}\}$ is a sequence in $[\kappa_i, \kappa)$ with $0 < \kappa_i < \kappa < 1$ for each $i \geq 1$, and $\{r_n\}$ is a sequence in (r, ∞) with $r > 0$. The sequence $\{x_n\}$ is generated by $x_1 = x \in C$ and*

$$\begin{aligned}
z_n &= \theta_n x_n + \sum_{i=1}^{n}(\theta_{i-1} - \theta_i)T_{r_i}x_n,\\
w_n &= \alpha_n x_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\left(\beta_{i,n}I + (1 - \beta_{i,n})T_i^n\right)z_n,\\
C_n &= \{v \in C : \|w_n - v\| \leq \|x_n - v\| + \lambda_n\},\\
D_n &= \bigcap_{j=1}^{n} C_j,\\
x_{n+1} &= P_{D_n}x, \quad \forall n \geq 1,
\end{aligned} \tag{3.9}$$

where $\{T_{r_i}\}$ is defined by (3.8) and

$$\lambda_n = (1 - \alpha_n)\gamma_{1,n}\Delta_n \longrightarrow 0 \quad (n \longrightarrow \infty), \qquad \Delta_n = \sup\{\|x_n - v\| : v \in \Omega\}. \tag{3.10}$$

Then $\{x_n\}$ converges strongly to $P_\Omega x$.

Proof. We show first that the sequence $\{x_n\}$ is well defined. Obviously, C_n is closed for all $n \geq 1$. Since

$$\|w_n - v\| \leq \|x_n - v\| + \lambda_n \tag{3.11}$$

is equivalent to

$$\|w_n - x_n\|^2 + 2\langle w_n - x_n, x_n - z\rangle \leq \lambda_n, \tag{3.12}$$

C_n is convex for all $n \geq 1$. So $D_n = \bigcap_{j=1}^n C_j$ is also closed and convex for all $n \geq 1$.

For each $n \geq 1$ and $i \geq 1$, put $S_{i,n} = \beta_{i,n} I + (1 - \beta_{i,n}) T_i^n$. Let $p \in \Omega$. Note that $\theta_0 = 1$, $\{\theta_n\}$ is strictly decreasing and each T_{r_i} is firmly nonexpansive. Hence we have

$$\begin{aligned}
\|z_n - p\| &\leq \theta_n \|x_n - p\| + \sum_{i=1}^n (\theta_{i-1} - \theta_i) \|T_{r_i} x_n - p\| \\
&\leq \theta_n \|x_n - p\| + \sum_{i=1}^n (\theta_{i-1} - \theta_i) \|x_n - p\| \\
&\leq \theta_n \|x_n - p\| + (1 - \theta_n) \|x_n - p\| \\
&= \|x_n - p\|, \quad \forall n \geq 1.
\end{aligned} \tag{3.13}$$

Since $\alpha_0 = 1$ and $\{\alpha_n\}$ is strictly decreasing, by (3.13) and Lemma 3.1, we have

$$\begin{aligned}
\|w_n - p\| &\leq \alpha_n \|x_n - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \|S_{i,n} z_n - p\| \\
&\leq \alpha_n \|x_n - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) \sqrt{1 + \gamma_{i,n}} \|z_n - p\| \\
&\leq \alpha_n \|x_n - p\| + \sum_{i=1}^n (\alpha_{i-1} - \alpha_i) (1 + \gamma_{1,n}) \|x_n - p\| \\
&\leq \|x_n - p\| + \lambda_n.
\end{aligned} \tag{3.14}$$

So we have $p \in C_n$ and hence $p \in D_n = \bigcap_{j=1}^n C_j$ for all $n \geq 1$. This shows that $\Omega \subset D_n$ for all $n \geq 1$. This implies that the sequence $\{x_n\}$ is well defined.

Since Ω is a nonempty closed convex subset of H, there exists a unique $z^* \in \Omega$ such that

$$z^* = P_\Omega x. \tag{3.15}$$

From $x_{n+1} = P_{D_n} x$, we have

$$\|x_{n+1} - x\| \leq \|z - x\|, \quad \forall z \in D_n. \tag{3.16}$$

Since $z^* \in \Omega \subset D_n$, we have

$$\|x_{n+1} - x\| \leq \|z^* - x\|, \quad \forall n \geq 1. \tag{3.17}$$

Therefore, $\{x_n\}$ is bounded. From (3.13) and (3.14), $\{z_n\}$ and $\{w_n\}$ are also bounded.

From $x_{n+1} = P_{D_n}x$ and $D_{n+1} \subset D_n$, one sees that $x_{n+2} = P_{D_{n+1}}x \in D_{n+1} \subset D_n$ for all $n \geq 1$. It follows that

$$\|x_{n+1} - x\| \leq \|x_{n+2} - x\|, \quad \forall n \geq 1. \tag{3.18}$$

Since $\{x_n\}$ is bounded, the sequence $\{\|x - x_n\|\}$ is bounded and nondecreasing. So there exists $c \in \mathbb{R}$ such that

$$c = \lim_{n \to \infty} \|x - x_n\|. \tag{3.19}$$

Since $x_{n+1} = P_{D_n}x \in D_n$, $x_{n+2} = P_{D_{n+1}}x \in D_{n+1} \subset D_n$ and $(x_{n+1} + x_{n+2})/2 \in D_n$, we have

$$\begin{aligned} \|x - x_{n+1}\|^2 &\leq \left\| x - \frac{x_{n+1} + x_{n+2}}{2} \right\|^2 \\ &= \left\| \frac{1}{2}(x - x_{n+1}) + \frac{1}{2}(x - x_{n+2}) \right\|^2 \\ &= \frac{1}{2}\|x - x_{n+1}\|^2 + \frac{1}{2}\|x - x_{n+2}\|^2 - \frac{1}{4}\|x_{n+1} - x_{n+2}\|^2. \end{aligned} \tag{3.20}$$

So we get

$$\frac{1}{4}\|x_{n+1} - x_{n+2}\|^2 \leq \frac{1}{2}\|x - x_{n+2}\|^2 - \frac{1}{2}\|x - x_{n+1}\|^2. \tag{3.21}$$

Since $\lim_{n \to \infty}\|x - x_{n+1}\| = \lim_{n \to \infty}\|x - x_{n+2}\| = c$, we obtain

$$\lim_{n \to \infty} \|x_{n+1} - x_{n+2}\| = 0, \tag{3.22}$$

that is,

$$\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0. \tag{3.23}$$

Now, for each $l \geq 1$, from (3.23) we get

$$\begin{aligned} \|x_{n+l} - x_n\| &\leq \|x_{n+l} - x_{n+l-1}\| + \cdots + \|x_{n+1} - x_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.24}$$

This implies that there exists an element $\widehat{x} \in C$ such that $x_n \to \widehat{x}$ as $n \to \infty$.

Next we show that $\widehat{x} \in \bigcap_{i=1}^{\infty} F(T_i)$ and $\widehat{x} \in \bigcap_{i=1}^{\infty} \mathrm{EP}(\Phi_i)$.

From $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|x_n - w_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \\ &\leq 2\|x_n - x_{n+1}\| + \lambda_n. \end{aligned} \tag{3.25}$$

By (3.10) and (3.23), we obtain

$$\lim_{n \to \infty} \|x_n - w_n\| = 0. \tag{3.26}$$

For $p \in \Omega$, we have, from Lemma 2.2,

$$\begin{aligned} \left\|T_{r_i} x_n - p\right\|^2 &= \left\|T_{r_i} x_n - T_{r_i} p\right\|^2 \\ &\leq \left\langle T_{r_i} x_n - T_{r_i} p, x_n - p \right\rangle \\ &= \left\langle T_{r_i} x_n - p, x_n - p \right\rangle \\ &= \frac{1}{2}\left(\left\|T_{r_i} x_n - p\right\|^2 + \left\|x_n - p\right\|^2 - \|x_n - T_{r_i} x_n\|^2\right), \end{aligned} \tag{3.27}$$

and hence

$$\left\|T_{r_i} x_n - p\right\|^2 \leq \left\|x_n - p\right\|^2 - \|x_n - T_{r_i} x_n\|^2, \quad \forall i \geq 1. \tag{3.28}$$

Therefore

$$\begin{aligned} \left\|z_n - p\right\|^2 &\leq \theta_n \left\|x_n - p\right\|^2 + \sum_{i=1}^{n} (\theta_{i-1} - \theta_i) \left\|T_{r_i} x_n - p\right\|^2 \\ &\leq \theta_n \left\|x_n - p\right\|^2 + \sum_{i=1}^{n} (\theta_{i-1} - \theta_i) \left(\left\|x_n - p\right\|^2 - \|x_n - T_{r_i} x_n\|^2\right) \\ &= \left\|x_n - p\right\|^2 - \sum_{i=1}^{n} (\theta_{i-1} - \theta_i) \|x_n - T_{r_i} x_n\|^2. \end{aligned} \tag{3.29}$$

By (3.29) and Lemma 3.1, we have

$$\begin{aligned}
\|w_n - p\|^2 &\le \alpha_n \|x_n - p\|^2 + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|S_{i,n} z_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(1 + \gamma_{1,n})^2 \|z_n - p\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)(1 + \gamma_{1,n})^2 \|z_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)(1 + \gamma_{1,n})^2 \left(\|x_n - p\|^2 - \sum_{i=1}^{n} (\theta_{i-1} - \theta_i) \|x_n - T_{r_i} x_n\|^2 \right) \\
&= \|x_n - p\|^2 + (1 - \alpha_n)\left(2\gamma_{1,n} + \gamma_{1,n}^2\right) \|x_n - p\|^2 \\
&\quad - (1 - \alpha_n)(1 + \gamma_{1,n})^2 \sum_{i=1}^{n} (\theta_{i-1} - \theta_i) \|x_n - T_{r_i} x_n\|^2,
\end{aligned} \tag{3.30}$$

and hence

$$\begin{aligned}
&(1 - \alpha_n)(1 + \gamma_{1,n})^2 \sum_{i=1}^{n} (\theta_{i-1} - \theta_i) \|x_n - T_{r_i} x_n\|^2 \\
&\quad \le \|x_n - p\|^2 - \|w_n - p\|^2 + (1 - \alpha_n)\left(2\gamma_{1,n} + \gamma_{1,n}^2\right) \|x_n - p\|^2 \\
&\quad \le \|x_n - w_n\| \left(\|x_n - p\| + \|w_n - p\| \right) + (1 - \alpha_n)\left(2\gamma_{1,n} + \gamma_{1,n}^2\right) \|x_n - p\|^2.
\end{aligned} \tag{3.31}$$

This shows that

$$\begin{aligned}
&(1 - \alpha_n)(1 + \gamma_{1,n})^2 (\theta_{i-1} - \theta_i) \|x_n - T_{r_i} x_n\|^2 \\
&\quad \le \|x_n - w_n\| \left(\|x_n - p\| + \|w_n - p\| \right) \\
&\qquad + (1 - \alpha_n)\left(2\gamma_{1,n} + \gamma_{1,n}^2\right) \|x_n - p\|^2, \quad \forall i \ge 1.
\end{aligned} \tag{3.32}$$

Since $\{\alpha_n\} \subset [0, a]$ with $0 < a < 1$, $\gamma_{1,n} \to 0$, $\{\theta_n\}$ is strictly decreasing and $\|x_n - w_n\| \to 0$, we get

$$\lim_{n \to \infty} \|x_n - T_{r_i} x_n\| = 0, \quad \forall i \ge 1. \tag{3.33}$$

Let $M_n = \sup_{i\geq 1}\{\|x_n - T_{r_i}x_n\|\}$ for each $n \geq 1$. Then $M_n \to 0$ as $n \to \infty$. Hence, from (3.33), one has

$$\begin{aligned}\|x_n - z_n\| &\leq \sum_{i=1}^{n}(\theta_{i-1} - \theta_i)\|T_{r_i}x_n - x_n\| \\ &\leq \sum_{i=1}^{n}(\theta_{i-1} - \theta_i)M_n = (1-\theta_n)M_n \\ &\longrightarrow 0.\end{aligned} \tag{3.34}$$

From (3.26) and (3.34), we obtain

$$\|z_n - w_n\| \leq \|z_n - x_n\| + \|x_n - w_n\| \longrightarrow 0. \tag{3.35}$$

Noting that

$$\begin{aligned}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(z_n - S_{i,n}z_n) &= \alpha_n x_n + (1-\alpha_n)z_n - w_n \\ &= \alpha_n(x_n - w_n) + (1-\alpha_n)(z_n - w_n),\end{aligned} \tag{3.36}$$

we have

$$\begin{aligned}&\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle z_n - S_{i,n}z_n, z_n - p\rangle \\ &\quad = \alpha_n\langle x_n - w_n, z_n - p\rangle + (1-\alpha_n)\langle z_n - w_n, z_n - p\rangle.\end{aligned} \tag{3.37}$$

By Lemma 3.1, we have

$$\begin{aligned}\|z_n - S_{i,n}z_n\|^2 &\leq \gamma_{i,n}\|z_n - p\|^2 + 2\langle z_n - S_{i,n}z_n, z_n - p\rangle \\ &\leq \gamma_{1,n}\|z_n - p\|^2 + 2\langle z_n - S_{i,n}z_n, z_n - p\rangle.\end{aligned} \tag{3.38}$$

Therefore, combining this inequality with (3.37), we get

$$\begin{aligned}&\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|z_n - S_{i,n}z_n\|^2 \\ &\quad\leq \gamma_{1,n}(1-\alpha_n)\|z_n - p\|^2 + 2\alpha_n\langle x_n - w_n, z_n - p\rangle \\ &\qquad + 2(1-\alpha_n)\langle z_n - w_n, z_n - p\rangle,\end{aligned} \tag{3.39}$$

and hence (noting that $\alpha_{i-1} > \alpha_i$ for each $i \geq 1$)

$$\|z_n - S_{i,n}z_n\|^2 \leq \frac{\gamma_{1,n}(1-\alpha_n)}{\alpha_{i-1}-\alpha_i}\|z_n - p\|^2 + \frac{2\alpha_n}{\alpha_{i-1}-\alpha_i}\langle x_n - w_n, z_n - p\rangle + \frac{2(1-\alpha_n)}{\alpha_{i-1}-\alpha_i}\langle z_n - w_n, z_n - p\rangle. \tag{3.40}$$

From (3.26), (3.35) and $\lim_{n\to\infty}\gamma_{1,n} = 0$, we have

$$\lim_{n\to\infty}\|z_n - S_{i,n}z_n\| = 0, \quad \forall i \geq 1. \tag{3.41}$$

From the definition of $S_{i,n}$ and (3.41), we have (noting that $\{\beta_{i,n}\} \subset [\kappa_i, \kappa) \subset (0,1)$)

$$\|z_n - T_i^n z_n\| \leq \frac{1}{1-\beta_{i,n}}\|z_n - S_{i,n}z_n\| \longrightarrow 0, \quad \forall i \geq 1. \tag{3.42}$$

We next show (3.42) implies that

$$\lim_{n\to\infty}\|z_n - T_i z_n\| = 0, \quad \forall i \geq 1. \tag{3.43}$$

As a matter of fact, from (3.23) and (3.34) we have

$$\|z_n - z_{n+1}\| \leq \|z_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - z_{n+1}\| \longrightarrow 0. \tag{3.44}$$

Now, (3.42), (3.44), and Lemma 3.2 imply (3.43).

Since each T_i is uniformly continuous and $z_n \to \hat{x}$ as $n \to \infty$, one get $\hat{x} \in F(T_i)$ for each $i \geq 1$ and hence $\hat{x} \in \bigcap_{i=1}^{\infty} F(T_i)$.

Now we show $\hat{x} \in \bigcap_{i=1}^{\infty} \mathrm{EP}(\Phi_i)$.

Since every T_{r_i} is nonexpansive, from (3.33) and $x_n \to \hat{x}$, we have $\hat{x} \in F(T_{r_i})$ and hence $\hat{x} \in \bigcap_{i=1}^{\infty} F(T_{r_i})$. Lemma 2.2 shows that $\hat{x} \in \bigcap_{i=1}^{\infty} \mathrm{EP}(\Phi_i)$.

Finally, we prove that $\hat{x} = P_\Omega x$. From $x_{n+1} = P_{D_n}x$, one sees

$$\langle x_{n+1} - z, x - x_{n+1}\rangle \geq 0, \quad \forall z \in D_n. \tag{3.45}$$

Since $\Omega \subset D_n$ for all $n \geq 1$, one arrives at

$$\langle x_{n+1} - z, x - x_{n+1}\rangle \geq 0, \quad \forall z \in \Omega. \tag{3.46}$$

Taking the limit for above inequality, we get

$$\langle \hat{x} - z, x - \hat{x}\rangle \geq 0, \quad \forall z \in \Omega. \tag{3.47}$$

Hence $\hat{x} = P_\Omega x$. This completes the proof. □

As direct consequences of Theorem 3.3, we can obtain the following corollaries.

Corollary 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{\Phi_n\}$ be a countable family of bifunctions from: $C\times C$ to $\mathbb{R}$ satisfying* (A1)–(A4). *Assume that $\Omega = \bigcap_{i=1}^{\infty} \mathrm{EP}(\Phi_i)$ is nonempty and bounded. Let $\{r_n\}$ be a sequence in (r,∞) with $r > 0$. Set $\theta_0 = 1$. The sequence $\{x_n\}$ is generated by $x_1 = x \in C$ and*

$$\begin{aligned} z_n &= \theta_n x_n + \sum_{i=1}^{n}(\theta_{i-1} - \theta_i)T_{r_i}x_n, \\ C_n &= \{v \in C : \|z_n - v\| \le \|x_n - v\|\}, \\ D_n &= \bigcap_{j=1}^{n} C_j, \\ x_{n+1} &= P_{D_n}x, \quad \forall n \ge 1, \end{aligned} \tag{3.48}$$

where $\{T_{r_i}\}$ is defined by (3.8) *and $\{\theta_n\}$ is a strictly decreasing sequence in* $(0,1)$. *Then $\{x_n\}$ converges strongly to $P_{\Omega}x$.*

Proof. Putting $T_i = I$ for all $i \ge 1$ and $\alpha_n = 0$ for all $n \ge 1$ in Theorem 3.3, we obtain Corollary 3.4. □

Corollary 3.5. *Let C be a nonempty closed subset of a Hilbert space H. Let T be an asymptotically κ-strict pseudo-contraction with sequence $\{\gamma_n\} \subset (0,\infty)$ satisfying $\gamma_n \to 0$ as $n \to \infty$ and $F(T) \ne \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and*

$$\begin{aligned} z_n &= \theta_n x_n + (1-\theta_n)P_C x_n, \\ w_n &= \alpha_n x_n + (1-\alpha_n)\big(\beta_n I + (1-\beta_n)T^n\big)z_n, \\ C_n &= \{v \in C : \|w_n - v\| \le \|x_n - v\|\}, \\ D_n &= \bigcap_{j=1}^{n} C_j, \\ x_{n+1} &= P_{D_n}x, \quad \forall n \ge 1, \end{aligned} \tag{3.49}$$

where $\{\theta_n\} \subset (0,1)$, $\{\alpha_n\} \subset [0,a]$ with $0 < a < 1$, and $\{\beta_n\} \subset [\kappa,\kappa')$ with $\kappa < \kappa' < 1$. Then $\{x_n\}$ converges strongly to $P_{F(T)}x$.

Proof. Put $\Phi_i(x,y) = 0$ for all $x,y \in C$ and set $r_n = 1$ for all $n \ge 1$ in Theorem 3.3. By Lemma 2.2, we have $T_{r_i}x_n = P_C x_n$ for each $i \ge 1$. Hence, by Theorem 3.3, we obtain Corollary 3.5. □

Remark 3.6. Our algorithms are of interest because the sequence $\{x_n\}$ in Theorem 3.3 is very different from the known manner. The proof is simple and different from those of others. The main results extend and improve those of Kim and Xu [3], Tada and Takahashi [8], and many others.

Remark 3.7. Put $\alpha_0 = 1$, $\theta_0 = 1$, $\kappa = 3/4$, $r = 1$, $\gamma_{i,n} = 1/4^{in}$, $\kappa_i = 1/4 + 1/(3+i)$, $\alpha_n = 1/(1+n)$, $\theta_n = 1/4 + 1/8n$, $\beta_{i,n} = 1/4 + 1/(3+i) + 1/8n$ for all $i \geq 1$ and all $n \geq 1$, $r_0 = 1$, and $r_n = 1 + 1/n$. Then these control sequences satisfy all the conditions of Theorem 3.3.

Acknowledgments

The authors thank the referees for useful comments and suggestions. This study was supported by research funds from Dong-A University.

References

[1] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 35, pp. 171–174, 1972.

[2] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[3] T.-H. Kim and H.-K. Xu, "Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 9, pp. 2828–2836, 2008.

[4] Q. Liu, "Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 11, pp. 1835–1842, 1996.

[5] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[6] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.

[7] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.

[8] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.

[9] L. C. Ceng, S. Schaible, and J. C. Yao, "Implicit iteration scheme with perturbed mapping for equilibrium problems and fixed point problems of finitely many nonexpansive mappings," *Journal of Optimization Theory and Applications*, vol. 139, no. 2, pp. 403–418, 2008.

[10] L. C. Ceng, A. Petruşel, and J. C. Yao, "Iterative approaches to solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings," *Journal of Optimization Theory and Applications*, vol. 143, no. 1, pp. 37–58, 2009.

[11] S.-S. Chang, Y. J. Cho, and J. K. Kim, "Approximation methods of solutions for equilibrium problem in Hilbert spaces," *Dynamic Systems and Applications*, vol. 17, no. 3-4, pp. 503–513, 2008.

[12] Y. J. Cho, X. Qin, and J. I. Kang, "Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 4203–4214, 2009.

[13] X. Qin, Y. J. Cho, and S. M. Kang, "Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 20–30, 2009.

[14] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[15] D. R. Sahu, H.-K. Xu, and J.-C. Yao, "Asymptotically strict pseudocontractive mappings in the intermediate sense," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 10, pp. 3502–3511, 2009.