Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 859795, 11 pages
doi:10.1155/2011/859795

Research Article

A Weak Convergence Theorem for Total Asymptotically Pseudocontractive Mappings in Hilbert Spaces

Xiaolong Qin,[1] Sun Young Cho,[2] and Shin Min Kang[3]

[1] *School of Mathematics and Information Sciences, North China University of Water Resources and Electric Power, Zhengzhou 450011, China*
[2] *Department of Mathematics, Gyeongsang National University, Jinju 660-701, Republic of Korea*
[3] *Department of Mathematics and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea*

Correspondence should be addressed to Shin Min Kang, smkang@gnu.ac.kr

Received 13 December 2010; Accepted 1 February 2011

Academic Editor: Yeol J. Cho

The modified Ishikawa iterative process is investigated for the class of total asymptotically pseudocontractive mappings. A weak convergence theorem of fixed points is established in the framework of Hilbert spaces.

1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. $\to$ and $\rightharpoonup$ are denoted by strong convergence and weak convergence, respectively. Let C be a nonempty closed convex subset of H and $T : C \to C$ a mapping. In this paper, we denote the fixed point set of T by $F(T)$.

T is said to be a *contraction* if there exists a constant $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

Banach contraction principle guarantees that every contractive mapping defined on complete metric spaces has a unique fixed point.

T is said to be a *weak contraction* if

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|), \quad \forall x, y \in C, \tag{1.2}$$

where $\psi : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0,\infty)$, $\psi(0) = 0$, and $\lim_{t\to\infty}\psi(t) = \infty$. We remark that the class of weak contractions was introduced by Alber and Guerre-Delabriere [1]. In 2001, Rhoades [2] showed that every weak contraction defined on complete metric spaces has a unique fixed point.

T is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad \forall n \ge 1,\ x, y \in C. \tag{1.4}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [3] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty closed convex bounded subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on C, then T has a fixed point.

T is said to be *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \le 0. \tag{1.5}$$

Observe that if we define

$$\xi_n = \max\left\{0, \sup_{x,y\in C}\left(\|T^n x - T^n y\| - \|x - y\|\right)\right\}, \tag{1.6}$$

then $\xi_n \to 0$ as $n \to \infty$. It follows that (1.5) is reduced to

$$\|T^n x - T^n y\| \le \|x - y\| + \xi_n, \quad \forall n \ge 1,\ x, y \in C. \tag{1.7}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [4] (see also [5]). It is known [6] that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense may not be Lipschitz continuous; see [5, 7].

T is said to be *total asymptotically nonexpansive* if

$$\|T^n x - T^n y\| \le \|x - y\| + \mu_n \phi(\|x - y\|) + \xi_n, \quad \forall n \ge 1,\ x, y \in C, \tag{1.8}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$ and $\{\mu_n\}$ and $\{\xi_n\}$ are nonnegative real sequences such that $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$. The class of mapping was introduced by Alber et al. [8]. From the definition, we see that the class of total asymptotically nonexpansive mappings includes the class of asymptotically nonexpansive mappings and the class of asymptotically nonexpansive mappings in the intermediate sense as special cases; see [9, 10] for more details.

T is said to be *strictly pseudocontractive* if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - Ty\| \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.9}$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [11] in a real Hilbert space. In 2007, Marino and Xu [12] obtained a weak convergence theorem for the class of strictly pseudocontractive mappings; see [12] for more details.

T is said to be an *asymptotically strict pseudocontraction* if there exist a constant $\kappa \in [0,1)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2, \quad \forall n \geq 1,\ x, y \in C. \tag{1.10}$$

The class of asymptotically strict pseudocontractions was introduced by Qihou [13] in 1996. Kim and Xu [14] proved that the class of asymptotically strict pseudocontractions is demiclosed at the origin and also obtained a weak convergence theorem for the class of mappings; see [14] for more details.

T is said to be an *asymptotically strict pseudocontraction in the intermediate sense* if there exist a constant $\kappa \in [0,1)$ and a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\|^2 - k_n\|x - y\|^2 - \kappa\|(I - T^n)x - (I - T^n)y\|^2 \right) \leq 0. \tag{1.11}$$

Put

$$\xi_n = \max\left\{ 0, \sup_{x,y\in C} \left(\|T^n x - T^n y\|^2 - k_n\|x - y\|^2 - \kappa\|(I - T^n)x - (I - T^n)y\|^2 \right) \right\}. \tag{1.12}$$

It follows that $\xi_n \to 0$ as $n \to \infty$. Then, (1.11) is reduced to the following:

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2 + \xi_n, \quad \forall n \geq 1,\ x, y \in C. \tag{1.13}$$

The class of mappings was introduced by Sahu et al. [15]. They proved that the class of asymptotically strict pseudocontractions in the intermediate sense is demiclosed at the origin and also obtained a weak convergence theorem for the class of mappings; see [15] for more details.

T is said to be *asymptotically pseudocontractive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\langle T^n x - T^n y, x - y \rangle \leq k_n\|x - y\|^2, \quad \forall n \geq 1,\ x, y \in C. \tag{1.14}$$

It is not hard to see that (1.14) is equivalent to

$$\|T^n x - T^n y\|^2 \le (2k_n - 1)\|x - y\|^2 + \|x - y - (T^n x - T^n y)\|^2, \quad \forall n \ge 1,\ x, y \in C. \tag{1.15}$$

The class of asymptotically pseudocontractive mapping was introduced by Schu [16] (see also [17]). In [18], Rhoades gave an example to showed that the class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings; see [18] for more details. Zhou [19] showed that every uniformly Lipschitz and asymptotically pseudocontractive mapping which is also uniformly asymptotically regular has a fixed point.

T is said to be an *asymptotically pseudocontractive mapping in the intermediate sense* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\langle T^n x - T^n y, x - y\rangle - k_n\|x - y\|^2\right) \le 0. \tag{1.16}$$

Put

$$\xi_n = \max\left\{0, \sup_{x,y\in C}\left(\langle T^n x - T^n y, x - y\rangle - k_n\|x - y\|^2\right)\right\}. \tag{1.17}$$

It follows that $\xi_n \to 0$ as $n \to \infty$. Then, (1.16) is reduced to the following:

$$\langle T^n x - T^n y, x - y\rangle \le k_n\|x - y\|^2 + \xi_n, \quad \forall n \ge 1,\ x, y \in C. \tag{1.18}$$

It is easy to see that (1.18) is equivalent to

$$\|T^n x - T^n y\|^2 \le (2k_n - 1)\|x - y\|^2 + \|x - y - (T^n x - T^n y)\|^2 + 2\xi_n, \quad \forall n \ge 1,\ x, y \in C. \tag{1.19}$$

The class of asymptotically pseudocontractive mappings in the intermediate sense was introduced by Qin et al. [20]. Weak convergence theorems of fixed points were established based on iterative methods; see [20] for more details.

In this paper, we introduce the following mapping.

Definition 1.1. Recall that $T : C \to C$ is said to be *total asymptotically pseudocontractive* if there exist sequences $\{\mu_n\} \subset [0, \infty)$ and $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$ such that

$$\langle T^n x - T^n y, x - y\rangle \le \|x - y\|^2 + \mu_n\phi(\|x - y\|) + \xi_n, \quad \forall n \ge 1,\ x, y \in C, \tag{1.20}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function with $\phi(0) = 0$.

It is easy to see that (1.20) is equivalent to the following:

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + 2\mu_n \phi(\|x - y\|) + \|x - y - (T^n x - T^n y)\|^2 + 2\xi_n, \quad \forall n \geq 1, \ x, y \in C. \tag{1.21}$$

Remark 1.2. If $\phi(\lambda) = \lambda^2$, then (1.20) is reduced to

$$\langle T^n x - T^n y, x - y \rangle \leq (1 + \mu_n)\|x - y\|^2 + \xi_n, \quad \forall n \geq 1, \ x, y \in C. \tag{1.22}$$

Remark 1.3. Put

$$\xi_n = \max\left\{0, \sup_{x,y \in C}\left(\langle T^n x - T^n y, x - y \rangle - (1 + \mu_n)\|x - y\|^2\right)\right\}. \tag{1.23}$$

If $\phi(\lambda) = \lambda^2$, then the class of total asymptotically pseudocontractive mappings is reduced to the class of asymptotically pseudocontractive mappings in the intermediate sense.

Recall that the modified Ishikawa iterative process which was introduced by Schu [16] generates a sequence $\{x_n\}$ in the following manner:

$$\begin{gathered} x_1 \in C, \\ y_n = \beta_n T^n x_n + (1 - \beta_n) x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad \forall n \geq 1, \end{gathered} \tag{1.24}$$

where $T : C \to C$ is a mapping, x_1 is an initial value, and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$.

If $\beta_n = 0$ for each $n \geq 1$, then the modified Ishikawa iterative process (1.24) is reduced to the following modified Mann iterative process:

$$x_1 \in C, \quad x_{n+1} = \alpha_n T^n x_n + (1 - \alpha_n) x_n, \quad \forall n \geq 1. \tag{1.25}$$

The purpose of this paper is to consider total asymptotically pseudocontractive mappings based on the modified Ishikawa iterative process. Weak convergence theorems are established in real Hilbert spaces.

In order to prove our main results, we also need the following lemmas.

Lemma 1.4. *In a real Hilbert space, the following inequality holds:*

$$\|ax + (1 - a)y\|^2 = a\|x\|^2 + (1 - a)\|y\|^2 - a(1 - a)\|x - y\|^2, \quad \forall a \in [0, 1], \ x, y \in C. \tag{1.26}$$

Lemma 1.5 (see [21]). *Let $\{r_n\}$, $\{s_n\}$, and $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \leq (1+s_n)r_n + t_n, \quad \forall n \geq n_0, \tag{1.27}$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists.

2. Main Results

Now, we are ready to give our main results.

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a uniformly L-Lipschitz and total asymptotically pseudocontractive mapping as defined in* (1.20). *Assume that $F(T)$ is nonempty and there exist positive constants M and M^* such that $\phi(\lambda) \leq M^*\lambda^2$ for all $\lambda \geq M$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{gathered} x_1 \in C, \\ y_n = \beta_n T^n x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad \forall n \geq 1, \end{gathered} \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Assume that the following restrictions are satisfied:

(a) $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \xi_n < \infty$,

(b) $a \leq \alpha_n \leq \beta_n \leq b$ *for some* $a > 0$ *and some* $b \in (0, L^{-2}[\sqrt{1+L^2}-1])$.

Then, the sequence $\{x_n\}$ generated in (2.1) *converges weakly to fixed point of T.*

Proof. Fix $x^* \in F(T)$. Since ϕ is an increasing function, it results that $\phi(\lambda) \leq \phi(M)$ if $\lambda \leq M$ and $\phi(\lambda) \leq M^*\lambda^2$ if $\lambda \geq M$. In either case, we can obtain that

$$\phi(\|x_n - x^*\|) \leq \phi(M) + M^*\|x_n - x^*\|^2. \tag{2.2}$$

In view of Lemma 1.4, we see from (2.2) that

$$\begin{aligned} \|y_n - x^*\|^2 &= \|\beta_n(T^n x_n - x^*) + (1-\beta_n)(x_n - x^*)\|^2 \\ &= \beta_n\|T^n x_n - x^*\|^2 + (1-\beta_n)\|x_n - x^*\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\leq \beta_n\Big(\|x_n - x^*\|^2 + 2\mu_n\phi(\|x_n - x^*\|) + 2\xi_n + \|x_n - T^n x_n\|^2\Big) \\ &\quad + (1-\beta_n)\|x_n - x^*\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2 \\ &\leq (1 + 2\beta_n\mu_n M^*)\|x_n - x^*\|^2 + \beta_n^2\|T^n x_n - x_n\|^2 + 2\beta_n\mu_n\phi(M) + 2\beta_n\xi_n \\ &\leq q_n\|x_n - x^*\|^2 + \beta_n^2\|T^n x_n - x_n\|^2 + 2\beta_n\mu_n\phi(M) + 2\beta_n\xi_n, \end{aligned} \tag{2.3}$$

where $q_n = 1 + 2\mu_n M^*$ for each $n \ge 1$. Notice from Lemma 1.4 that

$$\begin{aligned}\|y_n - T^n y_n\|^2 &= \|\beta_n(T^n x_n - T^n y_n) + (1-\beta_n)(x_n - T^n y_n)\|^2\\ &= \beta_n\|T^n x_n - T^n y_n\|^2 + (1-\beta_n)\|x_n - T^n y_n\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2\\ &\le \beta_n^3 L^2\|x_n - T^n x_n\|^2 + (1-\beta_n)\|x_n - T^n y_n\|^2 - \beta_n(1-\beta_n)\|T^n x_n - x_n\|^2.\end{aligned} \tag{2.4}$$

Since ϕ is an increasing function, it results that $\phi(\lambda) \le \phi(M)$ if $\lambda \le M$ and $\phi(\lambda) \le M^*\lambda^2$ if $\lambda \ge M$. In either case, we can obtain that

$$\phi(\|y_n - x^*\|) \le \phi(M) + M^*\|y_n - x^*\|^2. \tag{2.5}$$

This implies from (2.3) and (2.4) that

$$\begin{aligned}\|T^n y_n - x^*\|^2 &\le \|y_n - x^*\|^2 + 2\mu_n\phi(\|y_n - x^*\|) + 2\xi_n + \|y_n - T^n y_n\|^2\\ &\le q_n\|y_n - x^*\|^2 + \|y_n - T^n y_n\|^2 + 2\mu_n\phi(M) + 2\xi_n\\ &\le q_n^2\|x_n - x^*\|^2 - \beta_n\Big(1 - q_n\beta_n - \beta_n^2L^2 - \beta_n\Big)\|T^n x_n - x_n\|^2\\ &\quad + 2p_n + (1-\beta_n)\|x_n - T^n y_n\|^2,\end{aligned} \tag{2.6}$$

where $p_n = q_n\beta_n\mu_n\phi(M) + q_n\beta_n\xi_n + \mu_n\phi(M) + \xi_n$ for each $n \ge 1$. It follows that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(T^n y_n - x^*) + (1-\alpha_n)(x_n - x^*)\|^2\\ &= \alpha_n\|T^n y_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2\\ &\le q_n^2\|x_n - x^*\|^2 - \alpha_n\beta_n\Big(1 - q_n\beta_n - \beta_n^2L^2 - \beta_n\Big)\|T^n x_n - x_n\|^2 + 2\alpha_n p_n.\end{aligned} \tag{2.7}$$

From the restriction (b), we see that there exists n_0 such that

$$1 - q_n\beta_n - \beta_n^2L^2 - \beta_n \ge \frac{1 - 2b - L^2b^2}{2} > 0, \quad \forall n \ge n_0. \tag{2.8}$$

It follows from (2.7) that

$$\|x_{n+1} - x^*\|^2 \le (1 + (q_n + 1)2\mu_n M^*)\|x_n - x^*\|^2 + 2\alpha_n p_n, \quad \forall n \ge n_0. \tag{2.9}$$

Notice that $\sum_{n=1}^{\infty}(q_n+1)2\mu_n M^* < \infty$ and $\sum_{n=1}^{\infty} p_n < \infty$. In view of Lemma 1.5, we see that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. For any $n \geq n_0$, we see that

$$\begin{aligned}&\frac{a^2(1-2b-L^2b^2)}{2}\|T^n x_n - x_n\|^2\\&\quad\leq (q_n+1)2\mu_n M^*\|x_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n p_n,\end{aligned} \tag{2.10}$$

from which it follows that

$$\lim_{n\to\infty}\|T^n x_n - x_n\| = 0. \tag{2.11}$$

Note that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \alpha_n\left(\left\|T^n y_n - T^n x_n\right\| + \|T^n x_n - x_n\|\right)\\&\leq \alpha_n\left(L\left\|y_n - x_n\right\| + \|T^n x_n - x_n\|\right)\\&\leq \alpha_n(1+\beta_n L)\|T^n x_n - x_n\|.\end{aligned} \tag{2.12}$$

In view of (2.11), we obtain that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.13}$$

Note that

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + \left\|T^{n+1}x_{n+1} - T^{n+1}x_n\right\| + \left\|T^{n+1}x_n - Tx_n\right\|\\&\leq (1+L)\|x_n - x_{n+1}\| + \left\|x_{n+1} - T^{n+1}x_{n+1}\right\| + L\|T^n x_n - x_n\|.\end{aligned} \tag{2.14}$$

Combining (2.11) and (2.13) yields that

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0. \tag{2.15}$$

Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup \overline{x}$. Next, we claim that $\overline{x} \in F(T)$. Choose $\alpha \in (0, 1/(1+L))$ and define $y_{\alpha,m} = (1-\alpha)\overline{x} + \alpha T^m \overline{x}$ for arbitrary but fixed $m \geq 1$. From the assumption that T is uniformly L-Lipschitz, we see that

$$\begin{aligned}\|x_n - T^m x_n\| &\leq \|x_n - Tx_n\| + \left\|Tx_n - T^2 x_n\right\| + \cdots + \left\|T^{m-1}x_n - T^m x_n\right\|\\&\leq [1+(m-1)L]\|x_n - Tx_n\|.\end{aligned} \tag{2.16}$$

It follows from (2.15) that

$$\lim_{n\to\infty}\|x_n - T^m x_n\| = 0. \tag{2.17}$$

Since ϕ is an increasing function, it results that $\phi(\lambda) \le \phi(M)$ if $\lambda \le M$ and $\phi(\lambda) \le M^*\lambda^2$ if $\lambda \ge M$. In either case, we can obtain that

$$\phi\left(\left\|x_n - y_{\alpha,m}\right\|\right) \le \phi(M) + M^*\left\|x_n - y_{\alpha,m}\right\|^2. \tag{2.18}$$

This in turn implies that

$$\begin{aligned}
\left\langle \overline{x} - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\right\rangle &= \left\langle \overline{x} - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\right\rangle + \left\langle x_n - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\right\rangle \\
&= \left\langle \overline{x} - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\right\rangle + \left\langle x_n - y_{\alpha,m}, T^m x_n - T^m y_{\alpha,m}\right\rangle \\
&\quad - \left\langle x_n - y_{\alpha,m}, x_n - y_{\alpha,m}\right\rangle + \left\langle x_n - y_{\alpha,m}, x_n - T^m x_n\right\rangle \\
&\le \left\langle \overline{x} - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\right\rangle + \mu_m\phi\left(\left\|x_n - y_{\alpha,m}\right\|\right) + \xi_m \\
&\quad + \left\|x_n - y_{\alpha,m}\right\|\|x_n - T^m x_n\| \\
&\le \left\langle \overline{x} - x_n, y_{\alpha,m} - T^m y_{\alpha,m}\right\rangle + \mu_m\phi(M) + \mu_m M^*\left\|x_n - y_{\alpha,m}\right\|^2 + \xi_m \\
&\quad + \left\|x_n - y_{\alpha,m}\right\|\|x_n - T^m x_n\|.
\end{aligned} \tag{2.19}$$

Since $x_n \rightharpoonup \overline{x}$, we see from (2.17) that

$$\left\langle \overline{x} - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\right\rangle \le \mu_m\phi(M) + \mu_m M^*\left\|x_n - y_{\alpha,m}\right\|^2 + \xi_m. \tag{2.20}$$

On the other hand, we have

$$\left\langle \overline{x} - y_{\alpha,m}, (\overline{x} - T^m\overline{x}) - \left(y_{\alpha,m} - T^m y_{\alpha,m}\right)\right\rangle \le (1+L)\left\|\overline{x} - y_{\alpha,m}\right\|^2 = (1+L)\alpha^2\|\overline{x} - T^m\overline{x}\|^2. \tag{2.21}$$

Note that

$$\begin{aligned}
\|\overline{x} - T^m\overline{x}\|^2 &= \langle \overline{x} - T^m\overline{x}, \overline{x} - T^m\overline{x}\rangle \\
&= \frac{1}{\alpha}\left\langle \overline{x} - y_{\alpha,m}, \overline{x} - T^m\overline{x}\right\rangle \\
&= \frac{1}{\alpha}\left\langle \overline{x} - y_{\alpha,m}, (\overline{x} - T^m\overline{x}) - \left(y_{\alpha,m} - T^m y_{\alpha,m}\right)\right\rangle + \frac{1}{\alpha}\left\langle \overline{x} - y_{\alpha,m}, y_{\alpha,m} - T^m y_{\alpha,m}\right\rangle.
\end{aligned} \tag{2.22}$$

Substituting (2.20) and (2.21) into (2.22), we arrive at

$$\|\overline{x} - T^m\overline{x}\|^2 \le (1+L)\alpha\|\overline{x} - T^m\overline{x}\|^2 + \frac{\mu_m\phi(M) + \mu_m M^*\left\|x_n - y_{\alpha,m}\right\|^2 + \xi_m}{\alpha}. \tag{2.23}$$

This implies that

$$\alpha[1-(1+L)\alpha]\|\overline{x}-T^m\overline{x}\|^2 \le \mu_m\phi(M)+\mu_m M^*\left\|x_n-y_{\alpha,m}\right\|^2+\xi_m, \quad \forall m \ge 1. \tag{2.24}$$

Letting $m \to \infty$ in (2.24), we see that $T^m\overline{x} \to \overline{x}$. Since T is uniformly L-Lipschitz, we can obtain that $\overline{x} = T\overline{x}$.

Next, we prove that $\{x_n\}$ converges weakly to $\overline{x}$. Suppose the contrary. Then, we see that there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $\widehat{x} \in C$, where $\widehat{x} \ne \overline{x}$. It is not hard to see that that $\widehat{x} \in F(T)$. Put $d = \lim_{n\to\infty}\|x_n - \overline{x}\|$. Since H enjoys Opial property, we see that

$$\begin{aligned} d &= \liminf_{i\to\infty}\|x_{n_i}-\overline{x}\| < \liminf_{i\to\infty}\|x_{n_i}-\widehat{x}\| \\ &= \liminf_{j\to\infty}\left\|x_{n_j}-\widehat{x}\right\| < \liminf_{j\to\infty}\left\|x_{n_j}-\overline{x}\right\| \\ &= \liminf_{i\to\infty}\|x_{n_i}-\overline{x}\| = d. \end{aligned} \tag{2.25}$$

This derives a contradiction. It follows that $\widehat{x} = \overline{x}$. This completes the proof. □

Remark 2.2. Demiclosedness principle of the class of total asymptotically pseudocontractive mappings can be deduced from Theorem 2.1.

Remark 2.3. Since the class of total asymptotically pseudocontractive mappings includes the class of strict pseudocontractions, the class of asymptotically strict pseudocontractions, the class of pseudocontractive mappings, the class of asymptotically pseudocontractive mappings and the class of asymptotically pseudocontractive mappings in the intermediate sense as special cases, Theorem 2.1 improves the corresponding results in Marino and Xu [12], Kim and Xu [14], Sahu et al. [15], Schu [16], Zhou [19], and Qin et al. [20].

Remark 2.4. It is of interest to improve the main results of this paper to a Banach space.

Acknowledgment

The authors thank the referees for useful comments and suggestions.

References

[1] Ya. I. Alber and S. Guerre-Delabriere, "On the projection methods for fixed point problems," *Analysis*, vol. 21, no. 1, pp. 17–39, 2001.

[2] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 4, pp. 2683–2693, 2001.

[3] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 35, pp. 171–174, 1972.

[4] R. Bruck, T. Kuczumow, and S. Reich, "Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property," *Colloquium Mathematicum*, vol. 65, no. 2, pp. 169–179, 1993.

[5] W. A. Kirk, "Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type," *Israel Journal of Mathematics*, vol. 17, pp. 339–346, 1974.

[6] H. K. Xu, “Existence and convergence for fixed points of mappings of asymptotically nonexpansive type,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 16, no. 12, pp. 1139–1146, 1991.

[7] Z. Liu, J. K. Kim, and K. H. Kim, “Convergence theorems and stability problems of the modified Ishikawa iterative sequences for strictly successively hemicontractive mappings,” *Bulletin of the Korean Mathematical Society*, vol. 39, no. 3, pp. 455–469, 2002.

[8] Ya. I. Alber, C. E. Chidume, and H. Zegeye, “Approximating fixed points of total asymptotically nonexpansive mappings,” *Fixed Point Theory and Applications*, vol. 2006, Article ID 10673, 20 pages, 2006.

[9] C. E. Chidume and E. U. Ofoedu, “A new iteration process for approximation of common fixed points for finite families of total asymptotically nonexpansive mappings,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 615107, 17 pages, 2009.

[10] C. E. Chidume and E. U. Ofoedu, “Approximation of common fixed points for finite families of total asymptotically nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 128–141, 2007.

[11] F. E. Browder and W. V. Petryshyn, “Construction of fixed points of nonlinear mappings in Hilbert space,” *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[12] G. Marino and H.-K. Xu, “Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces,” *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.

[13] L. Qihou, “Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 11, pp. 1835–1842, 1996.

[14] T.-H. Kim and H.-K. Xu, “Convergence of the modified Mann’s iteration method for asymptotically strict pseudo-contractions,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 9, pp. 2828–2836, 2008.

[15] D. R. Sahu, H.-K. Xu, and J.-C. Yao, “Asymptotically strict pseudocontractive mappings in the intermediate sense,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 10, pp. 3502–3511, 2009.

[16] J. Schu, “Iterative construction of fixed points of asymptotically nonexpansive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 158, no. 2, pp. 407–413, 1991.

[17] J. Schu, “Weak and strong convergence to fixed points of asymptotically nonexpansive mappings,” *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[18] B. E. Rhoades, “Comments on two fixed point iteration methods,” *Journal of Mathematical Analysis and Applications*, vol. 56, no. 3, pp. 741–750, 1976.

[19] H. Zhou, “Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3140–3145, 2009.

[20] X. Qin, S. Y. Cho, and J. K. Kim, “Convergence theorems on asymptotically pseudocontractive mappings in the intermediate sense,” *Fixed Point Theory and Applications*, vol. 2010, Article ID 186874, 14 pages, 2010.

[21] K.-K. Tan and H. K. Xu, “Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process,” *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.