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Research Article

ZPC Matrices and Zero Cycles

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Let H be an $m \times n$ real matrix and let Z_i be the set of column indices of the zero entries of row i of H. Then the conditions $|Z_k \cap (\cup_{i=1}^{k-1} Z_i)| \leq 1$ for all k $(2 \leq k \leq m)$ are called the (row) Zero Position Conditions (ZPCs). If H satisfies the ZPC, then H is said to be a (row) ZPC matrix. If H^T satisfies the ZPC, then H is said to be a column ZPC matrix. The real matrix H is said to have a zero cycle if H has a sequence of at least four zero entries of the form $h_{i_1 j_1}, h_{i_1 j_2}, h_{i_2 j_2}, h_{i_2 j_3}, \ldots, h_{i_k j_k}, h_{i_k j_1}$ in which the consecutive entries alternatively share the same row or column index (but not both), and the last entry has one common index with the first entry. Several connections between the ZPC and the nonexistence of zero cycles are established. In particular, it is proved that a matrix H has no zero cycle if and only if there are permutation matrices P and Q such that PHQ is a row ZPC matrix and a column ZPC matrix.

1. Introduction

A matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern). For a real matrix B, $\mathrm{sgn}(B)$ is the sign pattern obtained by replacing each positive (respectively, negative, zero) entry of B by + (respectively, −, 0). For a sign pattern A, the *sign pattern class of* A is defined by

$$Q(A) = \{B : \mathrm{sgn}(B) = A\}. \tag{1.1}$$

Further information on sign patterns can be found in [1, 2].

For a real matrix $C = [c_{ij}]$ of size $m \times n$, the *bipartite graph* of C is the graph with vertex set $\{1, 2, \ldots, m\} \cup \{1', 2', \ldots, n'\}$ such that there is an edge between i and j' if and only if $c_{ij} \neq 0$. The *zero bipartite graph* of C is the complement of the bipartite graph of C.

In [3], the following result is proved.

Theorem 1.1. *Let A, B, and C be real matrices such that $AB = C$. Suppose that the zero bipartite graph of C (this is the same as the complement of the bipartite graph of C) is a forest. Then there exist rational perturbations $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ of A, B, and C, respectively, in the same corresponding sign pattern classes, such that $\widetilde{A}\widetilde{B} = \widetilde{C}$.*

The purpose of this note is to investigate when the zero bipartite graph of a matrix is a forest from a combinatorial point of view. This will be done in terms of ZPC matrices and zero cycles.

2. ZPC Matrices

Definition 2.1. Let Z_i be the set of column indices of the zero entries of row i $(1 \leq i \leq m)$ of a real matrix $H_{m\times n}$. Then the conditions

$$\left| Z_k \cap \left(\bigcup_{i=1}^{k-1} Z_i \right) \right| \leq 1 \quad \text{for all } k \ (2 \leq k \leq m) \tag{2.1}$$

are called the *(row) Zero Position Conditions (ZPCs)*. If H satisfies the ZPC, then we say that H is a (row) *ZPC matrix*. If H^T satisfies the ZPC, then we say that H is a *column ZPC matrix*.

Definition 2.2. A zero entry in a matrix is called a *covered zero*, if there is another zero entry above this entry in the same column. A zero entry that is not a covered zero is called an *uncovered zero*.

The proposition below follows directly from the definition of the ZPC.

Proposition 2.3. *A matrix $H_{m\times n}$ satisfies the ZPC if and only if each row of H has at most one covered zero.*

Remark 2.4. Permutation of columns preserves the ZPC property.

Proposition 2.5. *Let H be a ZPC matrix. Then there is a permutation matrix P such that each covered zero of HP is the leading zero in its row.*

Proof. We proceed by induction on the number of rows.

The result is trivially true for every matrix with only one row.

Assume that this result holds for ZPC matrices with m rows. Now consider any ZPC matrix H with $m + 1$ rows. Write $H = \left[\begin{smallmatrix} H_1 \\ H_2 \end{smallmatrix}\right]$, where H_1 has m rows. By induction hypothesis there is a permutation matrix P_1 such that each covered zero of H_1P_1 is the leading zero in its row.

Since HP_1 satisfies the ZPC, the last row of HP_1 has at most one covered zero. If the last row of H has no covered zero, then the result is already true based on the induction hypothesis.

Assume that the last row of H (or equivalently HP_1) has a covered zero entry. Then all the other zero entries in the last row are uncovered and hence all the entries directly above these zeros in the last row are all nonzero. Therefore, we may permute the columns of HP_1 to put the columns containing the uncovered zeros of the last row of HP_1 to the far right positions, resulting in HP_1P_2. As none of these columns moved to the far right contains covered zeros of the first m rows of HP_1, we see that HP_1P_2 has the property that each covered zero of HP_1P_2 is the leading zero in its row. □

The ZPC places severe restrictions on the location and number of zeros in the matrix. In particular, we have the following result on the number of zeros.

Proposition 2.6. *If $H_{m\times n}$ is a ZPC matrix, then H has at most $m+n-1$ zeros.*

Proof. We proceed by induction on the number of rows.

If $m=1$, then clearly H has at most $n=1+n-1$ zeros.

Assume that the result holds for ZPC matrices with $m-1$ rows. Now consider a ZPC matrix H with m rows.

Suppose that the last row of H has p zeros. If $p\le 1$, then the result follows immediately by applying the induction hypothesis on the submatrix of H consisting of the first $m-1$ rows.

Assume that $p>1$. Since the last row of H has at most one covered zero, the last $p-1$ zeros are not covered, and permuting the columns of H if necessary (to put the uncovered zeros of the last row to the far right), we may assume that H has the block form

$$H=\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}, \tag{2.2}$$

where H_1 is $(m-1)\times(n-(p-1))$, H_2 is $(m-1)\times(p-1)$ and has no zero entry, H_3 has one (possibly covered) zero, and H_4 consists of $p-1$ uncovered zeros.

The induction hypothesis applied on H_1 says that H_1 has at most $(m-1)+(n-(p-1))-1=m+n-p-1$ zeros. Hence, H has at most $(m+n-p-1)+p=m+n-1$ zeros. □

3. ZPC and Nonexistence of Zero Cycles

In this section we explore interesting connections between the Zero Position Conditions and the nonexistence of zero cycles.

Let H be a real matrix. We say that H has a *zero cycle* if H has a sequence of at least four zero entries of the form

$$h_{i_1j_1}, h_{i_1j_2}, h_{i_2j_2}, h_{i_2j_3}, \ldots, h_{i_kj_k}, h_{i_kj_1}, \tag{3.1}$$

in which the consecutive entries alternatively share the same row or column index (but not both), and the last entry has one common index with the first entry. For example,

$$h_{11}, h_{13}, h_{33}, h_{34}, h_{24}, h_{21} \tag{3.2}$$

would form a zero cycle of H if all these entries are zeros. The idea of a zero cycle (called a loop) was introduced in [4, 5].

The *zero bipartite graph* of a matrix $H_{m\times n}$, denoted $Z(H)$, has vertex sets $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ such that there is an edge between u_i and v_j iff $h_{ij} = 0$. It can be seen that H has a zero cycle iff the zero bipartite graph $Z(H)$ has a cycle.

Theorem 3.1. *If a matrix H has a zero cycle, then no (row or column) permutation of H is a ZPC matrix or a column ZPC matrix.*

Proof. Suppose that H has a zero cycle. For any two permutations P and Q of suitable orders, PHQ also has a zero cycle. Consider a cycle of PHQ. Let k be the largest row index involved in the cycle. Then all the zero entries (there are at least two such entries) in the cycle with row index k are covered zeros. Hence, PHQ is not a ZPC. A similar argument on the columns shows that PHQ is not a column ZPC matrix. □

We need the following basic fact from graph theory.

Lemma 3.2. *If the degree of every vertex of a graph G is at least 2, then G has a cycle.*

We are now ready to establish a main result in this section.

Theorem 3.3. *If a matrix H has no zero cycle, then there are permutation matrices P and Q such that PHQ is both a row ZPC matrix and a column ZPC matrix.*

Proof. We proceed by double induction on the number of rows (m) and the number of columns (n) of a matrix.

Since every $1 \times n$ or $m \times 1$ matrix is a row ZPC matrix and a column ZPC matrix, the result is trivially true for $m = 1$ or $n = 1$.

Assume that the result holds for matrices with $m - 1$ rows or $n - 1$ columns. Consider an $m \times n$ matrix H. Since the zero bipartite graph $Z(H)$ of H has no cycle, by Lemma 3.2, $Z(H)$ has a vertex of degree at most 1. Without loss of generality, we may assume that u_m has degree at most 1. This can be achieved by a row permutation on H and possibly taking the transpose (in the case when a vertex of degree at most 1 is in V). Since the submatrix H_{m-1} of H obtained from H by deleting the last row clearly has no zero cycle, by induction hypothesis, the rows and columns of H_{m-1} may be permuted to produce a matrix that is both a row ZPC matrix and a column ZPC matrix. Of course, the row permutation of H_{m-1} does not affect the last row of H, while the column permutation of H_{m-1} should also be applied to the last column of H. For convenience, we may assume that H_{m-1} is a row ZPC matrix and a column ZPC matrix.

We now show that with the above mentioned permutations and a possible transposition, the resulting matrix (also denoted by H for convenience) is a row ZPC matrix and a column ZPC matrix. Since the last row of H contains at most one zero entry, and hence at most one covered zero, while H_{m-1} is a row ZPC matrix, it follows immediately that H is a row ZPC matrix.

Observe that the last column of H^T (the transpose of the last row of H) contains at most one zero entry, and hence, there is no covered zero in the last column of H^T. Combined with the assumption that H_{m-1}^T is a row ZPC matrix, we see that each row of H^T has at most one covered zero and so H^T is a row ZPC matrix, namely, H is a column ZPC matrix. □

By Theorem 3.1, the converse of Theorem 3.3 is true. Thus, we also have the following theorem.

Theorem 3.4. *A matrix H has no zero cycle iff there are permutation matrices P and Q such that PHQ is a row ZPC matrix and a column ZPC matrix.*

We now come to the culminating result.

Theorem 3.5. *The following statements are equivalent.*

(i) *H has no zero cycle.*

(ii) *The zero bipartite graph of H has no cycle.*

(iii) *The zero bipartite graph of H is a forest.*

(iv) *There is a permutation matrix P such that PH is a row ZPC matrix.*

(v) *There is a permutation matrix Q such that HQ is a column ZPC matrix.*

(vi) *There are permutation matrices P and Q such that PHQ is both a row ZPC matrix and a column ZPC matrix.*

Proof. It can be seen that the first three statements are equivalent. From Theorem 3.4, statements (i) and (vi) are equivalent. Since permutation of columns (rows) preserves the row ZPC (column ZPC) property, it is clear that statement (vi) implies each of the statements (iv) and (v). Next, suppose that H has a zero cycle. Then PH has a zero cycle for any permutation matrix P. Hence, the last row of PH that has at least two zero entries of the above zero cycle of PH contains at least two covered zeros, so that PH is not row ZPC. Thus, statement (iv) implies statement (i). Similarly, statement (v) implies statement (i). The proof is now complete. □

We point out that Proposition 2.5 also follows from Theorem 3.5, since the number of edges of a forest with $m + n$ vertices is at most $m + n - 1$.

The results of this paper may be stated in terms of the positions of the nonzero entries of a matrix and the bipartite graph of the matrix. However, we chose to use the zero bipartite graph because the motivation [3] for this study naturally requires concentrating on the zero entries.

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