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Research Article

Ramsey Numbers for Theta Graphs

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The graph Ramsey number $R(F_1, F_2)$ is the smallest integer N with the property that any complete graph of at least N vertices whose edges are colored with two colors (say, red and blue) contains either a subgraph isomorphic to F_1 all of whose edges are red or a subgraph isomorphic to F_2 all of whose edges are blue. In this paper, we consider the Ramsey numbers for theta graphs. We determine $R(\theta_4, \theta_k)$, $R(\theta_5, \theta_k)$ for $k \geq 4$. More specifically, we establish that $R(\theta_4, \theta_k) = R(\theta_5, \theta_k) = 2k - 1$ for $k \geq 7$. Furthermore, we determine $R(\theta_n, \theta_n)$ for $n \geq 5$. In fact, we establish that $R(\theta_n, \theta_n) = (3n/2) - 1$ if n is even, $2n - 1$ if n is odd.

1. Introduction and Preliminaries

The graphs considered in this paper are finite, undirected, and have no loops or multiple edges. For a given graph G, we denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. The cardinalities of these sets are denoted by $\nu(G)$ and $\mathcal{E}(G)$, respectively. Throughout this paper a cycle on m vertices will be denoted by C_m, the complete graph on n vertices by K_n. Suppose that $V_1 \subseteq V(G)$ and V_1 is non-empty, the subgraph of G whose vertex set is V_1 and whose edge set is the set of those edges of G that have both ends in V_1 is called the subgraph of G induced by V_1, denoted by $G[V_1]$. Let C be a cycle in a graph G, an edge in $E(G[C])\backslash E(C)$ is called a chord of C. Further, a graph G has a θ_k-graph if G has a cycle C_k that has a chord in G. Let G be a graph and $u \in V(G)$. The degree of a vertex u in G, denoted by $d_G(u)$, is the number of edges of G incident to u. The neighbor-set of a vertex u of G in a subgraph H of G, denoted by $N_H(u)$, consists of the vertices of H adjacent to u. The circumference, $c(G)$, of the graph G is the length of the longest cycle of G. For vertex-disjoint subgraphs H_1 and H_2 of G we let $E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1),\ \ y \in V(H_2)\}$ and $\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|$.

The graph Ramsey number $R(F_1, F_2)$ is the smallest integer N with the property that any complete graph of at least N vertices whose edges are colored with two colors (say, red and blue) contains either a subgraph isomorphic to F_1 all of whose edges are red or a subgraph isomorphic to F_2 all of whose edges are blue.

It is well known that the problem of determining the Ramsey numbers for complete graphs is very difficult, and it is easier to deal with paths, trees, cycles, and theta graphs. See the updated bibliography by Radziszowski [1]. In this paper we study $R(F_1, F_2)$ in the case when both F_1 and F_2 are theta graphs.

The results concerning Ramsey numbers for cycles were established by Chartrand and Schuster [2] (for $k < 7$), by Bondy and Erdös [3] (for $n = k$ odd and for the case when k is much smaller than n), and for all the remaining values by Rosta [4] and by Faudree and Schelp [5], independently. These results are summarized in the following theorem.

Theorem 1.1. *Let $3 \leq m \leq n$ be integers. Then*

$$R(C_n, C_m) = \begin{cases} 6 & \textit{if } m = n = 3 \textit{ or } 4, \\ n + \dfrac{m}{2} - 1 & \text{if } n, m \textit{ are even}, \ (n, m) \neq (4, 4), \\ \max\left\{n + \dfrac{m}{2} - 1, 2m - 1\right\} & \textit{if } n \textit{ is odd}, \ m \textit{ is even}, \\ 2n - 1 & \textit{if } m \textit{ is odd}, \ (n, m) \neq (3, 3). \end{cases} \tag{1.1}$$

In order to prove our results, we need to state the following results.

Theorem 1.2 (see [6]). *Let G be a graph on n vertices with no cycles of length greater than k. Then $\mathcal{E}(G) \leq (1/2)k(n-1) - (1/2)r(k-r-1)$ where $r = (n-1) - (k-1)\lfloor (n-1)/(k-1) \rfloor$.*

Theorem 1.3 (see [7]). *Every non-bipartite graph G on n vertices with more than $\lfloor (n-1)^2/4 \rfloor + 1$ edges contains cycles of every length l, where $3 \leq l \leq c(G)$.*

Theorem 1.4 (see [8]). *Let G be a non-bipartite graph on $n \geq 7$ vertices and G contains no θ_4-subgraph. Then $\mathcal{E}(G) \leq \lfloor (n-1)^2/4 \rfloor + 2$.*

Theorem 1.5 (see [9]). *Let G be a non-bipartite graph on $n \geq 9$ vertices and G contains no θ_5-subgraph. Then $\mathcal{E}(G) \leq \lfloor (n-1)^2/4 \rfloor + 1$.*

In this paper, we consider Ramsey numbers for theta graphs. We determine $R(\theta_4, \theta_k)$, $R(\theta_5, \theta_k)$ for $k \geq 4$. More specifically, we establish that $R(\theta_4, \theta_k) = R(\theta_5, \theta_k) = 2k - 1$ for $k \geq 7$. Furthermore, we determine $R(\theta_n, \theta_n)$ for $n \geq 5$. In fact, we establish that

$$R(\theta_n, \theta_n) = \begin{cases} \left(\dfrac{3n}{2}\right) - 1 & \text{if } n \text{ is even}, \\ 2n - 1 & \text{if } n \text{ is odd}. \end{cases} \tag{1.2}$$

Throughout this paper (Figures 1–5), solid lines represent red edges and dashed lines represent blue edges.

2. Main Result

In the following lemma we determine the Ramsey number $R(\theta_4, C_3)$.

Lemma 2.1. *The Ramsey number* $R(\theta_4, C_3) = 7$.

Proof. First we show that $R(\theta_4, C_3) \geq 7$. Let K_6 be colored as follows: the vertex set $V(K_6)$ is the disjoint union of two subsets H_1 and H_2 each of order 3 and completely colored red. All edges between H_1 and H_2 are colored blue. This coloring contains neither a red θ_4-graph nor a blue C_3. So, we conclude that $R(\theta_4, C_3) \geq 7$.

It remains to show that $R(\theta_4, C_3) \leq 7$. Let a red-blue coloring of K_7 be given. By Theorem 1.1, K_7 has a red C_3 or a blue C_3. If K_7 has a blue C_3, then the result is obtained. So we need to consider the case when K_7 has a red C_3. Let $x_1, x_2, \ldots, x_7$ be the vertices of K_7 and assume x_1, x_2, x_3 are the vertices of the red C_3. Any vertex of the remaining vertices x_4, x_5, x_6, and x_7 is adjacent to the red C_3 by at least 2 blue edges as otherwise a red θ_4-graph is produced. Assume $K_7[x_4, x_5, x_6, x_7]$ has a blue edge, say x_4x_5, then $K_7[x_1, x_2, x_3, x_4, x_5]$ has a blue C_3. So, we need to consider that $K_7[x_4, x_5, x_6, x_7]$ has no blue edge. Thus, a red θ_4-graph is produced. This completes the proof. □

In the following lemma we determine the Ramsey number $R(\theta_4, \theta_4)$.

Lemma 2.2. *The Ramsey number* $R(\theta_4, \theta_4) = 10$.

Proof. First we show that $R(\theta_4, \theta_4) \geq 10$. Let K_9 be colored as follows: The vertex set $V(K_9)$ is the disjoint union of three subsets G_1, G_2, and G_3 each of order 3 and completely colored red and the red edges between G_1, G_2, and G_3 are shown in Figure 1. The remaining edges are colored blue.

This coloring contains neither a red nor a blue θ_4-graph. So, we conclude that $R(\theta_4, \theta_4) \geq 10$. It remains to show that $R(\theta_4, \theta_4) \leq 10$. Let a red-blue coloring of K_{10} be given. By Theorem 1.1, K_{10} contains a red or a blue C_3. Without loss of generality we assume that K_{10} has a red C_3. Let $x_1, x_2, \ldots, x_{10}$ be the vertices of K_{10} and assume $x_1x_2x_3x_1$ be the red C_3 in K_{10}. Observe that $H_1 = K_{10}[x_4, x_5, \ldots, x_{10}]$ has a red C_3 or a blue C_3. So, we consider the following two cases.

Case 1. Suppose H_1 has a red C_3, say $x_8x_9x_{10}x_8$. Let x_4, x_5, x_6, and x_7 be the remaining vertices in K_{10}. Suppose $H_2 = K_{10}[x_4, x_5, x_6, x_7]$ has no blue edge, then K_{10} has a red θ_4-graph. So, we need to consider the case when H_2 has a blue edge, say x_4x_5 is the blue edge. Observe that any vertex in H_2 must be adjacent to each of $K_{10}[x_1, x_2, x_3]$ and $K_{10}[x_8, x_9, x_{10}]$ by two blue edges as otherwise a red θ_4-graph is produced. Thus x_4 and x_5 incident with two blue edges that have a common vertex in $K_{10}[x_1, x_2, x_3]$ and incident with two blue edges that have a common vertex in $K_{10}[x_8, x_9, x_{10}]$ and so a blue θ_4-graph is produced.

Case 2. Now we need to consider the case when H_1 has a blue C_3, say $x_8x_9x_{10}x_8$. Observe that every vertex in the red C_3 is adjacent to the blue C_3 by two red edges as otherwise a blue θ_4-graph is produced. Further, every vertex in the blue C_3 is adjacent to the red C_3 by two blue edges as otherwise a red θ_4-graph is produced.

Thus, there are at least six red edges between the red C_3 and the blue C_3 and at least six blue edges between the blue C_3 and the red C_3. We know that, $\mathcal{E}(C_3, C_3) = 9$. This is a contradiction. □

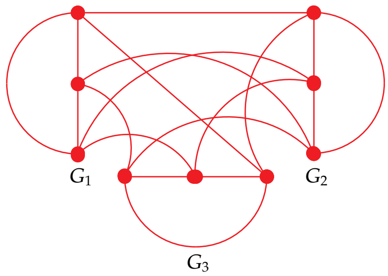

Figure 1: This figure represents the red edges in K_9.

Now, we begin with the following construction which will be used throughout our results. Let $n \geq 4$. Then, in K_{2n-2} color the edges of a complete bipartite graph $K_{n-1,n-1}$ with blue and all the remaining edges with red. Then this coloring contains neither a red θ_n-graph nor a blue θ_m-graph where $m = 4, 5$. Thus, $R(\theta_n, \theta_m) \geq 2n - 1$ where $m = 4, 5$. In the following lemma we determine the Ramsey number $R(\theta_5, \theta_5) = 9$.

Lemma 2.3. *The Ramsey number* $R(\theta_5, \theta_5) = 9$.

Proof. It is enough to show that $R(\theta_5, \theta_5) \leq 9$. Let a red-blue coloring of K_9 be given that contains neither a red θ_5-graph nor a blue θ_5-graph. By Theorem 1.1, K_9 must contain a blue C_5 or a red C_5. Without loss of generality we assume that K_9 has a red C_5. Let $x_1, x_2, \ldots, x_9$ be the vertices of K_9 and assume $x_1x_2x_3x_4x_5x_1$ is the red C_5. Define $H_1 = K_9[x_1, x_2, x_3, x_4, x_5]$ and $H_2 = K_9[x_6, x_7, x_8, x_9]$. Now we have the following observations.

(i) H_1 has no red chord as otherwise a red θ_5-graph is produced. Thus, H_1 contains a blue C_5.

(ii) Every vertex in H_2 is adjacent by at most 3 red (blue) edges to H_1 as otherwise a red (blue) θ_5-graph is produced.

(iii) If a vertex in H_2 is adjacent to H_1 by 3 red (blue) edges, then it must be adjacent to 3 consecutive vertices in H_1 with red (blue) color as otherwise K_9 would have a red (blue) θ_5-graph ((a, b, c) are consecutive with red (blue) if a is adjacent to b with a red (blue) edge and b is adjacent to c with a red (blue) edge).

(iv) Assume there are two vertices in H_2, say x_6 and x_7 are adjacent to H_1 by 3red (blue) edges each. Then x_6 and x_7 are adjacent to 3 consecutive vertices in H_1 and $|N_{H_1}(x_6) \cap N_{H_1}(x_7)| = 1$ as otherwise a red (blue) θ_5-graph is produced.

(v) There are exactly two vertices in H_2 adjacent to vertices of H_1 by exactly 3 red edges and two blue edges each, say x_6 and x_7, and so each of x_8 and x_9 is adjacent to vertices of H_1 by exactly 3 blue edges and two red edges.

To this end, one can notice from the above observations that if x_8 is adjacent to two nonadjacent vertices of C_5 by the red edges, then a red θ_5-graph is produced (Figure 2 depicts the situation), a contradiction. If x_8 is adjacent to two adjacent vertices of C_5 by the red edges, then x_8 adjacent to three non-consecutive vertices (of the internal blue cycle) by blue edges. And so a blue θ_5-graph is produced, a contradiction. This completes the proof. □

In the following lemma we determine the Ramsey number $R(\theta_4, \theta_5)$.

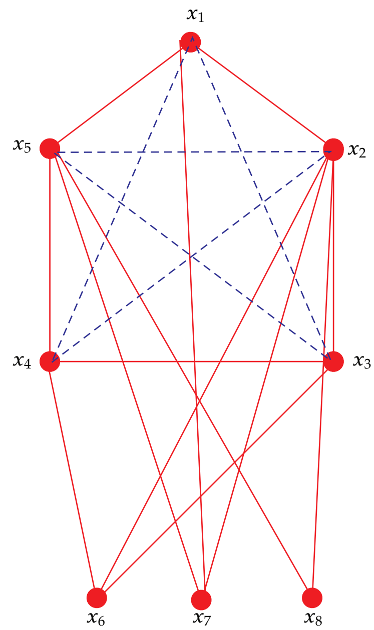

Figure 2: This figure depicts the case when x_8 is adjacent to two nonconsecutive vertices of C_5.

Lemma 2.4. *The Ramsey number* $R(\theta_4, \theta_5) = 9$.

Proof. It is enough to show that $R(\theta_4, \theta_5) \leq 9$. Let a red-blue coloring of K_9 be given. By Lemma 2.3, K_9 must contain a blue or a red θ_5-graph. If K_9 contains a blue θ_5-graph, then we are done. So, suppose K_9 has a red θ_5-graph. So, K_9 has a red triangle. Let $T_1 = x_1x_2x_3x_1$ be the red triangle. Let $y_1, y_2, \ldots, y_6$ be the remaining vertices. By Theorem 1.1, $H = K_9[y_1, y_2, \ldots, y_6]$ has a red or a blue C_3. We consider the following two cases.

Case 1. H contains a blue C_3. Let $T_2 = y_1y_2y_3y_1$ be the blue C_3. Every vertex in the blue C_3 is adjacent at least by two blue edges to T_1, as otherwise K_9 would has a red θ_4-graph. Let $d_{T_2}\text{blue}(x_i)$ denote the number of blue edges from x_i to T_2. We consider 3 subcases according to the number of blue edges of x_1, x_2, and x_3 to T_2.

Subcase 1.1. $d_{T_2}\text{blue}(x_1) = d_{T_2}\text{blue}(x_2) = d_{T_2}\text{blue}(x_3) = 2$, then a blue θ_5-graph is produced. Figure 3 depicts the situation.

Subcase 1.2. $d_{T_2}\text{blue}(x_1) = 3 = d_{T_2}\text{blue}(x_2)$ and $d_{T_2}\text{blue}(x_3) = 0$, then a blue θ_5-graph is produced. Figure 4 depicts the situation.

Subcase 1.3. $d_{T_2}\text{blue}(x_1) = 1, d_{T_2}\text{blue}(x_2) = 3,$ and $d_{T_2}\text{blue}(x_3) = 2$, then a blue θ_5-graph is produced. Figure 5 depicts the situation.

Case 2. H contains a red C_3, say $T_2 = y_1y_2y_3y_1$. Let y_4, y_5, y_6 be the remaining vertices of K_9. Observe that each vertex of y_4, y_5, y_6 is adjacent to at least two vertices of each T_1 and T_2 which are colored by blue, as otherwise, G contains a red θ_4-graph. Hence, if $K_9[y_4, y_5, y_6]$ has a blue edge, say y_5y_6, then a blue C_3 is produced. Hence, by the above case a blue θ_5-graph is produced. So, we need to consider the case when $K_9[y_4, y_5, y_6]$ has no blue edges.

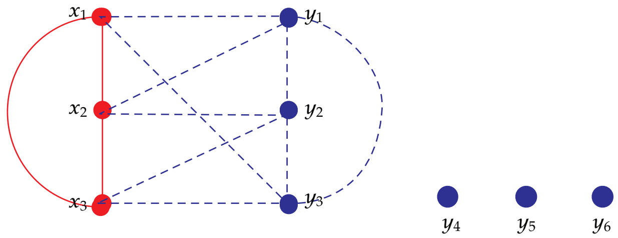

Figure 3: This figure represents the situation in Subcase 1.1.

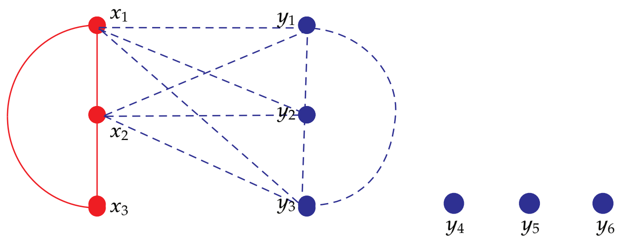

Figure 4: This figure represents the situation in Subcase 1.2.

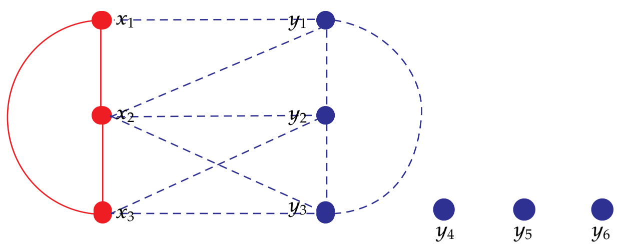

Figure 5: This figure represents the situation in Subcase 1.3.

Observe that the number of blue edges in K_9 is at least 18. Further, the induced graph by blue edges is non-bipartite. By Theorem 1.5, K_9 would have a blue θ_5-graph. This completes the proof. □

In the following lemma we determine the Ramsey number $R(\theta_4, \theta_6)$.

Lemma 2.5. *The Ramsey number* $R(\theta_4, \theta_6) = 11$.

Proof. It is enough to show that $R(\theta_4, \theta_6) \leq 11$. Let a red-blue coloring of K_{11} be given that contains neither a red θ_4 nor a blue θ_6. Suppose K_{11} has a blue cycle of length 6. Let $x_1, x_2, \ldots, x_{11}$ be the vertices of K_{11}, and assume $H_1 = x_1x_2x_3x_4x_5x_6x_1$ is the blue C_6. Then H_1 has no blue chord as otherwise a blue θ_6-graph is produced. So, $K_{11}[x_1, x_3, x_5]$ and $K_{11}[x_2, x_4, x_6]$ are red triangles. Every vertex of the remaining vertices must be adjacent to H_1 by at least 4 blue edges as otherwise a red θ_4-graph is produced. Now, let x_7 be a vertex of the remaining vertices that is adjacent to H_1 by 4 blue edges. We consider three cases.

Case 1. x_7 is adjacent to 4 consecutive vertices in H_1. Assume x_7 is adjacent to x_1, x_2, x_3, and x_4, then a blue θ_6-graph is produced.

Case 2. x_7 is adjacent to 3 consecutive vertices and a vertex separated in H_1. Assume x_7 is adjacent to x_2, x_3, x_4, and x_6, then a blue θ_6-graph is produced.

Case 3. x_7 is adjacent to a pair of 2 consecutive vertices separated from each other in H_1. Assume x_7 is adjacent to x_3, x_4 and x_1, x_6, then a blue θ_6-graph is produced.

So, we need to consider that K_{11} has no blue C_6. We need to prove that K_{11} has a red θ_4-graph. By contradiction, suppose K_{11} has no red θ_4-graph. By Theorem 1.1, K_{11} has a red C_3. So, the subgraph induced by red edges is a non-bipartite graph. By Theorem 1.4, the number of red edges is at most 27. So, the number of blue edges is at least 28. By Lemma 2.1, K_{11} has a blue C_3. Hence, the subgraph induced by blue edges is a non-bipartite graph. By Theorem 1.3, there is a blue C_6, this is a contradiction. This completes the proof. □

In the following theorem we determine the Ramsey number $R(\theta_4, \theta_k)$, for $k \geq 7$.

Theorem 2.6. *The Ramsey number* $R(\theta_4, \theta_k) = 2k - 1,\ k \geq 7$**.**

Proof. It is enough to show that $R(\theta_4, \theta_k) \leq 2k - 1,\ k \geq 7$. We prove it by contradiction. Let a red-blue coloring of K_{2k-1} be given. Suppose K_{2k-1} has a blue cycle of length k. Let $x_1, x_2, \ldots, x_{2k-1}$ be the vertices of K_{2k-1}, and assume $H = x_1 x_2 \cdots x_k x_1$ is the blue C_k. Then H has no blue chord as otherwise a blue θ_k-graph is produced. So, H contains a red θ_4-graph. This is a contradiction.

Now, we need to consider the case when K_{2k-1} has no blue cycle of length k. By Theorem 1.1, K_{2k-1} contains a red C_3. Let G_1 be the induced subgraph of the blue edges. Note that the subgraph induced by the red edges is a non-bipartite graph and contains no red θ_4. Hence, the number of red edges is at most $(2k-2)^2/4 + 2$. Thus, the number of blue edges is

$$
\begin{aligned}
\mathcal{E}(G_1) &\geq \frac{(2k-1)(2k-2)}{2} - \frac{(2k-2)^2}{4} - 2 \\
&= k^2 - k - 2 \\
&> k^2 - 2k + 3 \\
&\geq \frac{(2k-2)^2}{4} + 2.
\end{aligned}
\tag{2.1}
$$

Observe that G_1 is a non-bipartite graph ($R(\theta_4, C_3) = 7$ and K_{2k-1} does not contain a red θ_4-graph and so it contains a blue C_3, Lemma 2.1). If $c(G_1) \geq k$, then by Theorem 1.3, G_1 has a blue C_k, this is a contradiction. If $c(G_1) \leq k - 1$, then by Theorem 1.2

$$
\mathcal{E}(G_1) \leq k^2 - 2k + 1, \tag{2.2}
$$

which contradicts the inequality (2.1) for $k \geq 7$. This completes the proof. □

In the following theorem we determine the Ramsey number $R(\theta_5, \theta_k)$, for $k \geq 6$.

Theorem 2.7. *The Ramsey number* $R(\theta_5, \theta_k) = 2k - 1,\ \ k \geq 6$.

Proof. It is enough to show that $R(\theta_5, \theta_k) \leq 2k - 1,\ k \geq 7$. We prove it by contradiction. Let a red-blue coloring of K_{2k-1} be given. Suppose K_{2k-1} has a blue cycle of length k. Let $x_1, x_2, \ldots, x_{2k-1}$ be the vertices of K_{2k-1}, and assume $H = x_1 x_2 \cdots x_k x_1$ is the blue C_k. Then H

has no blue chord as otherwise a blue θ_k-graph is produced. So, H contains a red θ_5-graph. This a contradiction.

Now, we consider the case when K_{2k-1} has no blue C_k. Now, we have the following observations.

(i) $R(\theta_4, \theta_k) = 2k-1$, $k \geq 7$. Thus, the induced graph on the red edges is a non-bipartite graph.

(ii) $R(\theta_5, \theta_4) = 9$. Thus, the induced graph by the blue edges is a non-bipartite graph.

Let G_1 be the graph induced by the blue edges. Since K_{2k-1} has no red θ_5-graph, by Theorem 1.5 the number of red edges is at most $(2k-2)^2/4+1$. Thus, as in the above theorem,

$$\begin{aligned}\mathcal{E}(G_1) &\geq \frac{(2k-2)(2k-1)}{2} - \frac{(2k-2)^2}{4} - 1 \\ &= k^2 - k - 1\end{aligned} \tag{2.3}$$

edges. If $c(G_1) \geq k$, then by Theorem 1.3, there is a blue C_k. This is a contradiction. If $c(G_1) \leq k-1$, then

$$\mathcal{E}(G_1) \leq k^2 - 2k + 1. \tag{2.4}$$

This contradicts the inequality (2.3) for $k \geq 6$. □

Theorem 2.8. *For $n \geq 6$,*

$$R(\theta_n, \theta_n) = \begin{cases} \left(\dfrac{3n}{2}\right) - 1 & if\ n\ is\ even, \\ 2n-1 & if\ n\ is\ odd. \end{cases} \tag{2.5}$$

Proof. First we consider the case when n is odd. In K_{2n-2} color the edges of two vertex disjoint complete graphs of order $n-1$ with a red color and the remaining edges with a blue color. This coloring contains neither a red nor a blue θ_n-graph. We conclude that $R(\theta_n, \theta_n) \geq 2n-1$.

Let a red-blue coloring of K_{2n-1} be given that contains neither a red nor a blue θ_n. We know, by Theorem 1.1, that $R(C_n, C_n) = 2n-1$. Thus, K_{2n-1} contains either a red or a blue C_n. Without loss of generality, we suppose that K_{2n-1} has a blue C_n. Then there are no chords in C_n as otherwise a blue θ_n is produced. So, K_{2n-1} contains a red θ_n. This is a contradiction.

Now we consider the case when n is even. Let $K_{(3n/2)-2}$ $((3n/2)-2)$ be colored with two colors, say red and blue, as follows: the edges of vertex disjoint K_{n-1} and $K_{n/2-1}$ are colored blue and the remaining edges are colored red. This coloring contains neither a red nor a blue θ_n. We conclude that $R(\theta_n, \theta_n) \geq (3n/2)-1$.

To show that $R(\theta_n, \theta_n) \leq (3n/2)-1$, we follow, word by word, the above argument when n is odd by taking into account that $R(C_n, C_n) = (3n/2)-1$. The proof is complete. □

We conclude this paper by highlighting an interesting open problem. We begin with the following constructions.

Let $n \geq m \geq 6$. If θ_m contains an odd cycle, then in K_{2n-2} color the edges of a complete bipartite graph $K_{n-1,n-1}$ with blue and all the remaining edges with red. This coloring contains neither a red θ_n-graph nor a blue θ_m-graph. Thus, $R(\theta_n, \theta_m) \geq 2n - 1$.

Now, we consider the case when θ_m contains no an odd cycle. If n is even, then in $K_{n+(m/2)-2}$, color the edges of a complete bipartite graph $K_{n-1,(m/2)-1}$ with blue and all the remaining edges with red. This coloring contains neither a red θ_n-graph nor a blue θ_m-graph. Thus, $R(\theta_n, \theta_m) \geq n+m/2-1$. If n is odd, then in K_{2m-2}, color the edges of a complete bipartite graph $K_{m-1,m-1}$ with red and all the remaining edges with blue and in $K_{n+(m/2)-2}$, color the edges of a complete bipartite graph $K_{n-1,(m/2)-1}$ with blue and all the remaining edges with red. $K_{m-1,m-1}$ and $K_{n-1,(m/2)-1}$, respectively, provide examples for lower bounds when n is odd, respectively, when θ_m contains no an odd cycle. Thus, $R(\theta_n, \theta_m) \geq \max\{n+m/2-1, 2m-1\}$.

From the above construction we conjecture that for $n \geq m \geq 6$,

$$R(\theta_n, \theta_m) = \begin{cases} 2n-1 & \text{if } \theta_m \text{ contains odd cycle,} \\ \max\left\{n+\dfrac{m}{2}-1, 2m-1\right\} & \text{if } n \text{ is odd and } \theta_m \text{ does not contain an odd cycle,} \\ n+\dfrac{m}{2}-1 & \text{if } n \text{ is even and } \theta_m \text{ does not contain an odd cycle.} \end{cases} \tag{2.6}$$

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