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A NEW CLASS OF COMPOSITION OPERATORS

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ABSTRACT. A new class of composition operators P_ϕ: $H^2(T) \to H^2(T)$, with ϕ: $T \to \bar{D}$ is introduced. Sufficient conditions on ϕ for P_ϕ to be bounded and Hilbert-Schmidt are obtained. Properties of P_ϕ with $\phi(e^{it}) = ae^{it} + be^{-it}$ for different values of the parameters a and b have been investigated. This paper concludes with a discussion on the compactness of P_ϕ .

KEY WORDS AND PHRASES. H^p Space, Composition operator, Hilbert Schmidt operator, Compact operator.
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1. PRELIMINARIES.

For a complex valued function f analytic in $D = \{z: |z| < 1\}$ and for $1 \le p \le \infty$ set

$$M_p(r,f) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

and

$$M(r,f) = \sup_{0 < \theta < 2\pi} |f(re^{i\theta})|.$$

The function f is said to be in $H^p(D)$, if $\lim\limits_{r\to 1-} M_p(r,f) < \infty$. Similarly, let $H^p(T)$, $T = \{z: |z| = 1\}$, be the class of functions in $L^p(T)$ such that

$$\int_0^{2\pi} f(e^{i\theta})\, e^{in\theta}\, d\theta = 0 \ , \ n = 1,2,3 \ldots\ldots \ .$$

It is known [1,2] that for f in $H^p(D)$ $\lim\limits_{r\to 1-} f(re^{i\theta}) = f_*(e^{i\theta})$ exists for almost all θ and f_* belongs to $H^p(T)$. Conversely, the Poisson integral of a function in $H^p(T)$ is in $H^p(D)$. Also, if f in $H^p(D)$ has the sequence $\{a_n\}$ as its Taylor coefficients then, f_* has the same sequence as its Fourier coefficients and vice versa. This correspondence establishes an isometrical isomorphism between $H^p(D)$ and $H^p(T)$. Thus, these two spaces are interchangeably used and are usually referred to as the Hardy Space H^p[1,2].

In the sequel we came across another space familiarly known as the weighted Hardy space [3] . Let $\rho(n)$ be a sequence of positive numbers. An analytic function $f:D \to C$, given by $f(z) = \Sigma a_n z^n$, is said to be in the class $H^2(\rho)$, if $||f||_\rho = \Sigma |a_n|^2 \rho(n) < \infty$. Also we need the following definition. Let H be a Hilbert space and T be a bounded linear operator on H. Then, T is said to be Hilbert Schmidt if there exists an orthonormal basis $\{e_n\}$ in H such that $\Sigma ||Te_n||^2 < \infty$.

Throughout in the present paper we denote by e_n , $n = 0,1,2,\ldots$, the function $e_n(e^{it}) = e^{int}$. We note that $\{e_n\}$ forms an orthonormal basis for H^2.

2. A NEW CLASS OF CONPOSITION OPERATORS.

Let ϕ: $D \to D$ be analytic and let C_ϕ: $H^P(D) \to H^P(D)$ be defined by $(C_\phi f)(z) = f(\phi(z))$, z in D. The operator C_ϕ is known as a composition operator on $H^P(D)$ and is extensively studied in the literature [4]. In the present paper we introduce and study a new class of composition operators P_ϕ on $H^2(T)$ where ϕ: $T \to \overline{D}$ many be 'non-analytic' also. That is ϕ nonvanishing negative Fourier coefficients.

DEFINITION. Let ϕ: $T \to \overline{D}$ satisfy the following properties:

(a) for every set $E \subset T$, of linear measure zero, $\phi^{-1}(E) = \{z \in T: \phi(z) = w, w \in T\}$ is also a set of linear measure zero and

(b) for every f in $H^2(T)$, $fo\phi$ is in $L^2(T)$.

Then, define P_ϕ: $H^2(T) \to H^2(T)$ by $P_\phi f = P(fo\phi)$ where P is the projection of $L^2(T)$ into $H^2(T)$.

Here some explanations are in order. We observe that a function f in $H^2(T)$ can be extended analytically into D as described in Section 1. So with the condition (a), $fo\phi$ is defined almost everywhere on T . Futher, let f be represented by the Fourier series $\sum_{n=0}^{\infty} a_n e^{in\Theta}$. Then by the Weierstrass theorem, $\sum_{n=0}^{\infty} a_n(\phi(e^{i\Theta}))^n$ converges pointwise to $f(\phi(e^{i\Theta}))$ for all Θ such that $\phi(e^{i\Theta}) \in D$ and by a result of Carleson [5] $\sum_{n=0}^{\infty} a_n(\phi(e^{i\Theta}))^n$ converges pointwise to $f(\phi(e^{i\Theta}))$ for almost all Θ such that $\phi(e^{i\Theta}) \in T$. Hence $\sum_{n=0}^{\infty} a_n(\phi(e^{i\Theta}))^n$ converges pointwise almost everywhere on T to $f(\phi(e^{i\Theta}))$. Thus throughout in this paper we write $\sum_{n=0}^{\infty} a_n(\phi(e^{i\Theta}))^n$ in place of $f(\phi(e^{i\Theta}))$.

We note theat if ϕ satisfies the conditions of the definition, then by the Closed Graph Theorem, P_ϕ is a bounded operator. So a natural question is: under what conditions on ϕ, $fo\phi$ is in $L^2(T)$ for all f in $H^2(T)$. The present paper primarily deals with this question.

In the following sections we first obtain bounds for the norm of P_ϕ under suitable conditions on ϕ. Then we consider ϕ defined by $\phi(e^{it}) = ae^{it} + be^{-it}$ and study conditions on a and b such that $f\circ\phi \in L^2(T)$ for all f in $H^2(T)$. In the last section we have discussed the compactness of P_ϕ with the help of some examples.

3. NORM OF P_ϕ.

We have the following results.

THEOREM 1. Let $\phi: T \to \bar{D}$ be such that

$$\int_0^{2\pi} \frac{dt}{1 - |\phi(e^{it})|^2} = M(\phi) < \infty \tag{3.1}$$

Then, P_ϕ is Hilbert Schmidt and $||P_\phi|| \le (M(\phi)/2\pi)^{1/2}$.

PROOF. Let f in $H^2(T)$ be given by $f(z) = \Sigma a_n z^n$, z in T. Then,

$$|f(\phi(e^{i\theta})|^2 = | \Sigma a_n(\phi(e^{i\theta}))^n|^2 \le (\Sigma|a_n|^2)\ (\Sigma|\phi(e^{i\theta})|^{2n})$$

$$= ||f||^2 \frac{1}{1 - |\phi(e^{i\theta})|^2} \quad \text{a.e.}$$

So,

$$||P_\phi f||^2 \le ||f\circ\phi||_2^2 \le ||f||_2^2 \ \frac{M(\phi)}{2\pi}$$

and we get $||P_\phi|| \le (M(\phi)/2\pi)^{1/2}$.

Next, with the orthonormal basis e_n, n = 0,1,2,..., of $H^2(T)$, we have

$$\sum_{r=0}^{\infty} ||P_\phi(e_r)||^2 \le \sum_{n=0}^{\infty} ||e_n \circ \phi||^2 = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{it})|^{2n}\, dt = (M(\phi)/2\pi) < \infty.$$

Thus, P_ϕ is Hilbert Schmidt.

COROLLARY 2. If $\phi: T \to D$ is continuous then P_ϕ is Hilbert Schmidt.

PROOF. The condition (3.1) is trivially satisfied if ϕ is continuous.

By an example in the next section we will show that (3.1) is only a sufficient condition for P_ϕ to be Hilbert Schmidt. We need the following lemma due to Gabriel [6] for the proof of our next theorem.

LEMMA. Let Γ be a rectifiable convex curve in the closed unit disc. Then, for every f in H^2

$$\int_\Gamma |f(w)|^2\ |dw| \le 4\pi||f||_2^2$$

THEOREM 2. Let $\phi: T \to \bar{D}$ be such that

(i) ϕ describes a closed rectifiable convex curve in $\bar{D}$ and

(ii) $m = \inf|\phi'(e^{it})| > 0$, $0 \le t \le 2\pi$,

Then, $||P_\phi|| \le (2/m)^{1/2}$.

PROOF. By lemma and the condition (ii) we have

$$4\pi||f||_2^2 \ge \int_0^{2\pi} |f(\phi(e^{it}))|^2\ |\phi'(t)|\ dt \ge m \int_0^{2\pi} |(f\circ\phi)\ (e^{it})|^2 dt$$

so that

$$||P_\phi f||^2 \le \frac{1}{2\pi} \int_0^{2\pi} |(f\circ\phi)\ (e^{it})|^2\ dt \le \frac{2}{m}\ ||f||_2^2$$

The conditions (i) and (ii) in the above theorem are not necessary for P_ϕ to be bounded. As an example consider

$$\phi\,(t) = (e^{it}) = \begin{cases} e^{it} & 0 < t < \pi \\ 0 & \pi \le t \le 2\pi \end{cases}$$

so that ϕ does not satisfy any of the conditions (i) or (ii) of the theorem.
Now,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\phi)(e^{it}))|^2\,dt = \frac{1}{2\pi}\int_0^{\pi} |f(e^{it})|^2\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} |f(o)|^2 dt \le \;||f||_2^2 .$$

This shows that P_ϕ is bounded with $||P_\phi|| \le \sqrt{2}$.

4. A FAMILY OF COMPOSITION OPERATORS.

In this section we study the properties of P_ϕ for the particular family of functions $\phi : T \to \overline{D}$ given by

$$\phi(z) = az + b\overline{z}\;, \quad z \in T \tag{4.1}$$

where $|a| + |b| \le 1$. We note that if $|a| \ne |b|$ then the curve traced by ϕ is an ellipse containing the orgin in its interior. Also $m = \inf|ae^{-it} - b| \ge ||a| - |b||>0$. Hence by Theorem 2, P_ϕ is bounded. It turns out that P_ϕ has many interesting properties for different values of the parameters a and b. We need the following technical lemma.

LEMMA 1. For all n, k in Z_+

$$\binom{n+2k}{k} < 2^{n+2k} \tag{4.2}$$

PROOF. We shall prove (4.2) by method of induction on n. Let $n = 0$ so that we have to show

$$\binom{2k}{k} < 2^{2k} \quad \text{for} \quad k = 1,2,3,\ldots\ldots . \tag{4.3}$$

We establish (4.3), also by the process of induction on k. For $k = 1$, $\binom{2}{1} = 2 < 2^2$ is trivially true. Next, assume that

$$\binom{2k}{k} < 2^{2k}, \quad \text{i.e.} \quad \frac{(2k)!}{(k!)\,(k!)} < 2^{2k}$$

To complete induction on k we consider

$$\binom{2(k+1)}{k+1} = \frac{(2k+2)!}{(k+1)!(k+1)!} < 2^{2k}\; 2\,\frac{(2k+1)}{(k+1)} < 2^{2(k+1)} .$$

Thus, (4.3) is true for all $k = 1,2,3\ldots.$. Next, let $n = 1$. Then,

$$\binom{2k+1}{k} = \binom{2k}{k}\frac{(2k+1)}{(k+1)} < 2^k \cdot 2 = 2^{k+1} .$$

Now, assume that $\binom{n+2k}{k} < 2^{n+2k}$. To complete the induction we consider

$$\binom{n+1+2k}{k} = \frac{(n+2k)!}{(k!)(n+k)!}\,\frac{(n+1+2k)}{(n+1+k)} < 2^{n+2k} \cdot 2 < 2^{n+1+2k} .$$

Thus (4.3) is true for all $k = 1,2,\ldots..$ and $n = 0,1,2\ldots..$.

THEOREM 3. Let ϕ: $T \to \bar{D}$ be given by $\phi(z) = az + b\bar{z}$, z in T.

(i) If $|a| + |b| \leq 1$, $|a| \neq |b|$, $b \neq 0$ and $|a| < \frac{1}{2}$ then P_ϕ is Hilbert Schmidt.

(ii) If $|a| + |b| = \frac{1}{2}$ then $f \circ \phi$ need not be in L^2 for all f in H^2 so that P_ϕ is not defined on the whole of H^2.

(iii) The inequality $|a| < \frac{1}{2}$ in (i) is best possible.

PROOF. Consider the orthonormal basis e_n, $n = 0,1,2,\ldots$, for H^2. With respect to this basis P_ϕ has a matrix representation

$$t_{mn} = \begin{cases} 0 & \text{if } n < m \\ \binom{n+2k}{k} a^n (ab)^k & \text{if } n - m = 2k \\ 0 & \text{if } n - m = 2k + 1 \end{cases}$$

where m, n, and $k \in Z_+$. Now

$$\Sigma |t_{m,n}|^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{n+2k}{k}^2 |a|^{2n} |ab|^{2k}$$

$$\sum_{n=0}^{\infty} \left(|a_n|^{2n} + \sum_{k=1}^{\infty} \binom{n+2k}{k}^2 |a|^{2n} |ab|^{2k}\right)$$

$$= \frac{1}{1-|a|^2} + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \binom{n+2k}{k}^2 |a|^{2n} |ab|^{2k}$$

We use Lemma 1 to show that the second sum in the right hand side is convergent.

$$\sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \binom{n+2k}{k}^2 |a|^{2n} |ab|^{2k} \leq \Sigma\Sigma\, 2^{2n+4k} |a|^{2n} |ab|^{2n} = \Sigma\Sigma\, |2a|^{2n} |4ab|^{2k}$$

$$= \frac{1}{1-|2a|^2} \cdot \frac{1}{1-|4ab|^2} < \infty,$$

since $|a| < \frac{1}{2}$, which also implies $|ab| < \frac{1}{4}$. This proves that P_ϕ is Hilbert Schmidt. (ii) For the proof of (ii) consider the H^2 function $f(z) = (1-z)^{-\alpha}$, $0<\alpha<\frac{1}{2}$. For $a = b = \frac{1}{2}$, $\phi(e^{i\theta}) = \cos\theta$. Thus,

$$f(\phi(e^{i\theta})) = f(\cos\theta) = \frac{1}{(1-\cos\theta)^\alpha} = \frac{1}{(2^\alpha \sin^{2\alpha}\frac{\theta}{2})} \qquad \text{a.e.}$$

and

$$\int_0^{2\pi} |(f(\phi(e^{i\theta}))|^2\, d\theta = \int_0^{\pi/2} \frac{2^{2-2\alpha}}{\sin^{4\alpha}\theta}\, d\theta \geq \int_0^{\pi/2} \frac{2^{2-2\alpha}}{\theta^{4\alpha}}\, d\theta = \infty$$

if $\alpha > \frac{1}{4}$. In the above we have made use of the well known inequality $\frac{2\theta}{\pi} < \sin\theta < \theta$ for $0 < \theta < \frac{\pi}{2}$.

(iii) In view of (ii), for the proof of (iii), it is sufficient to show that P_ϕ is not Hilbert Schmidt if $a + b = 1$ and $a > b$. In fact, we show that under the above condition $\sum_m |t_{m,m}|^2 = \infty$.

Observe that

$$e_n(\phi(e^{i\theta})) = (ae^{i\theta} + be^{-i\theta})^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k e^{i(n-k)\theta} e^{-ik\theta} .$$

Since $(a + b)^n = 1$, we have $\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = 1$. Hence considering this sum as the inner product of two vectors

$$\left[\binom{n}{0} a^n b^0 , \binom{n}{1} a^{n-1} b^1, \ldots , \binom{n}{n} a^0 b^n \;\middle|\text{and}\middle|\; 1,1 \underbrace{\ldots\ldots\ldots\ldots}_{(n+1)\text{ terms}} 1 \right]$$

We see that, since $||[1,1,\ldots\ldots, 1]||^2 = (n+1)$, by Cauchy Schwarz inequality

$$\sum_{k=0}^{n} |\binom{n}{k} a^{n-k} b^k |^2 \geq \frac{1}{(n+1)} .$$

Further, we observe that if $a > b$ then $\binom{n}{r} a^{n-r} b^r > \binom{n}{n-r} a^r b^{n-r}$ so that over half of the above sum is from terms where $n-k \geq k$ and so $||P_\phi e_n||^2 \geq 1/2(n+1)$, leading us to $\sum_n |t_{n,n}|^2 = \infty$. This completes the proof of the theorem.

Also, with the help of the same function $\phi(z) = az + b\bar{z}$, we show that the condition (3.1) of Theorem 1 is not a necessary condition for P_ϕ to be a Hilbert Schmidt. For this take a, b in R, $ab > 0$ and $|a| + |b| = 1$. Then,

$$\int_0^{2\pi} \frac{dt}{1-|\phi(e^{it})|^2} = \int_0^{2\pi} \frac{dt}{(1-(a-b)^2) \sin^2 t} = \infty .$$

However, in view of Theorem 3, it follows that P_ϕ is Hilbert Schmidt.

In the following theorem we present a sufficient condition on f in H^2 to ensure that $f o\phi$ is in $L^2(T)$.

THEOREM 4. Let ϕ: $T\to\bar{D}$ be given by $\phi(z) = az + b\bar{z}$. $|a| = |b| = \frac{1}{2}$ and f in H^2 be given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Further if $a_n = 0(\frac{1}{n^\alpha})$ with $\alpha > \frac{3}{4}$, then $f o\phi \in L^2$.

PROOF. First let $a = b = \frac{1}{2}$, so that $\phi(e^{i\theta}) = \cos \theta$. Now,

$$|f(\phi(e^{i\theta}))|^2 \leq |\sum_{n=0}^{\infty} \frac{\cos^n \theta}{n^\alpha} |^2 \tag{4.4}$$

We know that

$$\frac{1}{(1-z)^{\beta+1}} = \sum_{n=0}^{\infty} \binom{n+\beta}{n} z^n \tag{4.5}$$

and [7]

$$\binom{n+\beta}{n} \sim \frac{n^\beta}{\Gamma(\beta+1)} . \tag{4.6}$$

Taking $\beta = -\alpha$ in (4.5) and (4.6), we get

$$n=0 \frac{\cos^n \theta}{n^\alpha} \sim \frac{\Gamma(-\alpha+1)}{(1-\cos\theta)^{(1-\alpha)}} .$$

Thus, to complete the proof, it is sufficient to show that

$$\int_0^{2\pi} (1-\cos\theta)^{(2\alpha-2)} d\theta = \int_0^{2\pi} \sin^{(4\alpha-4)} \frac{\theta}{2} \, d\theta < \infty .$$

However, this is ture because of the condition $\alpha > \frac{3}{4}$. To dispose of the general case we observe that if $2a = e^{i\alpha}$ and $2b = e^{i\gamma}$ then $\phi(e^{i\theta}) = e^{i(\alpha+\gamma)/2}\cos(\frac{\alpha-\gamma+2\theta}{2})$ and this leads to similar calculations as above.

Taking cue from the above theorem we next show that $f\circ\phi \in L^2$ for $f \in H^2(\rho(n))$ for a suitable choice of the sequence $\rho(n)$.

THEOREM 5. Let ϕ: $T \to \overline{D}$ be given by $\phi(e^{i\theta}) = ae^{i\theta} + be^{-i\theta}$, $|a| = |b| = \frac{1}{2}$ and $\rho(n) = n^{\beta}$. Then,

(i) $f\circ\phi$ is in L^2 for all f in $H^2(\rho(n))$ if $\beta > \frac{1}{2}$,

(ii) for each $\beta < \frac{1}{2}$ there is a function f_β in $H^2(\rho(n))$ such that $f_\beta\circ\phi$ is not in L^2

PROOF. As in the previous theorem we assume $a = b = \frac{1}{2}$ so that $f(\phi(e^{i\theta})) = f(\cos\theta)$. Let f in $H^2(\rho)$ be given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We have

$$|f(\phi(e^{i\theta}))|^2 = |\sum_{n=0}^{\infty} a_n \cos^n\theta|^2 \le (\sum_{n=0}^{\infty} |a_n|^2 \rho(n))\,(\sum_{n=0}^{\infty} \frac{\cos^{2n}\theta}{\rho(n)}).$$

Now using (4.5) and (4.6) as in the previous theorem it can be shown that

$$\sin^{2\beta-2}\theta \sim \sum_{n=0}^{\infty} \frac{n^{-\beta}}{\Gamma(-\beta+1)} \cos^{2n}\theta.$$

Thus $f\circ\phi$ is in L^2 if $\beta > \frac{1}{2}$.

For the proof of (ii) consider the function

$$f(z) = \frac{1}{(1-z)^{\alpha+1}} = \sum A_n^{\alpha} z^n .$$

By (4.6),

$$\Sigma |A_n^{\alpha}|^2\, n^{\beta} \sim \Sigma\, n^{2\alpha+\beta} .$$

The sum on the right hand converges if $2\alpha + \beta < -1$ i.e. $\alpha < -(\beta+1)/2$. Thus, f is in $H^2(\rho)$, $\rho(n) = n^{\beta}$ for $\alpha < -(\beta+1)/2$. However,

$$\int_0^{2\pi} |f(\phi(e^{i\theta}))|^2\, d\theta = \frac{1}{2^{2\alpha+1}} \int_0^{\pi} \frac{1}{\sin^{4\alpha+4}\theta}\, d\theta = \infty$$

if $\alpha \ge -\frac{3}{4}$. Thus, for given $\beta < \frac{1}{2}$, if we chose $\alpha = -(3+2\beta+2)/8$, f is in $H^2(\rho)$ but $f\circ\phi$ is not in L^2.

REMARK. The case $\rho(n) = n^{\frac{1}{2}}$, remains open in the above theorem. However, in the next theorem we prove the same result for a sequence $\rho(n)$ having faster rate of growth than $n^{1/2}$ but with slower rate than $n^{1/2+\varepsilon}$ for any $\varepsilon > 0$.

THEOREM 6. Let ϕ: $T \to \overline{D}$ be as in the previous theorem and $\rho(n) = n^{1/2}(\log n)^{\beta}$. Then,

(i) $f\circ\phi$ is in L^2 for all f in $H^2(\rho(n))$ if $\beta > 1$,

(ii) for each $\beta < 0$, there is a function f_β in $H^2(\rho(n))$ such that $f_\beta\circ\phi$ is not in L^2.

PROOF (i) Let f, given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, be in $H^2(\rho(n))$ so that

$$||f||_\rho = \Sigma\, n^{1/2}(\log n)^\beta |a_n|^2 < \infty .$$

By Cauchy Schwarz inequality,

$$|f(\phi(e^{i\theta}))|^2 \le \sum_{n=0}^{\infty} n^{1/2}(\log n)^\beta\, |a_n|^2)(\sum_{n=0}^{\infty} \frac{\cos^{2n}\theta}{n^{1/2}(\log n)^\beta})) . \tag{4.7}$$

It is known [7, p. 192] that if

$$\frac{1}{(1-z)^{\alpha+1}} (\log \frac{a}{1-z})^\beta = \sum_{n=0}^{\infty} A_n^{(\alpha,\beta)} z^n \tag{4.8}$$

then for $a > 2$, $\alpha \neq -1,-2,-3,\dots$, $\alpha, \beta \in R$

$$A_n^{(\alpha,\beta)} \sim \frac{n^\alpha}{\Gamma(\alpha+1)} (\log n)^\beta$$

so that

$$A_n^{(-1/2,-\beta)} \sim \frac{1}{\sqrt{\pi}\, n^{1/2}(\log n)^\beta}$$

i.e.

$$\frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(\cos^2\theta)^n}{n^{1/2}(\log n)^\beta} \sim \frac{1}{(1-\cos^2\theta)^{1/2}} (\log \frac{a}{1-\cos^2\theta})^{-\beta} = \frac{2^{-\beta}}{\sin\theta} (\log \frac{b}{\sin\theta})^{-\beta} ,$$

$$b = \sqrt{a} \tag{4.9}$$

In view of (4.7) and (4.9), in order to show that $fo\phi$ is in L^2, it is sufficient to show that

$$\int_0^{2\pi} \frac{1}{\sin\theta} (\log \frac{1}{\sin\theta})^{-\beta} d\theta < \infty .$$

Further, because of the inequality $(2\theta/\pi) < \sin\theta < \theta$, it is sufficient to show the integrability, in an interval $(0,\delta)$, of the function

$$h(\theta) = \frac{1}{\theta} (\log \frac{1}{\theta})^{-\beta}$$

Making the substitution $\log (\frac{1}{\theta}) = u$, we get

$$\lim_{\varepsilon\to 0} \int_\varepsilon^\delta \frac{1}{\theta}(\log \frac{1}{\theta})^{-\beta} d\theta = \lim_{\varepsilon\to 0} \frac{(\log \frac{1}{\varepsilon})^{1-\beta} - (\log \frac{1}{\delta})^{1-\beta}}{1-\beta}$$

Thus, the above integral converges if $\beta > 1$. This completes the proof of (i).

(ii) Let $\rho(n) = n^{1/2}(\log n)^{-\beta}$, $\beta > 0$. Now consider the function

$$f(z) = \frac{1}{(1-z)^{1/4}} (\log \frac{a}{1-z})^\gamma = \Sigma A_n^\gamma z^n$$

where $-1/2 > \gamma < (\beta-1)/2$. We first observe that f is in $H^2(\rho(n))$. In fact, a comparison of f with (4.8) shows that

$$A_n^\gamma \sim \frac{n^{-3/4}}{\Gamma(1/4)} (\log n)^\gamma .$$

Thus,

$$\Sigma |A_n^\gamma|^2 \rho(n) \sim \Sigma n^{-1} (\log n)^{2\gamma-\beta}$$

and the right hand side series converges because $\gamma < (\beta-1)/2$. Now,

$$\int_0^{2\pi} |f(\phi(e^{i\theta}))|^2 \, d\theta = 2^{\gamma-1/2}\, 4 \int_0^{\pi/2} \frac{1}{\sin\theta}\left(\log \frac{b}{\sin\theta}\right)^{2\gamma} d\theta$$

where $b = \frac{a}{2}$. The above integral diverges with the integral

$$\int_0^{\pi/2} \frac{1}{\theta}\left(\log \frac{1}{\theta}\right)^{2\gamma} d\theta$$

because of the condition $\gamma > \frac{1}{2}$. Thus, we prove that although $f \in H^2(\rho(n))$, $fo\phi$ is not in L^2.

REMARK. The case $\rho(n) = n^{\frac{1}{2}}(\log n)^{\beta}$, $0 \leq \beta < 1$ remains unsettled.

We conclude this section by showing that P_ϕ is an unbounded operator on H^2 for $\phi(e^{it}) = (e^{it} + e^{-it})/2 = \cos t$. This we do by exhibiting a sequence of function f_n in H^2 for which $\lim_{n\to\infty} ||P_\phi f_n|| = \infty$

Let $f_n(z) = \sum_{k=1}^{n} \frac{z^k}{k}$ and $f(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = \log \frac{1}{1-z}$ so that

$$g_n(t) = f_n(\phi(e^{it})) = \sum_{k=1}^{n} \frac{\cos^k t}{k} \quad \text{and} \quad g(t) = f(\phi(e^{it})) = \sum_{k=1}^{\infty} \frac{\cos^k t}{k}.$$

Observe that g is in L^1 but is not in L^2. Let a_k and $a_n^{(n)}$ respectively be the k^{th} Fourier coefficients of g and g_n. Since

$$\lim_{n\to\infty} \int_0^{2\pi} |g_n(t) - g(t)| \, dt = 0$$

we get

$$\lim_{n\to\infty} a_n^{(n)} = a_k. \quad \text{Now,}$$

$$\lim_{n\to\infty} ||P_\phi||_2^2 = \lim_{n\to\infty} \sum_{k=-n}^{n} |a_k^{(n)}|^2 = \sum_{k=-\infty}^{\infty} |a_k|^2 \quad \frac{1}{2\pi} \int |g(e^{it})|^2 \, dt = \infty.$$

5. COMPACTNESS OF P_ϕ

In this section we discuss some examples illustrating cases when P_ϕ is compact and when it is not. Let $\phi_1, \phi_2, \phi_3 : T \to \overline{D}$, be defined by

(i) $\phi_1(e^{it}) = ae^{-it}$, $\quad |a| = 1 \quad 0 \leq t \leq 2\pi$

(ii) $$\phi_2(e^{it}) \begin{cases} e^{it}, & 0 \leq t < \pi \\ 0, & \pi \leq t \leq 2\pi \end{cases}$$

(iii) $$\phi_3(e^{it}) = \begin{cases} e^{it}, & 0 \leq t \leq \pi \\ ae^{it} + be^{-it}, & \pi \leq t \leq 2\pi \quad a+b = 1 \quad a \neq b, \quad a,b \geq 0 \end{cases}$$

(i) $P_{\phi 1}$ is a finite rank, hence a compact, operator. For, if f in H^2 is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $(P_{\phi_1} f)(e^{it}) = P\left(\sum_{n=0}^{\infty} a_n a^n e^{-int}\right) = a_o$.

For composition operators with analytic ϕ Schwartz [8] has shown that if C_ϕ: $H^p(D) \to H^p(D)$ is compact then $|\phi(e^{it})| < 1$ a.e. where $\phi(e^{it})$ is the radial limit of $\phi(z)$. We observe in this example that $P_{\phi 1}$ deviates in behaviour from C_ϕ.

(ii) We have shown at the end of Section 3 that $||P_{\phi_2}|| < \sqrt{2}$. We show here that P_{ϕ_2} is not compact.

By Riemann-Lebesgue Lemma the sequence e_n, $n = 0,1,2,\ldots$ converges to zero weakly in H^2. However, $P_{\phi_2}(e_n) = P(e_n o \phi_2)$, does not converge strongly to zero. For, if the Fourier series of $e_n o \phi_2$ is given by $(e_n o \phi_2)(e^{it}) = \sum_{m=-\infty}^{\infty} a_m e^{imt}$, then by direct computation it can be seen that

$$a_m = \begin{cases} \dfrac{i}{\pi(n-m)} & \text{if } n-m \text{ is odd} \\ 0 & \text{if } n-m \text{ is even} \\ \dfrac{1}{2} & \text{if } n=m \end{cases}$$

and $||P_{\phi_2}(e_n)||_2^2 = \sum_{m=0}^{\infty} |a_m|^2 > \frac{1}{4}$. Thus, P_{ϕ_2} is not compact.

By a similar argument as in (ii) it can be shown that P_{ϕ_3} is bounded but not a compact operator.

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