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STRONG LAWS OF LARGE NUMBERS FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS

ROBERT LEE TAYLOR

Department of Statistics
University of Georgia
Athens, GA 30602 U.S.A.

TIEN-CHUNG HU

Department of Mathematics
National Tsing-Hua University
Hsin-chu, Taiwan R.O.C.

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ABSTRACT. Let $\{X_{nk}\}$ be an array of rowwise independent random elements in a separable Banach space of type $p + \delta$ with $EX_{nk} = 0$ for all k, n. The complete convergence (and hence almost sure convergence) of $n^{-1/p}\Sigma_{k=1}^{n} X_{nk}$ to 0, $1 \leq p < 2$, is obtained when $\{X_{nk}\}$ are uniformly bounded by a random variable X with $E|X|^{2p} < \infty$. When the array $\{X_{nk}\}$ consists of i.i.d. random elements, then it is shown that $n^{-1/p}\Sigma_{k=1}^{n} X_{nk}$ converges completely to 0 if and only if $E\|X_{11}\|^{2p} < \infty$.

KEY WORDS AND PHRASES. Random elements, Strong laws of large numbers, Complete Convergence, Rademacher type $p + \delta$ spaces.
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1. INTRODUCTION AND PRELIMINARIES.

Let $(E, \|\ \|)$ be a real separable Banach space. Let (Ω, A, P) denote a probability space. A random element X in E is a function from Ω into E which is A-measurable with respect to the Borel subsets $\mathcal{B}(E)$. The p^{th} absolute moment of a random element X is $E\|X\|^p$ where E is the expected value of the random variable $\|X\|^p$. The expected value of X is defined to be the Bochner integral (when $E\|X\| < \infty$) and is denoted by EX. The concepts of independence and identical distributions have direct extensions to E. A separable Banach space is said to be of (Rademacher) type p, $1 \leq p \leq 2$, if there exists a constant C such that

$$E \left\| \sum_{k=1}^{n} X_k \right\|^p \leq C \sum_{k=1}^{n} E\|X_k\|^p$$

for all independent random elements $X_1, \ldots, X_n$ with zero means and finite p^{th} moments. Every separable Hilbert space and finite-dimensional Banach space is of type 2. Every separable Banach space is at least type 1 while the ℓ^p and L^p spaces are of type $\min\{2,p\}$ for $p \geq 1$.

Throughout this paper $\{X_{nk}: 1 \leq k \leq n, n \geq 1\}$ will denote rowwise independent random elements in E such that

$$EX_{nk} = 0 \quad \text{for all n and k} \tag{1.1}$$

and such that $\{X_{nk}\}$ are uniformly bounded by a random variable X with

$$E|X|^{2p} < \infty \quad \text{for some } 1 \leq p < 2. \tag{1.2}$$

Recall that an array $\{X_{nk}\}$ of random elements is said to be uniformly bounded by a random variable X if for all n and k and for every real number $t > 0$

$$P[\|X_{nk}\| > t] \leq P[|X| > t]. \tag{1.3}$$

Note that i.i.d. random elements are uniformly bounded by $\|X_{11}\|$. The major results of this paper show that

$$\frac{1}{n^{1/p}} \Sigma_{k=1}^{n} X_{nk} \to 0 \text{ completely} \tag{1.4}$$

where complete convergence is defined (as in Hsu and Robbins [1]) by

$$\Sigma_{n=1}^{\infty} P[\| \frac{1}{n^{1/p}} \Sigma_{k=1}^{n} X_{nk}\| > \varepsilon] < \infty \tag{1.5}$$

for each $\varepsilon > 0$.

Erdös [2] showed that for an array of i.i.d. random variables $\{X_{nk}\}$, (1.4) holds if and only if $E|X_{11}|^{2p} < \infty$. Jain [3] obtained a uniform strong law of large numbers for sequences of i.i.d. random elements in separable Banach spaces of type 2 which would yield (1.4) with $p = 1$ for an array of i.i.d. random elements $\{X_{nk}\}$ in a type 2 space. Woyczynski [4] showed that

$$\frac{1}{n^{1/p}} \Sigma_{k=1}^{n} X_{k} \to 0 \text{ completely} \tag{1.6}$$

for any sequence $\{X_n\}$ of independent random elements in a type $p + \delta$, $1 \leq p < 2$ and $\delta > 0$, with $EX_n = 0$ for all n which is uniformly bounded by a random variable X satisfying $E|X|^p < \infty$. Móricz, Hu and Taylor [5] showed that Erdös' result could be obtained by replacing the i.i.d. condition by the uniformly bounded condition (1.3). In addition, they showed that Jain's result for i.i.d. random elements with $p = 1$ did not require the space to be type 2 but held in all separable Banach spaces. In this paper, (1.4) is established in type $p + \delta$ spaces, $1 \leq p < 2$ and $\delta > 0$, for uniformly bounded rowwise independent random elements. For i.i.d. random elements in type $p + \delta$ spaces, it is shown that (1.4) holds if and only if $E\|X_{11}\|^{2p} < \infty$. Thus, no sharper moment conditions are possible.

2. MAJOR RESULTS.

Many authors (starting with Beck [6]) have related the strong law of large numbers for non-identically distributed, independent random elements in separable Banach spaces to the necessity of the space being of type $p + \delta$ for $1 \leq p < 2$ and some $\delta > 0$. Consequently, attention is restricted to type $p + \delta$ spaces in this paper. Three lemmas will be used in obtaining the major results. They are stated here without proof. Lemma 1 with $r = 1$ is in most textbooks while Lemma 2 is accomplished using integration by parts. Lemma 3 is in Woyczynski [4].

LEMMA 1. For any $r \geq 1$, $E|X|^r < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{r-1} P[|X| > n] < \infty.$$

More precisely, $r2^{-r} \sum_{n=1}^{\infty} n^{r-1} P[|X| > n]$

$$\leq E|X|^r \leq 1 + r2^r \sum_{n=1}^{\infty} n^{r-1} P[|X| > n].$$

LEMMA 2. If $r \geq 1$, then for any $p > 0$

$$E\left(|X|^r I_{[\,|X| \leq n^{1/p}\,]}\right) \leq r \int_0^{n^{1/p}} t^{r-1} P[\,|X| > t\,]\,dt$$

and

$$E\left(|X|\, I_{[\,|X| > n^{1/p}\,]}\right) = n^{1/p} P[\,|X| > n^{1/p}\,] + \int_{n^{1/p}}^{\infty} P[\,|X| > t\,]\,dt.$$

LEMMA 3. Let $1 \leq p \leq 2$ and $q \geq 1$. The following properties are equivalent:

(i) E is of type p.

(ii) There exists a C such that for all independent random elements $X_1, \ldots, X_n$ in E with $EX_k = 0, k = 1, \ldots, n$,

$$E \left\| \sum_{k=1}^{n} X_k \right\|^q \leq C\, E\left(\sum_{k=1}^{n} \| X_k \|^p\right)^{q/p}.$$

THEOREM 4. If $\{X_{nk}\}$ is an array of rowwise independent random elements in a type $p + \delta$ space, $1 \leq p < 2$ and $\delta > 0$, which are uniformly bounded by a random variable X such that (1.1) and (1.2) holds, then

$$\frac{1}{n^{1/p}} \Sigma_{k=1}^{n} X_{nk} \to 0 \text{ completely.}$$

PROOF. Define

$$Y_{nk} = X_{nk} I_{[\,\|X_{nk}\| \leq n^{1/p}\,]} \qquad 1 \leq k \leq n,\ n \geq 1. \tag{2.1}$$

Then, by Lemma 1 (with r = 2),

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} P[\,X_{nk} \neq Y_{nk}\,] = \sum_{n=1}^{\infty} \sum_{k=1}^{n} P[\,\|X_{nk}\| > n^{1/p}\,]$$
$$\leq \sum_{n=1}^{\infty} nP[\,|X| > n^{1/p}\,]$$
$$= \sum_{n=1}^{\infty} nP[\,|X|^p > n\,] \leq 2E|X|^{2p} < \infty.$$

Next, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left[\,\left\|\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} - \frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk}\right\| > \varepsilon\,\right]$$

$$\leq \sum_{n=1}^{\infty} P\left[\,\bigcup_{k=1}^{n} [X_{nk} \neq Y_{nk}]\,\right]$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{n} P[\,X_{nk} \neq Y_{nk}\,] < \infty.$$

Therefore,

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk} - \frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk} \right\| \to 0 \quad \text{completely,}$$

and it sufficies to prove that

$$\left\| \frac{1}{n^{1/p}} \sum_{k=1}^{n} Y_{nk} \right\| \to 0 \text{ completely.} \tag{2.2}$$

To this end, let

$$Z_{nk} = Y_{nk} - EY_{nk} \quad (k=1,2,\ldots,n; \ n=1,2,\ldots).$$

Then for $1 \le q \le 2p$ it follows by Hölder's inequality that

$$(E\|Z_{nk}\|^q)^{1/q} \le 2(E\|Y_{nk}\|^q)^{1/q}$$
$$\le 2(E\|Y_{nk}\|^{2p})^{1/(2p)} \le 2(E|X|^{2p})^{1/(2p)},$$

so that

$$E\|Z_{nk}\|^q \le 2^q(E|X|^{2p})^{q/(2p)}$$
$$\le 2^{2p}(1 + E|X|^{2p}) = C_1, \text{ say.} \tag{2.3}$$

Furthermore,

$$\|Z_{nk}\| \le \|Y_{nk}\| + \|EY_{nk}\| \le 2n^{1/p}. \tag{2.4}$$

Following the techniques of Taylor [7] in expanding a high power of a sum, let $r = p + \delta$ and ν be chosen so that

$$s = \frac{\nu}{r} \text{ is an integer and } \nu > \left(\frac{1}{p} - \frac{1}{r}\right)^{-1}. \tag{2.5}$$

It is readily seen that $E\left(\sum_{k=1}^{n} \|Z_{nk}\|\right)^{\nu} < \infty$, so that, by Lemma 3,

$$E\left(\left\|\sum_{k=1}^{n} Z_{nk}\right\|\right)^{\nu} \le C\, E\left(\sum_{k=1}^{n} \|Z_{nk}\|^r\right)^{s}.$$
$$= C \sum_{k_1,\ldots,k_s} E\left(\prod_{j=1}^{s} \|Z_{nk_j}\|^r\right) \tag{2.6}$$

where the sum is extened for all s-tuples $(k_1,\ldots,k_s)$ with $k_j = 1, 2, \ldots, n$ for each j. The general term to be considered then will have

$$q_1 \text{ of the k's} = \xi_1, \ \ldots, \ q_m \text{ of the k's} = \xi_m;$$

$$r_1 \text{ of the k's} = \eta_1, \ \ldots, \ r_\ell \text{ of the k's} = \eta_\ell;$$

where

$$r \le rq_i \le 2p, \ rr_j > 2p, \text{ and} \tag{2.7}$$

$$\sum_{i=1}^{m} q_i + \sum_{j=1}^{\ell} r_j = s. \tag{2.8}$$

Clearly, $q_i = 1$. Then, using (2.3) and (2.4), we can conclude that

$$E\left(\prod_{i=1}^{m} \|Z_{n\xi_i}\|^{rq_i} \prod_{j=1}^{\ell} \|Z_{n\eta_j}\|^{rr_j}\right) \tag{2.9}$$

$$= \prod_{i=1}^{m} E\|Z_{n\xi_i}\|^{rq_i} \prod_{j=1}^{\ell} E\left(\|Z_{n\eta_j}\|^{2p}\|Z_{n\eta_j}\|^{rr_j-2p}\right)$$

$$\le C_1^{m+\ell} \prod_{j=1}^{\ell} (2n^{1/p})^{rr_j-2p}$$

$$= C_1^{m+\ell}\, 2^{\sum_{j=1}^{\ell}(rr_j - 2p)}\, n^{\sum_{j=1}^{\ell}(rr_j/p)-2\ell}$$

$$\leq C_1^{\nu}\, 2^{\nu}\, n^{\sum_{j=1}^{\ell} (rr_j/p)-2\ell}$$

$$= C_2\, n^{\sum_{j=1}^{\ell} (rr_j/p)-2\ell}, \text{ say.}$$

Combining all possible terms of form (2.9), we can write

$$E\left(\sum_{k=1}^{n} \|Z_{nk}\|^{r}\right)^{s} \tag{2.10}$$

$$\leq C_3 \sum_{q_1,\ldots,q_m;r_1,\ldots,r_\ell}^{*} \; \sum_{\xi_1,\ldots,\xi_m;\eta_1,\ldots,\eta_\ell}^{**} E\left(\prod_{i=1}^{m} \|Z_{n\xi_i}\|^{rq_i} \prod_{j=1}^{\ell} \|Z_{n\eta_i}\|^{rr_j}\right)$$

$$= C_3 \sum_{q_1,\ldots,q_m;r_1,\ldots,r_\ell}^{*} S_{q_1,\ldots,q_m;r_1,\ldots,r_\ell}, \quad \text{say,}$$

where Σ^{*} is extended over all m-tuples $(q_1,\ldots,q_m)$ and ℓ-tuples $(r_1,\ldots,r_\ell)$ such that Conditions (2.7) and (2.8) are satisfied (the cases $m = 0$ or $\ell = 0$ may also occur), while Σ^{**} is extended over all $(m + \ell)$ - tuples $(\xi_1,\ldots,\xi_m;\ \eta_1,\ldots,\eta_\ell)$ of different integers between 1 and n and C is a constant independent of n. Let $m + \ell = t$. Obviously, $1 \leq t \leq s$. We distinguish two cases according to $t \geq 2$ or $t = 1$.

<u>Case</u> $t \geq 2$. By (2.9)

$$S_{q_1,\ldots,q_m;r_1,\ldots,r_\ell}$$

$$\leq C_2 \sum_{\xi_1,\ldots,\xi_m;\eta_1,\ldots,\eta_\ell}^{**} n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell}$$

$$\leq C_2\, n^{\sum_{j=1}^{\ell} (rr_j/p) - 2\ell + t}. \tag{2.11}$$

Now, the power to which n is raised here can be estimated by means of (2.8) and $q_i = 1$ as follows

$$\frac{1}{p} \sum_{j=1}^{\ell} rr_j - 2\ell + t$$

$$= \frac{1}{p}\left(rs - \sum_{i=1}^{m} rq_i\right) - 2(t - m) + t$$

$$= \frac{\nu}{p} - \frac{rm}{p} - 2(t - m) + t$$

$$= \frac{\nu}{p} - t - m\left(\frac{r}{p} - 2\right). \tag{2.12}$$

We distinguish two further subcases according to $m = t$ or $m \leq t - 1$.

<u>Subcase</u> $m = t$. By assumption $1 \leq p < 2$. Also, $q_i = 1$ for each i.

Thus, $m = s$ and, by (2.5),

$$t + m\left(\frac{r}{p} - 2\right) = s\left(\frac{r}{p} - 1\right)$$

$$= \nu\left(\frac{1}{p} - \frac{1}{r}\right) > 1. \tag{2.13}$$

Subcase $m \le t - 1$. Then $t - m \ge 1$ and even $t - m \ge 2$ in the particular case where $m = 0$. Thus, again

$$t + m\left(\frac{r}{p} - 2\right) = (t - m) + m\left(\frac{r}{p} - 1\right) > 1. \tag{2.14}$$

Now we turn to

Case $t = 1$. In this case necessarily $m = 0$ and $\ell = 1$, consequently $r_1 = s$ and

$$S_{q_1,\ldots,q_m;r_1,\ldots,r_\ell} = S_{0;s} = \sum_{k=1}^{n} E\|Z_{nk}\|^{rs}.$$

Using Lemma 2, we obtain that

$$\Sigma_1 = \sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} \sum_{k=1}^{n} E\|Z_{nk}\|^{\nu}$$

$$\le 2^{\nu} \sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} \sum_{k=1}^{n} E\|Y_{nk}\|^{\nu}$$

$$\le 2^{\nu} \sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} \sum_{k=1}^{n} \nu \int_0^{n^{1/p}} t^{\nu-1} P[\|X_{nk}\| > t]\,dt$$

$$\le 2^{\nu} \sum_{n=1}^{\infty} \frac{1}{n^{\nu/p}} \nu\, n \int_0^{n^{1/p}} t^{\nu-1} P[\,|X| > t\,]\,dt.$$

Letting $t = n^{1/p} s^{1/\nu}$ and applying Lemma 1 (with $r = 2$), it follows that

$$\Sigma_1 \le 2^{\nu} \sum_{n=1}^{\infty} n \int_0^{1} P[\,|X| > n^{1/p} s^{1/\nu}\,]\,ds$$

$$= 2^{\nu} \int_0^{1} \sum_{n=1}^{\infty} nP[\,|s^{-1/\nu} X|^{p} > n\,]\,ds \tag{2.15}$$

$$\le 2^{\nu+1} \int_0^{1} s^{-2p/\nu} E|X|^{2p}\,ds$$

$$= 2^{\nu+1} \frac{\nu}{\nu-2p} E|X|^{2p} < \infty.$$

Using Markov's inequality, (2.7) and (2.10) - (2.15) we have, for any $\varepsilon > 0$,

$$\Sigma_2(\varepsilon) = \sum_{n=1}^{\infty} P[\,\|\frac{1}{n^{1/p}} \sum_{k=1}^{n} Z_{nk}\| > \varepsilon\,]$$

$$\le \sum_{n=1}^{\infty} \frac{1}{(\varepsilon n^{1/p})^{\nu}} E\left(\|\sum_{k=1}^{n} Z_{nk}\|\right)^{\nu}$$

$$\le \sum_{n=1}^{\infty} \frac{C}{(\varepsilon n^{1/p})^{\nu}} E\left(\sum_{k=1}^{n} \|Z_{nk}\|^{r}\right)^{s}$$

$$\leq \frac{CC_3}{\varepsilon^\nu}\Big[\sum_{n=1}^{\infty}\frac{1}{n^{\nu/p}}\sum_{k=1}^{n}E\|Z_{nk}\|^\nu$$

$$+\sum_{n=1}^{\infty}\frac{C_2}{n^{\nu/p}}\sum_{t=2}^{s}\sum_{q_1,\ldots,q_m;r_1,\ldots,r_\ell}^{(t)} n^{\frac{\nu}{p}-t-m(\frac{r}{p}-2)}\Big]$$

$$=\frac{CC_3}{\varepsilon^\nu}\Big[\Sigma_1+C_2\sum_{t=2}^{s}\sum_{q_1,\ldots,q_m;r_1,\ldots,r_\ell}^{(t)}\sum_{n=1}^{\infty} n^{-t-m(\frac{r}{p}-2)}\Big],$$

where $\Sigma^{(t)}$ means that the sum is extended over all m-tuples $(q_1,\ldots,q_m)$ and ℓ-tuples $(r_1,\ldots,r_\ell)$ with Conditions (2.7) and (2.8) such that $m+\ell=t$. Since the number of terms in each of $\Sigma^{(t)}$ is finite and the exponent of n is less than -1, for every $\varepsilon>0$, we have $\Sigma_2(\varepsilon)<\infty$. Thus, we have proved that

$$\Big\|\frac{1}{n^{1/p}}\sum_{k=1}^{n}Z_{nk}\Big\|\to 0 \quad \text{completely } (n\to\infty).$$

In order to prove (2.2), we need to establish

$$\Sigma_3=\sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\sum_{k=1}^{n}\|EY_{nk}\|<\infty. \tag{2.16}$$

To achieve this goal, we will proceed as follows. By (2.1),

$$Y_{nk}=X_{nk}\,I_{[\,\|X_{nk}\|\leq n^{1/p}\,]}$$

$$=X_{nk}-X_{nk}\,I_{[\,\|X_{nk}\|>n^{1/p}\,]}.$$

Since $EX_{nk}=0$, hence

$$\|E\,Y_{nk}\|\leq E\Big(\|X_{nk}\|\,I_{[\,\|X_{nk}\|>n^{1/p}\,]}\Big).$$

Thus, using Lemma 2,

$$\Sigma_3\leq\sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\sum_{k=1}^{n}E\Big(\|X_{nk}\|\,I_{[\,\|X_{nk}\|>n^{1/p}\,]}\Big)$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^{1/p}}\sum_{k=1}^{n}\Big(n^{1/p}P[\,\|X_{nk}\|>n^{1/p}\,]+\int_{n^{1/p}}^{\infty}P[\,\|X_{nk}\|>t\,]\,dt\Big)$$

$$\leq\sum_{n=1}^{\infty}\Big(n\,P[\,|X|>n^{1/p}\,]+\frac{n}{n^{1/p}}\int_{n^{1/p}}^{\infty}P[\,|X|>t\,]\,dt\Big).$$

Letting $t=n^{1/p}s$ and applying Lemma 1, we can conclude that

$$\Sigma_3\leq\sum_{n=1}^{\infty}n\,P[\,|X|^p>n\,]+\sum_{n=1}^{\infty}n\int_1^{\infty}P[\,|X|>n^{1/p}s\,]\,ds$$

$$\leq 2\,E|X|^{2p}+\int_1^{\infty}\sum_{n=1}^{\infty}P[\,|s^{-1}X|^p>n\,]\,ds$$

$$\leq 2E|X|^{2p} + \int_1^\infty s^{-2p} E|X|^{2p} ds$$

$$= \frac{4p-1}{2p-1} E|X|^{2p} < \infty,$$

proving (2.2) through (2.16), and thereby completing the proof of Theorem 4. ///

Note that if $\sup_{nk} E\|X_{nk}\|^{2p+\alpha} < \infty$, for some $\alpha > 0$, then there exists a r.v. X such that $\{X_{nk}\}$ are uniformly bounded by X and $E|X|^{2p} < \infty$. Therefore, Corollary 5 follows.

COROLLARY 5. Let E be a type $p+\delta$ separable Banach space for $1 \leq p < 2$ and $\delta > 0$. If $\sup_{nk} E\|X_{nk}\|^{2p+\alpha} < \infty$ for some $\alpha > 0$, then

$$\|\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk}\| \to 0 \text{ completely.}$$

For type $1 + \delta$ spaces, Taylor [7] obtained

$$\Sigma_{k=1}^{\infty} a_{nk} X_{nk} \to 0 \text{ completely} \tag{2.17}$$

where $\{X_{nk}\}$ is uniformly bounded by X with $E|X|^{1+1/r} < \infty$ and $\{a_{nk}\}$ are Toeplitz weights with $\max_k |a_{nk}| = O(n^{-r})$. In the special case of uniform weights $a_{nk} = \frac{1}{n}$, $1 \leq k \leq n$, then $r = 1$ and Theorem 4 can be thought of as an extension of this result. Extension of Theorem 4 to infinite arrays and general weights $\{a_{nk}\}$ are possible but the detailed verification of their proofs are not included here. However, it will be shown next that the moment condition $E|X|^{2p} < \infty$ cannot be reduced in Theorem 4. In particular, for an array $\{X_{nk}\}$ of i.i.d. random elements in a type $p+\delta$ space with $EX_{11} = 0$, it will be shown that the SLLN holds if and only if $E\|X_{11}\|^{2p} < \infty$.

THEOREM 6. Let $\{X_{nk}\}$ be an array of i.i.d. random elements in a type $p+\delta$ space, $1 \leq p < 2$ and $\delta > 0$, with $EX_{11} = 0$. Then $E\|X_{11}\|^{2p} < \infty$ if and only if

$$\frac{1}{n^{1/p}} \Sigma_{k=1}^{n} X_{nk} \to 0 \quad \text{completely.} \tag{2.18}$$

PROOF: From Theorem 4, we know that $E\|X_{11}\|^{2p} < \infty$ implies (2.18) since the array $\{X_{nk}\}$ is uniformly bounded by $\|X_{11}\|$.

Now, assume that (2.18) holds. Since $\{X_{nk}\}$ are i.i.d., for every n and $\varepsilon > 0$

$$P[\|\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{kk}\| > \varepsilon] = P[\|\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk}\| > \varepsilon].$$

By (2.18), for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P[\|\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{kk}\| > \varepsilon] < \infty, \tag{2.19}$$

which says $\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{kk} \to 0$ a.s..

As a consequence,

$$\frac{1}{n^{1/p}} X_{nn} = \frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{kk} - \left(\frac{n-1}{n} \frac{1}{n-1}\right)^{1/p} \sum_{k=1}^{n-1} X_{kk} \to 0 \quad \text{a.s..}$$

Let $\varepsilon = 1$. It follows from Lemma 1 (with $r = 1$) and the Borel-Cantelli lemma that

$$E\|X_{11}\|^p \leq 1 + 2 \sum_{n=1}^{\infty} P[\,\|X_{11}\|^p > n\,]$$

$$= 1 + 2 \sum_{n=1}^{\infty} P[\,\|\frac{1}{n^{1/p}} X_{nn}\| > 1\,] < \infty.$$

Hence,

$$n P[\,\|X_{11}\|^p > n\,] \to 0. \tag{2.20}$$

By (2.18),

$$P[\,\|\sum_{k=1}^{n} X_{nk}\| < n^{1/p}\,] \to 1. \tag{2.21}$$

Therefore, from (2.20) and (2.21) there exists N such that if $n \geq N$ then

$$n P[\,\|X_{11}\|^p > n\,] < \frac{1}{4} \quad \text{and} \quad P[\,\|\sum_{k=1}^{n} X_{nk}\|^p < n\,] > \frac{1}{2}. \tag{2.22}$$

Next, define the events

$$A_{nk} = [\max_{1\leq i<k} \|X_{ni}\| \leq 2n^{1/p},\ \|X_{nk}\| > 2n^{1/p},\ \text{and}\ \|\sum_{\substack{i=1\\ i\neq k}}^{n} X_{ni}\| < n^{1/p}\,]$$

$$(k = 1, 2, \ldots, n\colon n = 1, 2, \ldots).$$

Clearly, $\{A_{nk}\colon k = 1, 2, \ldots, n\}$ are disjoint subsets of the event $[\,\|\sum_{k=1}^{n} X_{nk}\| > n^{1/p}\,]$ for each $n = 1, 2, \ldots$. A familiar reasoning yields that

$$P[\,\|\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk}\| > 1\,] \geq \sum_{k=1}^{n} P(A_{nk})$$

$$= \sum_{k=1}^{n} P[\,\|X_{nk}\| > 2n^{1/p}\,]\ P\left[\bigcap_{i=1}^{k-1} [\,\|X_{ni}\| \leq 2n^{1/p}\,] \cap [\,\|\sum_{\substack{i=1\\ i\neq k}}^{n} X_{ni}\| < n^{1/p}\,]\right]$$

$$\geq \sum_{k=1}^{n} P[\,\|X_{nk}\| > 2n^{1/p}\,]\left(P[\,\|\sum_{\substack{i=1\\ i\neq k}}^{n} X_{ni}\| < n^{1/p}\,] - P[\bigcup_{i=1}^{n} [\,\|X_{ni}\| > 2n^{1/p}]\,]\right)$$

$$\geq \sum_{k=1}^{n} P[\,\|X_{11}\| > 2n^{1/p}\,]\left(P[\,\|\sum_{i=1}^{n-1} X_{ni}\| < (n-1)^{1/p}\,] - nP[\,\|X_{11}\| > 2n^{1/p}]\right).$$

Hence, by (2.22), for $n \geq N$,

$$P[\,\|\frac{1}{n^{1/p}} \sum_{k=1}^{n} X_{nk}\| > 1\,] \geq \frac{1}{4}\, nP[\,\|X_{11}\|^p > 2^p n\,].$$

Therefore, $\sum_{n=1}^{\infty} n P[\,\|X_{11}\|^p > 2^p n\,] < \infty$. Thus, Lemma 1 yields $E\|X_{11}\|^{2p} < \infty$. ///

CONCLUDING REMARKS.

1. It should be noted that the case p = 1 in Theorem 6 is obtainable in a type 1 space (cf: Theorem 4 of Hu, Moricz and Taylor [5]). In which case type 1 + δ is not needed.

2. For sequences of independent random elements which are uniformly bounded by a random variable X with $E|X|^p < \infty$, (1.6) holding necessitates the space being of type $p + \delta$ (cf: Woycyznski [4] and Maurey and Pisier [8]). Thus, the necessity of type $p + \delta$ follows for Theorem 4.

3. Theorem 6 shows that Theorem 4 is the best possible moment condition when no conditions on possible relations between the rows of the array are assumed.

4. In [4] it is mentioned that $n^{-1/p}\Sigma_{k=1}^{n} X_k \to 0$ a.s. for i.i.d. random elements $\{X_n\}$ with $EX_1 = 0$ and $E\|X_1\|^p < \infty$ apparently is equivalent to the space being of type p. Thus, it is interesting to conjecture whether Theorem 6 remains valid for only type p spaces $1 \leq p < 2$. Certainly, the "if part" is true for type p spaces, and Remark 1 indicates that it is true p = 1.

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