Internat. J. Math. & Math. Sci.
VOL. 11 NO. 1 (1988) 87-94

ON STAR POLYNOMIALS, GRAPHICAL PARTITIONS AND RECONSTRUCTION

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(Received April 29, 1986)

ABSTRACT. It is shown that the partition of a graph can be determined from its star polynomial and an algorithm is given for doing so. It is subsequently shown (as it is well known) that the partition of a graph is reconstructible from the set of node-deleted subgraphs.

KEY WORDS AND PHRASES. star, star polynomial, star cover, graphical partition, reconstruction.
AMS SUBJECT CLASSIFICATION CODES. 05A99, 05C99.

1. INTRODUCTION.

The graphs here will be finite, undirected, and will have no loops or multiple edges. We define an *m-star* S_m to be a tree consisting of a node of valency m (called the centre of S_m) joined to m other nodes. A *0-star* is a node and a *1-star* is an edge.

Let G be a graph. A *star-cover* (or simply, a *cover*) of G is a spanning subgraph whose components are all stars. Let us associate with each m-star in G, an indeterminate or *weight* w_{m+1}; and with each star cover C consisting of S_{m_1}, S_{m_2}, ..., S_{m_k}; the weight

$$w(C) = \prod_{i=1}^{k} w_{m_i}.$$

Then the *star polynomial* of G (relative to the given weight assignment) is

$$E(G;\underline{w}) = \Sigma w(C),$$

where the summation is taken over all the star covers in G and $\underline{w} = (w_1, w_2, \ldots)$ is a general weight vector. The basic results on star polynomials are given in the introductory paper by Farrell [1].

We will show that Π_G - the partition of a graph G can be obtained from $E(G;\underline{w})$. This will then be used to show that Π_G is node-reconstructible, a result that can be established by more elementary means (see Tutte [2]).

For brevity, we will write E(G) for $E(G;\underline{w})$, since the same weight vector $\underline{w}$ will be used throughout the paper. Also, in partitions, we will use r^k to denote the occurence of k r's. Finally, we will assume that $\binom{n}{r} = 0$, for all $r > n$.

2. STAR POLYNOMIALS AND GRAPHICAL PARTITIONS.

First of all, we state a lemma which can be easily proved.

LEMMA 1. Let v be a node of valency d in G . Then G contains $\binom{d}{m}$ m-stars with centre v .

DEFINITION. Let G be a graph with p nodes. A *simple m-cover* of G is a cover consisting of an m-star and p-m-1 isolated nodes.

It is clear that a simple m-cover in G will have weight $w_1^{p-m-1}w_{m+1}$ in E(G). A monómial of this form in E(G) will be referred to as a *simple term*, and its coefficient, which will be denoted by c_m - a *simple coefficient*. c_m will be the number of simple m-covers in G. Note that the term w_p will also be a simple term.

LEMMA 2. Let G be a graph with p nodes. Let $\Pi_G = (n^{b_n}, \ldots, 2^{b_2}, 1^{b_1}, 0^{b_0})$ $(0 \le n \le p-1)$. Then for $m > 1$,

$$c_m = \sum_{r=m}^{n} \binom{r}{m} b_r.$$

PROOF. This is straightforward.

The following lemma can be easily proved.

LEMMA 3. Let n be the largest valency of a node in G . Then E(G) contains all the terms $w_1^{p-r-1}w_{r+1}$ $(0 \le r \le n)$ with non-zero coefficient. i.e $c_r \neq 0$ for for $(0 \le r \le n)$.

Suppose that we put n = 1 in the above Lemma. Then G will consist of a set of disjoint edges and possibly isolated nodes. Clearly then the partition of G will be given by

$$\Pi_G = (1^{c_1}, 0^{p-2c_1}). \tag{2.1}$$

Hence Π_G can be found from E(G).

If $n > 1$, then from Lemma 2, we get

$$c_k = \sum_{r=k}^{n} \binom{r}{k} b_r = b_k + \sum_{r=k+1}^{n} \binom{r}{k} b_r, \quad \text{for } k > 1.$$

$$\Rightarrow \quad b_k = c_k - \sum_{r=k+1}^{n} \binom{r}{k} b_r. \tag{2.2}$$

Therefore b_k can be obtained from c_k (which is defined in E(G)) and all the higher b's i.e. $b_{k+1}, b_{k+2}, \ldots, b_n$. For k = 1 and 0, we can find b_1 and b_0 as follows: c_1 is the number of edges in G. Therefore the sum of the valencies of the nodes of G is $2c_1 = \sum_{k=1}^{n} kb_k = b_1 + \sum_{k=2}^{n} kb_k$.

$$b_1 = 2c_1 - \sum_{k=2}^{n} kb_k. \tag{2.3}$$

b_0 is the number of isolated nodes in G . Therefore

$$b_0 = p - \sum_{i=1}^{n} b_i. \tag{2.4}$$

Our discussion yields the following theorem.

THEOREM 1. Let G be a graph with p nodes and let $\Pi_G = (n^{b_n},\ldots,2^{b_2},1^{b_1},0^{b_0})$, where $0 \le n \le p-1$. Then

$$b_k = c_k - \sum_{r=k+1}^{n} \binom{r}{k} b_r, \quad \text{for } 1<k \le n.$$

$$b_1 = 2c_1 - \sum_{r=2}^{n} rb_r$$

and

$$b_0 = p - \sum_{r=1}^{n} b_r.$$

For $n = 1$, Π_G is given by Equation (2.1). For $n = 0$, the result is trivial.

Theorem 1 yields an algorithm for obtaining Π_G from E(G). This algorithm is illustrated in the following example:

EXAMPLE 1. Let G be a graph such that

$$E(G) = w_1^6 + 6w_1^4w_2 + 7w_1^3w_3 + w_1^2w_4 + 8w_1^2w_2^2 + 12w_1w_2w_3 + w_2^3 + w_2w_4 + 3w_3^2 \sim$$

First of all, we observe from the term w_1^6, that G has 6 nodes i.e. $p = 6$. The simple terms in E(G) are $6w_1^4w_2$, $7w_1^3w_3$ and $w_1^2w_4$. Therefore $c_1 = 6$, $c_2 = 7$ and $c_3 = 1$. Since the largest k for which w_k occurs in E(G) is 4, it follows that $n = 3$. Now

$$b_3 = c_3 = 1; \quad b_2 = c_2 - \sum_{r=3}^{3} \binom{r}{2} b_r = 7 - 3b_3 = 7-3 = 4.$$

$$b_1 = 2(6) - \sum_{k=2}^{3} kb_k = 12 - 2b_2 - 3b_3 = 12-8-3 = 1.$$

$$b_0 = p - \sum_{r=1}^{3} b_r = 6-1-4-1 = 0.$$

Hence $\Pi_G = (3^1 2^4 1^1)$.

It would be nice to be able to obtain G itself from E(G). From Theorem 1, Π_G can be obtained. However there can be several graphs with the same partition. Since only the simple terms in E(G) are used to obtain Π_G, it is not surprising that G itself is not clearly defined. Should G itself be clearly defined by the simple terms, then it would mean that the remaining terms of E(G) are useless as far as the characterization of G is concerned. It would be interesting to investigate the nature of these 'useless terms'.

Suppose that Π_G is unigraphic (i.e. there is only one graph with partition Π_G), then G could be uniquely constructed from Π_G. Hence we have the following theorem.

THEOREM 2. Let Π_G be unigraphic. Then G can be constructed from E(G).

PROOF. This follows from Theorem 1 and the discussion above.

THEOREM 3. Let G and H be two graphs with p nodes. Then $\Pi_G = \Pi_H$ if and only if $E(G)$ and $E(H)$ have the same simple coefficients.

PROOF. Suppose that the simple coefficients in $E(G)$ and $E(H)$ are equal. Then from Theorem 1, G and H must have the same partition. Conversely, suppose that $\Pi_G = \Pi_H$. Then from Lemma 2, the coefficients c_r $(r > 1)$ in $E(G)$ and $E(H)$ must be equal. Finally, $c_0 = 1$, for all graphs. Hence $E(G)$ and $E(H)$ have the same simple coefficients. The result therefore follows.

3. STAR POLYNOMIALS AND RECONSTRUCTION.

The following theorem is analogous to the result for circuit polynomials given in Lemma 3 of Farrell and Grell [3], with $i = 1$. We suspect that the general result holds for all F-polynomials (see Farrell [4]). Here $G-x$ denotes the graph obtained from G by removing node x. $V(G)$ is the node set of G.

THEOREM 4.

$$\frac{\partial E(G)}{\partial w_1} = \sum_{x \in V(G)} E(G-x;\underline{w})$$

PROOF. Let us write $E(G) = \sum_j A_j w_1^{n_{1,j}} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}$, where p is the number of nodes in G. Then

$$\frac{\partial E(G)}{\partial w_1} = \sum_j n_{1,j} A_j w_1^{n_{1,j}-1} w_2^{n_{2,j}} \dots w_p^{n_{p,j}} . \qquad (3.1)$$

It is clear that the monomial $w_1^{n_{1,j}-1} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}$ is the weight of a cover with one isolated node less than the corresponding cover in G. It is therefore the weight of a cover in $G-x$, for some node x in G. Hence it is a monomial of the polynomial $\sum E(G-x;\underline{w})$. Conversely, every cover of $G-x$ can be extended to a cover of G by adding an isolated node. Therefore every monomial m in $\sum E(G-x;\underline{w})$ yields a corresponding monomial $w_1 m$ in G. The derivative of $w_1 m$ with respect to w_1 yields a term with monomial m. It follows that $\frac{\partial E(G)}{\partial w_1}$ and $\sum E(G-x;w)$ have the same monomials.

Since A_j is the coefficient of $w_1^{n_{1,j}} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}$, G has A_j covers consisting of $n_{1,j}$ isolated nodes $n_{2,j}$ edges,..., $n_{p,j}(p-1)$ - stars. Suppose that node x is removed from G. Then $G-x$ will have a similar cover but with $n_{1,j}-1$ isolated nodes. Since node x could be any of the $n_{1,j}$ isolated nodes in the cover, it follows that each such cover in G gives rise to $n_{1,j}$ covers with one less isolated node. Hence the coefficient of $w_1^{n_{1,j}-1} w_2^{n_{2,j}} \dots w_p^{n_{p,j}}$ in $\sum E(G-x;\underline{w})$ is $n_{1,j}A_j$. From Equation (3.1) it follows that the monomials occur in $\sum E(G-x;\underline{w})$ and $\frac{\partial E(G)}{\partial w_1}$ with equal coefficients. Hence the result follows.

Throughout the rest of this section, we will assume that the graph has at least three nodes. Also 'reconstructible' would mean node-reconstructible. By the *deck*

D_G a graph G we would mean the set $\{G-x: x \in V(G)\}$.

It is well known (see Harary [5], Kelly [6], and Chatrand and Kronk [7]) that disconnected graphs are reconstructible. It follows that Π_G is reconstructible if G is disconnected, and therefore the number of isolated nodes in G is reconstructible. We can however, prove the latter independently. The following lemma gives a connection between the number of isolated nodes and D_G.

LEMMA 4. Let G be a graph with p nodes. Then G has r (>0) isolated nodes if and only if D_G has exactly r graphs with $r-1$ isolated nodes and $p-r$ graphs with r or more isolated nodes.

PROOF. Suppose that G has r isolated nodes. Then D_G must be of the form described. Conversely, suppose that D_G is as described in the theorem. Let k be the number of isolated nodes in G. Since every element of D_G has at least $r-1$ isolated nodes, G cannot have less than $r-1$ isolated nodes or it would mean that the removal of each node from G yields at least one new isolated node, and this is impossible unless G is a matching (in which case the result holds). Therefore $k \geq r-1$. Since D_G contains no elements with less that $r-1$ isolated nodes, $k \neq r-1$. Hence $k > r-1$. But exactly r elements of D_G has $r-1$ isolated nodes and only one node can be removed at a time from G to form an element of D_G. Therefore G has exactly r nodes each of whose removal reduces k to $r-1$. These nodes must be themselves isolated nodes. $\Rightarrow$ $k \geq r$. Clearly $k \not> r$. Therefore $k = r$ and the result follows.

From the above lemma, we see that b_0 (of Theorem 1) can be obtained from D_G. It follows that if D_G is given, then we can find Π_G - provided that all the remaining b_r's ($0 < r \leq n$) can be determined. The following result is well known (see Tutte [2]). We will give different derivation using star polynomials.

THEOREM 5. Π_G is reconstructible.

PROOF. Let G be a graph with p nodes. From Theorem 4, we have, by integrating both sides with respect to w_1,

$$E(G) = \int(\Sigma E(G-x;\underline{w})) + C(w_2,w_3,\ldots,w_p),$$

where $C(w_2,w_3,\ldots,w_p)$ is a polynomial in the weights $w_2,w_3,\ldots,w_p$.

Suppose that D_G is given. Then $\Sigma E(G-x;\underline{w})$ can be found. Hence $\int \Sigma E(G-x;\underline{w})$ can be found. But this polynomial contains all the simple terms except w_p. Therefore all the simple coefficients of $E(G)$, except c_{p-1} can be immediately found. We will consider two cases (i) $c_{p-2} = 0$ and (ii) $c_{p-2} \neq 0$.

If $c_{p-2} = 0$, then from Lemma 3, $c_{p-1} = 0$. Therefore all the simple coefficients will be defined. It follows from Theorem 1, that Π_G can be found. If $c_{p-2} \neq 0$ then c_{p-1} will be unknown. However b_0 will be known from D_G (Lemma 4).

Therefore the system of equations in Theorem 1 will have p-1 equations and p-1 unknowns. It can be solved to find $b_1, b_2, \ldots, b_{p-1}$. Hence Π_G can be found.

In the proof of Theorem 5, we did not give any useful practical method for finding Π_G, when $c_{p-2} \neq 0$. We shall do so now.

The equations of Theorem 1 can be written as follows:

$$p = b_{p-1} + b_{p-2} + \ldots + b_2 + b_1 + b_0$$

$$2c_1 = \binom{p-1}{1} b_{p-1} + \binom{p-2}{1} b_{p-2} + \ldots + 2b_2 + b_1$$

$$c_2 = \binom{p-1}{2} b_{p-1} + \binom{p-2}{2} b_{p-2} + \ldots + b_2$$

$$\vdots$$

$$c_{p-2} = \binom{p-1}{p-2} b_{p-1} + \binom{p-2}{p-2} b_{p-2} .$$

By adding and subtracting alternate equations, we get

$$p-2c_1 + c_2 - c_3 \ldots + (-1)^p c_{p-2} = b_0 + (-1)^p b_{p-1}. \tag{3.2}$$

Let us denote the L.H.S. of this equation by S. Then

$$S = b_0 + (-1)^p b_{p-1}.$$

$$\Rightarrow \quad b_{p-1} = (-1)^p (S-b_0) \tag{3.3}$$

Since S can be found from the simple coefficients c_r $(r \geq p-2)$ and p, and b_0 can be found (from Lemma 5), b_{p-2} can be found from Equation (3.3). Hence all the b's can be found by using the algorithm suggested by Theorem 1.

The following example illustrates the technique.

EXAMPLE 2. Let G be a graph with D_G given below in Figure 1.

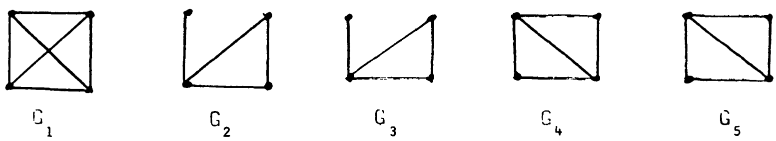

Figure 1.

It can be easily confirmed that

$$E(G_1) = w_1^4 + 6w_1^2w_2^2 + 12w_1w_3 + 3w_2^2 + 4w_4$$

$$E(G_2) = E(G_3) = w_1^4 + 4w_1^2w_2 + 5w_1w_3 + w_2^2 + w_4$$

$$E(G_4) = E(G_5) = w_1^4 + 5w_1^2w_2 + 8w_1w_3 + 2w_2^2 + 2w_4 \quad .$$

Therefore

$$\sum_{i=1}^{5} E(G_i) = \frac{\partial E(G)}{\partial w_1} = 5w_1^4 + 24w_1^2w_2 + 38w_1w_3 + 9w_2^2 + 10w_4.$$

By integrating with respect to w_1, we get

$$E(G) = w_1^5 + 8w_1^3w_2 + 19w_1^2w_3 + 10w_1w_4 + C(w_2,w_3,w_4,w_5).$$

Therefore

$$p = 5, \; c_1 = 8, \; c_2 = 19 \text{ and } c_3 = 10.$$

$$\Rightarrow \; S = 5 - 16 + 19 - 10 = -2 .$$

Since no element of D_G has an isolated node, it follows from Lemma 5 that $b_0 = 0$. From Equation (3.3), we get

$$b_4 = (-1)^5 \, (-2-0) = 2.$$

$$b_3 = c_3 - \sum_{r=4}^{4} \binom{r}{3} b_r = c_3 - 4b_4 = 10 - 8 = 2.$$

$$b_2 = c_2 - \sum_{r=3}^{4} \binom{r}{2} b_r = c_2 - \binom{3}{2} b_3 - \binom{4}{2} b_4 = 19 - 6 - 12 = 1.$$

$$b_1 = 2c_1 - \sum_{r=2}^{4} \binom{r}{1} b_r = 2c_1 - 2b_2 - 3b_3 - 4b_4 = 16 - 2 - 6 - 8 = 0.$$

Hence we get

$$\Pi_G = (4^3, 3^2, 2^1).$$

Suppose that G is unigraphic. Then G is uniquely determined from Π_G. But Π_G is reconstructible from D_G (Theorem 5). It follows that G is uniquely reconstructible from D_G. Thus we have the following theorem, which can otherwise be proved by more elementary methods.

THEOREM 6. The Reconstruction Conjecture holds for all graphs with unigraphic partitions.

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