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ON A CLASS OF FUNCTIONS UNIFYING THE CLASSES OF PAATERO, ROBERTSON AND OTHERS

S. BHARGAVA and S. NANJUNDA RAO

Department of Mathematics
University of Mysore
Manasagangothri
Mysore-570 006, India

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ABSTRACT. We study a class $M_k^\lambda(\alpha, \beta, b, c)$ of analytic functions which unifies a number of classes studied previously by Paatero, Robertson, Pinchuk, Moulis, Mocanu and others. Thus our class includes convex and starlike functions of order β, spirallike functions of order β and functions for which zf' is spirallike of order β, functions of boundary rotation utmost $k\pi$, α-convex functions etc. An integral representation of Paatero and a variational principle of Robertson for the class V_k of functions of bounded boundary rotation, yield some representation theorems and a variational principle for our class. A consequence of these basic theorems is a theorem for this class $M_k^\lambda(\alpha,\beta,b,c)$ which unifies some earlier results concerning the radii of convexity of functions in the class $V_k^\lambda(\beta)$ of Moulis and those concerning the radii of starlikeness of functions in the classes U_k of Pinchuk and $U_2(\beta)$ of Robertson etc. By applying an estimate of Moulis concerning functions in $V_k^\lambda(0)$, we obtain an inequality in the class $M_k^\lambda(\alpha,\beta,b,c)$ which will contain an estimate for the Schwarzian deriative of functions in the class $V_k^\lambda(\beta)$ and in particular the estimate of Moulis for the Schwarzian of functions in $V_k^\lambda(0)$.

KEY WORDS AND PHRASES. Convex functions, starlike functions, spirallike functions, functions of bounded boundary rotation, α-convex functions.
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1. INTRODUCTION.

Let N denote the set of all regular functions f on the unit disc E: $|z|< 1$ such that $f(0) = 0$, $f'(0) = 1$ and let, for such a function f,

$$J_{f(z)} = J_{f(z)}(\alpha,b,c) = \alpha\left[1 + \frac{z f''(z)}{b f'(z)}\right] + (1-\alpha)\left[1 - \frac{1}{c} + \frac{z f'(z)}{c f(z)}\right] \tag{1.1}$$

where α, $b\neq 0$ and $c\neq 0$ are complex numbers. Let $M_k^\lambda(\alpha,\beta,b,c)$ denote the class of all functions f in N such that

$$\int_0^{2\pi} |\mathrm{Re}\, e^{i\lambda} J_f - \beta\cos\lambda|\, d\theta \leq k\pi(1-\beta)\cos\lambda \tag{1.2}$$

where $0 \leqq \beta < 1$, $-\pi/2 < \lambda < \pi/2$, $z = re^{i\theta}$, $0 \leqq r < 1$ and $k \geqq 2$.

This class unifies and generalizes various classes studied earlier by V. Paatero [1-2], M.S. Robertson [3-4], E.J. Moulis [5-6], B. Pinchuk [7-9], K.S. Padmanabhan and R. Parvatham [10-11], P.N. Chichra [12], M.A. Nasr [13], G. Lakshma Reddy [14] and P.T. Mocanu [15]. In particular, convex functions, convex functions of order β, starlike functions, starlike functions of order β, λ-spirallike functions, functions f for which zf' are λ-spirallike, functions of bounded boundary rotations, functions f for which $\mathrm{Re}[\alpha(1+\frac{zf''}{f'})+(1-\alpha)\frac{zf'}{f}]>0$ are all contained in the class $M_k^\lambda(\alpha,\beta,b,c)$. In this note, we study several properties of functions in the new class, thereby giving a unified approach to proving and at the same time generalizing many of the results of the earlier authors.

We use the following additional notations throughout our work. If f is N, let

$$\hat{J}_{f(z)} = \hat{J}_{f(z)}(\alpha,b,c) = \frac{J_{f(z)}-1}{z} = \frac{\alpha}{b}\frac{f''(z)}{f'(z)} + \frac{1-\alpha}{c}\left[\frac{f'(z)}{f(z)} - \frac{1}{z}\right] \tag{1.3}$$

with $\hat{J}_{f(0)}$ defined by continuity,

$$N_{f(z)}(\nu) = N_{f(z)}(\alpha,b,c,\nu) = \hat{J}'_{f(z)} - \nu\hat{J}^2_{f(z)} \quad (\nu:\ \text{complex}) \tag{1.4}$$

(so that $N_{f(z)}(1,1,1,\frac{1}{2})$ is the Schwarzian of f)

$$d = d(\lambda,\beta) = e^{i\lambda}\sec\lambda\,(1-\beta)^{-1} \tag{1.5}$$

$$H_f^\lambda = H_f^\lambda(\alpha,\beta,b,c) = [(f')^{\alpha/b}(f/z)^{(1-\alpha)/c}]^d. \tag{1.6}$$

2. SOME PRELIMINARY RESULTS.

The following Lemmas are immediate from the definitions (1.1)-(1.6) above and will be used later.

LEMMA 2.1. If $f(z) = z + A_2z^2 + A_3z^3 + \ldots$ is in N, then

$$\frac{(H_f^\lambda)'}{(H_f^\lambda)} = d\,\hat{J}_f \tag{2.1}$$

$$\hat{J}_{f(0)} = u_2A_2,\quad \hat{J}'_{f(0)} = u_3A_3 - uA_2^2 \tag{2.2}$$

$$N_{f(0)}(\nu) = u_3(A_3 - \hat{u}A_2^2),\quad \hat{u} = \frac{u+\nu u_2^2}{u_3} \tag{2.3}$$

where

$$u = \frac{4\alpha}{b} + \frac{1-\alpha}{c},\quad u_2 = \frac{2\alpha}{b} + \frac{1-\alpha}{c} \quad\text{and}\quad u_3 = \frac{6\alpha}{b} + \frac{2(1-\alpha)}{c}. \tag{2.4}$$

PROOF. Differentiating (1.6) logarithmically we have (2.1). $\hat{J}_{f(0)}$ and $\hat{J}'_{f(0)}$ are respectively the constant term and the coefficient of z in the power series expansion of $\hat{J}_{f(z)}$ and to obtain this power series it is enough to substitute the power series for f in (1.3). Thus (2.2) follows. Substituting (2.2) in (1.4) we have (2.3).

LEMMA 2.2. If $f(z) = z + A_2z^2 + A_3z^3 + \ldots$ and $g(z) = z + A_2'z^2 + A_3'z^3 + \ldots$ are in N, then

$$H_f^\lambda(\alpha,\beta,b,c) = H_g^{\lambda'}(\alpha',\beta',b',c') \tag{2.5}$$

if and only if

$$d\,\hat{J}_f(\alpha,b,c) = d'\hat{J}_g(\alpha',b',c'),\quad d' = d'(\lambda',\beta'). \tag{2.6}$$

Further, if f and g satisfy (2.5) or (2.6), then

$$\mathrm{Re}\,\frac{\sec\lambda(e^{i\lambda}J_f - \beta\cos\lambda)}{1-\beta} = \mathrm{Re}\,\frac{\sec\lambda'(e^{i\lambda'}J_g - \beta'\cos\lambda')}{1-\beta'} \tag{2.7}$$

$$dN_f(\alpha,b,c,\nu) = d'N_g(\alpha',b',c',\nu') \tag{2.8}$$

where ν and ν' are complex numbers satisfying $\nu/d = \nu'/d'$,

$$du_2A_2 = d'u_2'A_2', \quad d(u_3A_3 - uA_2^2) = d'(u_3'A_3' - u'A_2'^2) \tag{2.9}$$

$$du_3\,(A_3 - \hat{u}A_2^2) = d'u_3'\,(A_3' - \hat{u}'A_2'^2) \tag{2.10}$$

where $\hat{u}$ and $\hat{u}'$ are complex numbers satisfying

$$d'u_2'^2(\hat{u}u_3 - u) = du_2^2(\hat{u}'u_3' - u).$$

Here u', u_2' and u_3' are as given by (2.4) with α,b,c replaced by α',b',c' respectively.

PROOF. Equivalence of (2.5) and (2.6) are immediate from (2.1). On taking real parts in (2.8) and using (1.3) we have (2.7). We get (2.8) on differentiating (2.6) and using (1.4). Again from (2.6) and (2.2) we have (2.9). Finally (2.10) is obtained on using (2.3) in (2.8).

LEMMA 2.3. If f and g are in N and if

$$H_g^{\lambda'}(\alpha',\beta',b',c') = \text{Constant.}\ (1+\bar{a}z)^{-2}\, H_{f(\xi)}^{\lambda}(\alpha,\beta,b,c) \tag{2.11}$$

where $\xi = \dfrac{z+a}{1+\bar{a}z}$, $|a| < 1$, then

$$d'\hat{J}_g = -\frac{2\bar{a}}{1+\bar{a}z} + \frac{d(1-|a|^2)}{(1+\bar{a}z)^2}\,\hat{J}_{f(\xi)} \tag{2.12}$$

$$\frac{d(1-|a|^2)^2}{(1+\bar{a}z)^4}\,N_{f(\xi)}(\alpha,b,c,\nu) = d'N_{g(z)}(\alpha',b',c',\nu')$$
$$+\frac{2\,\bar{a}d(1-2\,\nu/d)}{a(1+\bar{a}z)^2}\left[\frac{(1-|a|^2)\,a\hat{J}_{f(\xi)}}{1+\bar{a}z} - \frac{|a|^2}{d}\right] \tag{2.13}$$

where $\nu/d = \nu'/d'$.

PROOF. Logarithmic differentiation of (2.11) yields (2.12). Differentiating (2.12) and subtracting from the resulting equation ν'/d' times the square of (2.12) and simplifying we get (2.13).

Let B_k $(k \geq 2)$ denote the set of all real valued functions $m(t)$ of bounded variation on $[0,2\pi]$ such that

$$\int_0^{2\pi} dm(t) = 2 \quad\text{and}\quad \int_0^{2\pi} |dm(t)| \leq k. \tag{2.14}$$

The following is a result of Paatero [1] reformulated in terms of our notations.

LEMMA 2.4. A function g is in $M_k^o(1,0,1,1)$ $(=V_k)$ if and only if there exists $m(t)$ in B_k such that

$$H_g^o(1,0,1,1) = \exp - \int_0^{2\pi} \log(1-ze^{-it})\, dm(t) . \tag{2.15}$$

3. SOME BASIC RESULTS IN $M_k^\lambda(\alpha,\beta,b,c)$.

The following representation theorem is a simple consequence of the definition of the class $M_k^\lambda(\alpha,\beta,b,c)$.

THEOREM 3.1. Let $(\lambda,\alpha,\beta,b,c)$, $(\lambda',\alpha',\beta',b',c')$ and k be given. If f is in $M_k^\lambda(\alpha,\beta,b,c)$ and if g is defined by

$$H_g^{\lambda'}(\alpha',\beta',b',c') = H_f^\lambda(\alpha,\beta,b,c)$$

then g is in $M_k^{\lambda'}(\alpha',\beta',b',c')$.

PROOF. The hypothesis implies (2.7). Taking absolute values on both sides of (2.7) and integrating with respect to θ between 0 and 2π, we have the result immediately.

The above theorem contains as special cases (i) Theorem 3 in [5] concerning the class $V_k^\lambda = M_k^\lambda(1,0,1,1)$, (ii) Theorem 5 in [11] concerning the class $V_k(\beta)$, (iii) Lemmas 2, 3 and 4 in [6] concerning the class $V_k^\lambda(\beta)$ and (iv) Lemmas 2,3,4,5,6 and 9 in [14] concerning the classes $V_k^\lambda(\beta,b)$ and $U_k^\lambda(\beta)$.

COROLLARY 3.1. If f is in $M_k^\lambda(\alpha,\beta,b,c)$ then

$$|d\hat{J}_{f(0)}| \leqq k \tag{3.1}$$

with equality for f satisfying

$$H_f^\lambda(\alpha,\beta,b,c) = \frac{(1-z)^{\frac{k}{2}-1}}{(1+z)^{\frac{k}{2}+1}} \tag{3.2}$$

PROOF. By Theorem 3.1, there exists a g in $M_k^o(1,0,1,1) = V_k$ for which (2.6) holds with $\lambda' = \beta' = 0$, $\alpha' = b' = 1$. (3.1) now follows from (2.2) and (2.9) and on using the Pick's estimate [16] namely $|A_2'| \leqq k/2$. Now, if g' is given by the right side of (3.2) and $g(0) = 0$, then g is in $V_k = M_k^o(1,0,1,1)$ [6] and therefore, by Theorem 3.1, f given by (3.2) is in $M_k^\lambda(\alpha,\beta,b,c)$. That (3.1) is equality for this f follows easily.

The following representation theorem is a direct consequence of Paatero's Lemma stated in Lemma 2.4 above.

THEOREM 3.2. If f is in $M_k^\lambda(\alpha,\beta,b,c)$ then there exists $m(t)$ in B_k such that

$$H_f^\lambda(\alpha,\beta,b,c) = \exp - \int_0^{2\pi} \log(1-ze^{-it})\, dm(t). \tag{3.3}$$

Conversely, given $m(t)$ in B_k and if f is defined by (3.3), then f is in $M_k^\lambda(\alpha,\beta,b,c)$.

PROOF. If f is in $M_k^\lambda(\alpha,\beta,b,c)$ let us define g by

$$H_g^o(1,0,1,1) = H_f^\lambda(\alpha,\beta,b,c). \tag{3.4}$$

Then, by Theorem 3.1, g is in $M_k^o(1,0,1,1)$. Hence by Lemma 2.4 we get (2.15) which when read with (3.4) yields (3.3).

Conversely, given $m(t)$ in B_k and f as in (3.3), we have by Lemma 2.4 that and g satisfy (3.4) for g in V_k given by (2.15). By Theorem 3.1, (3.4) implies that f is in $M_k^\lambda(\alpha,\beta,b,c)$.

The above Theorem contains various integral representation theorems obtained previously, for example, Theorem 1 and the integral representation in Corollary 4 in [11], Theorem 1 in [6], Theorems 1 and 5 in [14]. Further, if f is in the Mocanu class $M(\alpha) = M_2^o(\alpha,0,1,1)$, [15], then we have the representation

$$f'^{\alpha}\ (f/z)^{1-\alpha} = \exp - \int_0^{2\pi} \log(1-ze^{-it})\ dm(t) \text{ for } m(t) \text{ in } B_2.$$

THEOREM 3.3. Let $(\lambda_j,\alpha_j,\beta_j,b_j,c_j)$, $j = 1,2,3$ and k be given. Then f_1 is in $M_k^{\lambda_1}(\alpha_1,\beta_1,b_1,c_1)$ if and only if there are functions f_j in $M_2^{\lambda_j}(\alpha_j,\beta_j,b_j,c_j)$, $j = 2,3$ such that

$$H_{f_1}^{\lambda_1}(\alpha_1,\beta_1,b_1,c_1) = \frac{H_{f_2}^{\lambda_2}(\alpha_2,\beta_2,b_2,c_2)^{(k+2)/4}}{H_{f_3}^{\lambda_3}(\alpha_3,\beta_3,b_3,c_3)^{(k-2)/4}}.$$

PROOF. The Theorem follows immediately from Theorem 3.2 on using the representation (Brannan [17]) that $m(t) = ((k+2)/4)m_1(t) - ((k-2)/4)m_2(t)$ where $m(t)$ is in B_k and $m_1(t)$ and $m_2(t)$ are in B_2.

Special cases of the above theorem are Theorem 4 in [5] and its corollary, Corollary 1 in [11], Theorem 2 in [6] and Theorem 2 in [14]. Theorem 6 in [14] is also a special case of the above theorem on using the fact that the functions $S(z)$ and $z^{it}S(z)$ are together starlike of order β if t is real.

COROLLARY 3.2. If f is in $M_k^\lambda(\alpha,\beta,b,c)$, then

$$\frac{(1-r)^{(k-2)/2}}{(1+r)^{(k+2)/2}} \le |H_f(\alpha,\beta,b,c)| \le \frac{(1+r)^{(k-2)/2}}{(1-r)^{(k+2)/2}} \tag{3.5}$$

$$|\arg H_f^\lambda(\alpha,\beta,b,c)| \le k \sin^{-1} r. \tag{3.6}$$

PROOF. Theorem (3.3) gives that there are convex functions g and h such that

$$H_f^\lambda(\alpha,\beta,b,c) = \frac{(g')^{(k+2)/4}}{(h')^{(k-2)/4}}.$$

Using in this, well known distorsion and rotation theorems concerning convex functions ϕ namely

$$(1+r)^{-2} \le |\phi'(z)| \le (1-r)^{-2}$$

$$|\arg \phi'(z)| \le 2 \sin^{-1} r, \qquad 0 \le |z| = r < 1,$$

we immediately have (3.5) and (3.6).

One can easily check that (3.5) becomes equality on taking the function defined by (3.2) and then putting $z = +r$ and $z = -r$ respectively.

Inequalities (3.5) contain as special cases some of the known distorsion and rotation theorems. For example those found in [8], [9], [11], and [14].

4. A VARIATIONAL FORMULA AND TWO MAIN INEQUALITIES.

THEOREM 4.1. Let $|a| < 1$ and $\xi(z) = (z+a)/(1+\bar{a}z)$, $|z| < 1$. Let f be in $M_k^{\lambda}(\alpha,\beta,b,c)$. Then $F(z)$ defined by

$$F(0) = 0, \quad H_{F(z)}^{\lambda'}(\alpha',\beta',b',c') = \frac{H_{f(\xi)}^{\lambda}(\alpha,\beta,b,c)}{(1+\bar{a}z)^2 H_{f(a)}^{\lambda}(\alpha,\beta,b,c)} \tag{4.1}$$

is in $M_k^{\lambda'}(\alpha',\beta',b',c')$.

PROOF. By Theorem 3.1, there exists g in $M_k^{o}(1,0,1,1) = V_k$ given by (3.4). Let $G(z)$ be defined by

$$G(0) = 0, \quad G(z) = \frac{g(\xi(z)) - g(a)}{(1-|a|^2)\; g'(a)} . \tag{4.2}$$

Then by a variational principle of Robertson [4], $G(z)$ is in V_k. By Theorem 3.1 again, there exists $F(z)$ in $M_k^{\lambda'}(\alpha',\beta',b',c')$ such that

$$H_{F(z)}^{\lambda'}(\alpha',\beta',b',c') = H_{G(z)}^{o}(1,0,1,1) = G'(z).$$

Using (4.2) and then (3.4) in this gives (4.1).

As special cases, the above Theorem contains Lemma 3 of [11], Theorem 6 of [5], Lemma 5 of [6] and Lemma 7 of [14].

The following Theorem contains as special cases many earlier results concerning radii of convexity and those concerning radii of starlikeness.

THEOREM 4.2. If f is in $M_k^{\lambda}(\alpha,\beta,b,c)$, then

$$\left|(1 - |z|^2)\; z\hat{J}_{f(z)} - 2\;|z|^2 d^{-1}\right| \leq k\;|zd^{-1}| \tag{4.3}$$

and for any nonzero complex number ζ ,

$$\operatorname{Re} J_{f(z)}(\alpha,b/\zeta,c/\zeta) \geq \frac{Q(r)}{1 - r^2} > 0 \tag{4.4}$$

when $r = |z| < R$, where

$$Q(r) = 1 - a_1 r + a_2 r^2 \tag{4.5}$$

with $a_1 = k|d^{-1}\zeta|$, $a_2 = 2\operatorname{Re}(\zeta d^{-1}) - 1$ and

$$1 \geq R = \begin{cases} \dfrac{a_1 - \sqrt{a_1^2 - 4a_2}}{2a_2}, & 2\operatorname{Re}(\zeta d^{-1}) \neq 1 \\[2ex] k^{-1}\;|d\zeta^{-1}|, & 2\operatorname{Re}(\zeta d^{-1}) = 1. \end{cases} \tag{4.6}$$

PROOF. We get (4.3) on choosing g to satisfy (2.11), putting $z = 0$ in (2.12), using (3.1) and finally changing a to z. Now, we have from (1.3),

$$J_{f(z)}(\alpha,b/\zeta,c/\zeta) = 1 + z\zeta\hat{J}_{f(z)}(\alpha,b,c), \quad \zeta \neq 0$$

and therefore,

$$\operatorname{Re} J_{f(z)}(\alpha,b/\zeta,c/\zeta) = 1 + \operatorname{Re} z\zeta\hat{J}_{f(z)}(\alpha,b,c)$$

which yields the first part of (4.4) on using (4.3). The second part of (4.4) follows from the standard result for positivity of a quadratic form in r. Lastly, it is easy to check that the property $R \leqq 1$ is equivalent to

$$2 \operatorname{Re}(\zeta d^{-1}) = 1 + a_2 \leqq a_1 = k|\zeta d^{-1}| ,$$

which is true.

REMARKS 4.2. (i) The sharp radius of convexity of f in $V_2^\lambda(\beta)$ given by Theorem 2 of [12] is a special case of (4.6). The radius of convexity of f in $V_k^\lambda(\beta,b) = M_k^\lambda(1,\beta,b,1)$ given by (4.6) with $\alpha = 1$, $\zeta = b$ is better than the one found in Theorem 3 of [14]. In fact, $Q(r) \geqq \hat{Q}(r) = 1 - a_1 r - a_2' r^2$, where a_1 is as in (4.5) and $a_2' = 2|\zeta d^{-1}| + 1$, the radius of positivity of $\hat{Q}(r)$ being

$$\frac{-a_1 + \sqrt{a_1^2 + 4\, a_2'}}{2a_2'}$$

which gives the result of Theorem 3 of [14] on putting $\alpha = 1$, $\zeta = b$. Clearly the radius of positivity of $Q(r)$ is bigger than that of $\hat{Q}(r)$. (ii) The Sharp radius of convexity of f in $V_k(\beta)$ found in Theorem 3 of [11] is given by (4.6) with $\alpha = \zeta = b = 1$ and $\lambda = 0$. In fact, more generally, it is easy to see in (4.4) that

$$\operatorname{Re} J_{f(z)}(\alpha, b/\zeta, c/\zeta) = 0$$

for f given by (3.2) and $z = R$ provided ζd^{-1} is positive real. (iii) The radius of starlikeness of f in $U_k^\lambda(\beta,c) = M_k^\lambda(0,\beta,1,c)$ is given by (4.6) with $\alpha = 0$, $\zeta = c$. Putting $c = 1$ further, we get an improvement over Theorem 7 of [14]. (iv) We can get the radius of "α-convexity" of functions in the Mocanu class $M(\alpha) = M_2^0(\alpha,0,1,1)$ on putting $k = 2$, $\lambda = \beta = 0$, $b = c = \zeta = 1$ in (4.6). (v) Inequality (4.3) contains ($\alpha = b = 1$) Theorem 3 of [6].

In proving the inequality (4.7) below, we use an estimate of Moulis [5] for $|A_3 - \hat{u}A_2^2|$, $\hat{u} > 2/3$, related to functions $f(z) = z + A_2 z^2 + A_3 z^3 + \ldots$ in the class $V_k^\lambda = M_k^\lambda(1,0,1,1)$ and thereby obtain a corresponding estimate for functions in the more general class $M_k^\lambda(\alpha,\beta,b,c)$. The inequality (4.7) is then applied to obtain the estimate (4.9) which contains as special cases Moulis's estimate [5] for the Schwarzianderivative of functions in $V_k(\beta)$ (Theorem 4 of [11]).

LEMMA 4.1. If f is in $M_k^\lambda(\alpha,\beta,b,c)$, then

$$|N_{f(0)}(\alpha,b,c,\nu)| \leqq 6(1-\beta)\ \tilde{J}(\lambda,k,\tfrac{2}{3}(1+\nu(1-\beta))), \qquad (\nu > 0) \tag{4.7}$$

where $\tilde{J}(\lambda,k,\hat{u})$ is (in slightly different notation) the J-function of Moulis given by

$$\tilde{J}(\lambda,k,\hat{u}) = \begin{cases} (1/3)\cos\lambda\,\{(3\hat{u}-2)(k^2\cos\lambda)/4 + (k/2)\ |\sin\lambda| - \cos\lambda\}, & k > 4/(3\hat{u}-2) \\ (1/3)\cos\lambda\,\{(k-1)\ \cos\lambda + (k/2)\ |\sin\lambda|\ \}, & k \leq 4/(3\hat{u}-2) \end{cases} \tag{4.8}$$

PROOF. By Theorem 3.1 we can find g in $M_k^\lambda(1,0,1,1)$ such that (2.5) holds with $\alpha' = b' = 1$, $\lambda' = \lambda$ and $\beta' = 0$. Thus by (2.8) of Lemma 2.2 and 2.3 of Lemma 2.1, we have

$$\begin{aligned} |N_{f(0)}(\alpha,b,c,\nu)| &= (1-\beta)\ |N_{g(0)}(1,1,1,\nu(1-\beta))| \\ &= 6(1-\beta)\ |A_3' - \hat{u}' A_2'^2|, \quad \hat{u}' = (2/3)(1+\nu(1-\beta)), \end{aligned}$$

which gives (4.7) on using Moulis's result [5] namely,

$$|A_3' - \hat{u}'A_2'^2| \leq \tilde{J}(\lambda,k,\hat{u}'), \quad \hat{u}' > 2/3$$

where $\tilde{J}$ is as in (4.8).

THEOREM 4.3. If f is in $M_k^\lambda(\alpha,\beta,b,c)$, then

$$|N_{f(z)}(\alpha,b,c,\nu)| \leq \frac{6(1-\beta)\ \tilde{J}(\lambda,k,(2/3)(1+\nu(1-\beta))) + 2|z/d|\,|1-2\nu/d|\ (k+|z|)}{(1-|z|^2)^2} \tag{4.9}$$

where $\nu > 0$ and $\tilde{J}$ is as given by (4.8) above.

PROOF. From (4.3) we easily have

$$|(1-|a|^2)\ a\hat{J}_{f(a)} - \frac{|a|^2}{d}| \leq |ad^{-1}|\ (k + |a|).$$

Using this in (2.13) with $z = 0$, $\lambda' = \lambda$, $\beta' = \beta$, $\alpha' = \alpha$, $b' = b$, $c' = c$ and then using Lemma 4.1 and finally replacing a by z we have (4.9).

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