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THE MACKEY CONVERGENCE CONDITION FOR SPACES WITH WEBS

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ABSTRACT. If each sequence converging to 0 in a locally convex space is also Mackey convergent to 0, that space is said to satisfy the Mackey convergence condition. The problem of characterizing those locally convex spaces with this property is still open. In this paper, spaces with compatible webs are used to construct both a necessary and a sufficient condition for a locally convex space to satisfy the Mackey convergence condition.

KEY WORDS AND PHRASES. Compatible web, fast complete space, Mackey convergent sequence, inductive limit.

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1. *INTRODUCTION.*

In a locally convex space, it sometimes happens that every convergent sequence is also convergent with respect to a normed topology on some subspace. Since normed spaces have many tangible properties, it is important to know for which locally convex spaces this condition holds. Such spaces satisfy the so-called Mackey convergence condition.

Throughout this paper, E will denote a Hausdorff locally convex topological vector space. An absolutely convex set in E will be called a **disk**. If B is a disk in E, we equip its linear hull E_B with the semi-normed topology generated by the Minkowski functional of B. If B is bounded, the Minkowski functional of B generates a normed topology on E_B. If E_B is a Banach space, B is called a **Banach disk**. E is **fast complete** if each bounded set in E is contained in a bounded

Banach disk. Every sequentially complete locally convex space is fast complete.

In Jarchow ([1], 10.1) it is stated that no concise description of locally convex spaces which satisfy the Mackey convergence condition exists. This problem is still open. In this paper we will examine this characterization problem within the context of spaces possessing webs.

DEFINITION 1: A sequence (x_n) in a locally convex space E is **Mackey convergent to** x if there is a bounded disk $B \subset E$ such that $x_n \to x$ in the norm topology of E_B. If $x = 0$, we say that (x_n) is a **Mackey null (or locally null) sequence.**

Since the norm topology $\mathcal{T}_B$ on E_B is finer than the induced topology on E_B, every Mackey null sequence is a null sequence for the original topology on E. This prompts the following definition of the converse, due to Jarchow [1].

DEFINITION 2: A locally convex space satisfies the **Mackey convergence condition** if each of its null sequences is a Mackey null sequence.

Mackey convergence is defined by Mackey in [2]. In DeWilde [3] (Prop. III.1.10), it is shown that every Fréchet space satisfies the Mackey convergence condition. Several results concerning the Mackey convergence condition are obtained in Jarchow and Swart [4]. In their paper, it is shown that E satisfies the Mackey convergence condition if and only if $\mathcal{T}_n(E', E)$ is a Schwartz topology on E', where $\mathcal{T}_n(E', E)$ is the topology of uniform convergence on all null sequences of E. They also investigate the Mackey convergence condition for spaces which are fast complete and bornological. Specifically, the following is obtained (see [4]):

THEOREM 3: Let E be fast complete. Then the following are equivalent:
(a) E is bornological and satisfies the Mackey convergence condition.
(b) $E = \operatorname{ind\,lim}_{\alpha \in A} E_\alpha$, where each E_α is a separable Banach space, and each null sequence in E is a null sequence in some E_α.

2. *SEQUENTIALLY WEBBED SPACES.*

We will now examine the Mackey convergence condition for spaces with webs. The reader is referred to Robertson [5] for a description of webs in a topological vector space. Further information concerning webs may be found in Robertson and Robertson [6], and in DeWilde [3]. DeWilde originally used webs to obtain several generalized versions of the Closed Graph Theorem; see [3]. In this paper we will work only with locally convex spaces and we will assume, as in [6], that each member of a web is absolutely convex. The following aspects of webs will also be useful.

DEFINITION 4: A **strand** of a web $\mathcal{W}$ is a collection of members of $\mathcal{W}$, one from each layer, with the $k+1$ member of the strand contained in the the k-th member. Strands will be denoted by (W_k).

DEFINITION 5: A web $\mathcal{W}$ on a locally convex space is **compatible with** E if for each 0-neighborhood U in E and for each strand (W_k) of $\mathcal{W}$, there is a $k_0 \in \mathbf{N}$ such that $W_{k_0} \subset U$.

REMARK: It is to be noted here that we will assume (in accordance with Robertson [5]) that for each strand (W_k) and for each $k \in \mathbf{N}$

$$W_{k+1} \subset \frac{1}{2} W_k. \tag{1.1}$$

Suppose now that $x_n \to 0$ in E, and suppose that for large enough n, (x_n) is contained in some finite collection of strands from $\mathcal{W}$. Since Definition 5 implies that the members of $\mathcal{W}$ are in some sense smaller than the 0-neighborhoods in E, this is actually enough to coerce (x_n) to be a Mackey null sequence. We have now motivated the following definition.

DEFINITION 6: A locally convex space E is **sequentially webbed** if E has a compatible web $\mathcal{W}$ such that for each null sequence (x_n) in E, there is a finite collection of strands,

$$\{(W_k^{(1)}), \ldots, (W_k^{(m)})\},$$

from $\mathcal{W}$ such that for each $k \in \mathbf{N}$ there exists an $N_k \in \mathbf{N}$ such that for each $n \geq N_k$,

$$x_n \in \bigcup_{i=1}^{m} W_k^{(i)}.$$

Let us find some sequentially webbed spaces.

PROPOSITION 7: Every metrizable locally convex space is sequentially webbed.

PROOF: If $\{U_k : k \in \mathbf{N}\}$ is a base of absolutely convex 0-neighborhoods of the metrizable space E, such that $(\forall k \in \mathbf{N})$,

$$U_{k+1} \subset \frac{1}{2} U_k,$$

then $\{U_k : k \in \mathbf{N}\}$ is easily seen to be a compatible web on E (see [5] or [6]). Certainly, every (x_n) in E such that $x_n \to 0$ is eventually contained in the (only) strand (U_k). □

The following definition may be found in Floret [7].

DEFINITION 8: Let $E = \operatorname{ind\,lim}_n E_n$ be the inductive limit of the locally convex spaces E_n, where $E_1 \subset E_2 \subset \ldots$, and the injection $id : E_n \to E_{n+1}$ is continuous for each n. Then E is called **sequentially retractive** if each sequence converging in E is convergent to the same point in some E_n.

PROPOSITION 9: A sequentially retractive inductive limit of sequentially webbed spaces is sequentially webbed.

PROOF: Let $E = \operatorname{ind\,lim}_n E_n$, and let $\mathcal{W}^{(n)}$ denote the web on E_n for each n. In E, we define a compatible web $\mathcal{W}$ as follows: Let the first layer be the collection of all the first layers of the webs $\mathcal{W}^{(n)}$. Define the second layer of $\mathcal{W}$ to be the collection of all the second layers of the webs $\mathcal{W}^{(n)}$, and so forth. Certainly, $\mathcal{W}$ is a countable collection of absolutely convex sets, and since we have used all webs $\mathcal{W}^{(n)}$ simultaneously, the other properties of webs (as in [6]) are easily verified for $\mathcal{W}$.

As for compatibility, let U be a 0-neighborhood in E, and let (W_k) be a strand from $\mathcal{W}$. Note that W_1 is in the first layer of E_{n_0}, for some $n_0 \in \mathbf{N}$. Since $W_{k+1} \subset W_k$ for each k, then $W_k \subset E_{n_0}$ for each k. Hence, since $E_1 \subset E_2 \subset \ldots$, it follows that for each $k \in \mathbf{N}$, W_k is a member of some $\mathcal{W}^{(j)}$, where $1 \leq j \leq n_0$. If any $W_k \in \mathcal{W}^{(j')}$, where $j' < n_0$, then all succeeding members of (W_k) are in $E_{(j')}$, since we must have $W_{k+1} \subset W_k$ for all k. In fact, for large enough k, all succeeding members of (W_k) must be in $\mathcal{W}^{(j_0)}$ for some j_0; i.e., there is a $k_0 \in \mathbf{N}$ such that

$$W_k \in \mathcal{W}^{(j_0)},$$

for all $k \geq k_0$. Without loss of generality, assume $j_0 = 1$. Then since $U \cap E_1$ is a 0- neighborhood in E_1 and $\mathcal{W}^{(1)}$ is compatible with E_1, there is a $k \in \mathbf{N}$ such that

$$W_k \subset U \bigcap E_1 \subset U.$$

This makes $\mathcal{W}$ a compatible web on E.

Finally, given some $n \in \mathbf{N}$, a member of the kth layer of $\mathcal{W}^{(n)}$ is also a member of the kth layer of $\mathcal{W}$, and this holds for each $k \in \mathbf{N}$. Hence, each strand from the web $\mathcal{W}^{(n)}$ is also a strand of $\mathcal{W}$.

Now assume all the spaces E_n are sequentially webbed, and that E is sequentially retractive. If $x_m \to 0$ in E, then $x_m \to 0$ in some E_n. Thus, (x_m) is contained in a finite union of strands of the web $\mathcal{W}^{(n)}$. Since these are also strands of $\mathcal{W}$, it follows that E is sequentially webbed. □

COROLLARY 10: Every strict inductive limit of sequentially webbed spaces is sequentially webbed.

PROOF: Every strict inductive limit of sequentially webbed spaces is sequentially retractive since the topology of the inductive limit induces the original topology on each of the constituent spaces. □

COROLLARY 11: Every (LF)-space is sequentially webbed. □

3. *A NECESSARY CONDITION.*

We are now ready to describe a collection of locally convex spaces which satisfy the Mackey convergence condition.

THEOREM 12: Every sequentially webbed locally convex space satisfies the Mackey convergence condition.

PROOF: Assume E is sequentially webbed. Let $x_n \to 0$ in E. By Köthe [8], §28.3, (x_n) is a Mackey null sequence if and only if there is a sequence (r_n) in $(0, \infty)$ such that $r_n \to \infty$ as $n \to \infty$, and $r_n x_n \to 0$ in E. We seek to find such a sequence (r_n). Assume that we have found strands $(W_k^{(1)}), \ldots, (W_k^{(m)})$ from the web $\mathcal{W}$ such that for each $k \in \mathbf{N}$ there is an $N_k \in \mathbf{N}$ such that $(\forall n \geq N_k)$,

$$x_n \in \bigcup_{i=1}^{m} W_k^{(i)}.$$

Notice that by (1.1), for each i and for each $k \in \mathbf{N}$

$$W_{k+1}^{(i)} \subset \frac{1}{2}W_k^{(i)},\ W_{k+2}^{(i)} \subset \frac{1}{2}W_{k+1}^{(i)} \subset \frac{1}{2^2}W_k^{(i)},\ldots.$$

Hence, $(\forall l \in \mathbf{N})$

$$W_{k+l}^{(i)} \subset \frac{1}{2^l}W_k^{(i)}.$$

For $k \in \mathbf{N}$, we may find $l_{k_1} \in \mathbf{N}$ such that

$$\frac{1}{2^{l_{k_1}}} < \frac{1}{k}.$$

Then, by defining

$$W_{l_{k_1}}^{(1)} \equiv W_{k+l_{k_1}}^{(1)},$$

we have

$$W_{l_{k_1}}^{(1)} \subset \frac{1}{2^{l_{k_1}}}W_k^{(1)} \subset \frac{1}{k}W_k^{(1)}.$$

Similarly, there is an $l_{k_2} \in \mathbf{N}$ such that

$$W_{l_{k_2}}^{(2)} \subset \frac{1}{2^{l_{k_2}}}W_k^{(2)} \subset \frac{1}{k}W_k^{(2)},$$

and so forth.

Let

$$l_k \equiv max\{l_{k_i} : i = 1,\ldots,m\}.$$

Then there is an $N_{l_k} \in \mathbf{N}$ such that for each $n \geq N_{l_k}$

$$x_n \in \bigcup_{i=1}^{m} W_{l_k}^{(i)} \subset \bigcup_{i=1}^{m} \frac{1}{2^{l_k}}W_k^{(i)} \subset \bigcup_{i=1}^{m} \frac{1}{k}W_k^{(i)} \subset \frac{1}{k}\bigcup_{i=1}^{m} W_k^{(i)}.$$

Thus, $(\forall n \geq N_{l_k})$

$$kx_n \in \bigcup_{i=1}^{m} W_k^{(i)}. \tag{3.1}$$

Futhermore, there is an $N_{l_{k+1}} \in \mathbf{N}$ such that $N_{l_{k+1}} > N_{l_k}$, and $(\forall n \geq N_{l_{k+1}})$

$$(k+1)x_n \in \bigcup_{i=1}^{m} W_{k+1}^{(i)}, \tag{3.2}$$

and this continues for all $k \in \mathbf{N}$.

Now define (r_n) by letting

$$r_n = k, \text{ for } N_{l_k} \leq n < N_{l_{k+1}}.$$

By (3.2),

$$\lim_{n\to\infty} r_n = \lim_{k\to\infty} k = \infty.$$

It remains to be shown that $r_n x_n \to 0$ in E. To prove this, let U be a 0-neighborhood in E. By the compatibility of $\mathcal{W}$, there are positive integers $K_1, \ldots, K_m$ such that

$$W_{K_1}^{(1)} \subset U,\ W_{K_2}^{(2)} \subset U, \ldots, W_{K_m}^{(m)} \subset U.$$

Choosing

$$K = max\{K_i : i = 1, \ldots, m\},$$

we have

$$\bigcup_{i=1}^{m} W_K^{(i)} \subset U.$$

Moreover, we may find $N_{l_K} \in \mathbf{N}$ such that for $n \geq N_{l_K}$,

$$r_n x_n \in \bigcup_{i=1}^{m} W_K^{(i)} \subset U,$$

by (3.1). Hence, $r_n x_n \to 0$ in E, and (x_n) is a Mackey null sequence. □

REMARK: Notice that by Proposition 9, each sequentially retractive inductive limit of sequentially webbed spaces satisfies the Mackey convergence condition. In particular, each (LF)-space satisfies the Mackey convergence condition.

If $E = \operatorname{ind}\lim_n E_n$ as in Definition 8, then E is **regular** if each set bounded in E is contained in and bounded in some E_n. Floret [9] proved that a regular inductive limit E of Fréchet spaces is sequentially retractive if and only if E satisfies the Mackey convergence condition. This provides us with the following:

COROLLARY 13: Let E be a regular inductive limit of Fréchet spaces. Then E is sequentially retractive if and only if E is sequentially webbed. □

In order to present an example, we need the following fact.

LEMMA 14: $(E, \mathcal{T})$ is fast complete if and only if $(E, \mathcal{T}')$ is fast complete, where $\mathcal{T}'$ is any topology which is compatible with the duality $\langle E, E' \rangle$.

PROOF: If B is a bounded Banach disk in E with respect to the topology $\mathcal{T}$, then B is bounded in E with respect to any compatible topology $\mathcal{T}'$. Hence, B is a bounded Banach disk in E with respect to $\mathcal{T}'$. □

The following simple example shows that, in addition to the bornological spaces in Theorem 3, there are fast complete, non-bornological spaces which satisfy the Mackey convergence condition.

EXAMPLE 15: Consider l_1 with its weak topology $\sigma = \sigma(l_1, l_\infty)$. Evidently, l_1 under its norm topology $\mathcal{T}$ is bornological; hence, $(l_1, \mathcal{T})$ is a Mackey space. $\sigma \neq \mathcal{T}$, since otherwise l_1 would have finite dimension (see [10], page 10). Thus, (l_1, σ) is not a Mackey space; hence, (l_1, σ) is not bornological. However, (l_1, σ) is fast complete by Lemma 14, and the following shows that (l_1, σ) is sequentially webbed.

Since $(l_1, \mathcal{T})$ is a Fréchet space, it has a neighborhood web consisting of the strand (U_k), as in Proposition 7. Furthermore, $\sigma \subset \mathcal{T}$, so every weak 0-neighborhood V contains a strong 0-neighborhood U. Thus, for the neighborhood strand (U_k), of the web $\mathcal{W}$, there is a $k_0 \in \mathbf{N}$ such that

$$U_{k_0} \subset U \subset V,$$

so $\mathcal{W}$ is also compatible with σ. Now, if $x_n \to 0$ in (l_1, σ), then $x_n \to 0$ in $(l_1, \mathcal{T})$ by Shur's Theorem ([10], Chapter VII, page 85). That is, for each $k \in \mathbf{N}$ there is an $N_k \in \mathbf{N}$ such that for every $n \geq N_k$,

$$x_n \in U_k\,;$$

hence, (l_1, σ) is sequentially webbed.

4. *A SUFFICIENT CONDITION.*

With some slight restrictions on E, we may obtain a converse to Theorem 12. We start with a definition.

DEFINITION 16: A topological vector space E has a **strict web** if E has a compatible web $\mathcal{W}$ such that for each strand (W_k) of $\mathcal{W}$ and each series $\sum_{k=1}^{\infty} x_k$ with $x_k \in W_k$ for each $k \in \mathbf{N}$,

$\sum_{k=1}^{\infty} x_k$ is convergent in E, and further,

$$\sum_{r=k+1}^{\infty} x_r \in \mathcal{W}_{k-1},$$

for each $k \geq 2$. A space with a strict web is also called a **strictly webbed space**.

If the last condition, that

$$\sum_{r=k+1}^{\infty} x_r \in \mathcal{W}_{k-1}$$

is dropped, then E is said to have a **completing web**. Basically, strictly webbed spaces and spaces with completing webs are those which have compatible webs and have an additional mild form of completeness. The reader is referred to [1], [3], or [5] for more details concerning strict webs. It should be noted that in [5] strict webs are called *tight* webs.

There are plenty of strictly webbed spaces. In particular, Fréchet spaces, (LF)- spaces, and strong duals of such spaces are strictly webbed; see sections 10 and 11 of [5] for proofs.

The definition of strictly webbed spaces is suggestive of that for sequentially webbed spaces. A connection between these two will be revealed in Theorem 18 below. First, some preliminary remarks are in order. Strictly webbed spaces also originated in attempts to generalize the Closed Graph Theorem. The most well known of such results is usually called the Localization Theorem; see 5.6.3 of [1] or section 11 of [5]. We will utilize the following special case of this theorem, which is listed as Corollary 2 of Theorem 19, section 11 of [5]:

LEMMA 17: Let E and F be Hausdorff topological vector spaces such that F is strictly webbed. Let $t : E \longrightarrow F$ be linear with a closed graph. Then for each bounded sequentially closed disk B in E, there is a strand (W_k) in F such that for each $k \in \mathbf{N}$ there exists a number α_k such that

$$t(B) \subset \alpha_k W_k.$$

THEOREM 18: Let E be fast complete and strictly webbed. If E satisfies the Mackey convergence condition, then E is sequentially webbed.

PROOF: Let $\mathcal{W}$ be a strict web on E. Let $x_n \to 0$ in E. By assumption, (x_n) is a Mackey null sequence; therefore, by Köthe [8], §28.3, there is a sequence $(r_n) \subset (0, \infty)$, with $r_n \to \infty$ as

$n \to \infty$ and $r_n x_n \to 0$ in E. Let

$$A = \{r_n x_n : n \in \mathbf{N}\}.$$

Then A is bounded, so it is contained in a bounded Banach disk B. Furthermore, A is bounded in the Banach space E_B and by letting C denote the E_B-closure of the convex, balanced hull of A, then clearly C is a bounded, sequentially closed disk in E_B. Notice also that the injection $id : E_B \to E$ is continuous, therefore it has a closed graph. It follows now by Lemma 17 that there exists a strand (W_k) of $\mathcal{W}$ such that for every $k \in \mathbf{N}$ there is a number α_k such that

$$id(C) = C \subset \alpha_k W_k ;$$

therefore, for every $n \in \mathbf{N}$,

$$r_n x_n \in \alpha_k W_k .$$

Since $r_n \to \infty$ as $n \to \infty$, for a fixed $k \in \mathbf{N}$ there is an $N_k \in \mathbf{N}$ such that for every $n \geq N_k$,

$$\frac{|\alpha_k|}{r_n} < 1.$$

Thus, for every $n \geq N_k$

$$x_n \in \frac{|\alpha_k|}{r_n} W_k \subset W_k ,$$

since we assume that each W_k is balanced. □

As was mentioned before Lemma 17, the strong dual of a Fréchet space is strictly webbed. Moreover, by Corollary 2 of Proposition 1, Chapter VI of [6], the strong dual of a Fréchet space is complete, hence fast complete. This allows us to make the following statement.

COROLLARY 19: The strong dual of a Fréchet space satisfies the Mackey convergence condition if and only if it is sequentially webbed. □

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