Internat. J. Math. & Math. Sci.
VOL. 12 NO. 1 (1989) 89-98

ABELIAN GROUPS IN A TOPOS OF SHEAVES: TORSION AND ESSENTIAL EXTENSIONS

KIRAN R. BHUTANI
Department of Mathematics
The Catholic University of America
Washington, D.C. 20064

(Received September 24, 1987)

ABSTRACT. We investigate the properties of torsion groups and their essential extensions in the category AbShL of Abelian groups in a topos of sheaves on a locale. We show that every torsion group is a direct sum of its p-primary components and for a torsion group A, the group [A,B] is reduced for any BϵAbShL. . We give an example to show that in AbShL the torsion subgroup of an injective group need not be injective. Further we prove that if the locale is Boolean or finite then essential extensions of torsion groups are torsion. Finally we show that for a first countable hausdorff space X essential extensions of torsion groups in AbShO(X) are torsion iff X is discrete.

KEYWORDS AND PHRASES. Locale, Abelian groups in a topos, Sheaves on a locale.
1980 AMS CLASSIFICATION CODE. 18E15

1. INTRODUCTION.

In [1] the author discusses the notion of injectivity and injective hulls of abelian groups in a topos of sheaves on a locale where as in [2] the notion of injectivity, injective hulls and the role played by the initial Boolean algebra in a topos is discussed. Our purpose here is to show how torsion groups and their essential extensions behave in the category AbShL of abelian groups in the topos ShL of sheaves on a locale L. We show that torsion is a local property (Theorem 3.1) but not a global one (3.2), that is, A torsion in AbShL does not necessarily imply that AE is a torsion group in Ab. However if L has ACC, then torsion implies global torsion. We prove number of results about torsion groups in AbShL which are analogous to their counterparts in Ab, in particular, we show that every torsion group is a direct sum of its p-primary components (Theorem 3.5), and for a torsion group A the group [A,B] is reduced for all B ϵ AbShL (Proposition 3.10). Recall that in the category Ab, the torsion subgroup of an injective group is injective. We show by giving an example that this does not hold in AbShL for an arbitrary L (3.11).

In section 4 we show that in AbShL, essential extensions of torsion groups are torsion iff every injective group splits into a direct sum of a torsion group and a torsion free group (Proposition 4.2). For a Boolean locale and any finite locale the above result holds (4.3) and (4.4) respectively). We also give an example to show that the converse of (4.3) does not hold. In (4.7) we give an example of a space X and a torsion group in AbShX with a non torison essential extension. After proving some more results about essential extensions of torsion groups, we conclude our paper by showing that for a first countable Hausdorff space X, essential extensions of torsion groups in AbShX are torsion groups iff X is discrete (Theorem 4.8). For basic facts about about abelian groups with which this paper is concerned see [3] and [4]. Details concerning presheaves and sheaves on a locale can be found in [5], category theory in [6] and topos theory in [7].

2. BACKGROUND

(2.1) Recall that a locale denoted by L is a complete lattice satisfying the following distribution law;

$$U \wedge \bigvee_{i \in I} U_i = \bigvee_{i \in I} (U \wedge U_i)$$

for all U, and any family $\{U_i\}_{i \in I}$ in L. The zero (= bottom) of L will be denoted by 0, and the unit (= top) of L by E. A <u>morphism</u> of locales h: $L \to M$ (also called local lattice homomorphism) is a map which preserves arbitrary joins and finite meets (hence preserves the zero and the unit).

An obvious example of a locale is the topology OX (that is the lattice of open sets) of any topolocial space X with joins as unions and meets as intersections.

REMARKS. A locale L satisfies both the Ascending and Descending Chain Conditions iff L is finite. To prove the non-trivial implication ($\to$) note that such an L is spatial [8] and if L = O(X) and X is T_0, one has the following observations concerning X: Each $x \in X$ has a smallest open neighbourhood W_x and for the partial order $\leq$ given, such that $x \leq y (x, y \in X)$ iff $O(x) \subseteq O(y)$ (hence iff $W_y \subseteq W_x$), $W_x = \uparrow x = \{y \mid y \geq x\}$. Moreover DCC for L then implies that $\uparrow x$ is finite, and since X is compact by ACC, X itself is finite. It follows that L is also finite.

(2.2) ABELIAN GROUPS IN A CATEGORY. If E is any finitely complete category then by AbE one means a category with objects as abelian groups in E and maps as homomorphisms between them [9]. For $A \in$ AbE and $0 \neq n \in N$,

(i) The diagonal map $\Delta_A: A \to A^n$ is a unique map such that

$$A \xrightarrow{\Delta_A} A^n \xrightarrow{p_i} A = 1_A$$ for all i=1,2,...n, where $p_i: A^n \to A$ is the i^{th} projection

(ii) The sum $A^n \xrightarrow{+_A} A$ is the unique map such that $A \xrightarrow{q_i} A^n \xrightarrow{+_A} A = 1_A$ where $q_i: A \to A^n$ is the i^{th} injection for i=1,2...n. The composition $+_A \Delta_A: A \to A^n \to A$ is denoted by n_A and the kernel of n_A shall be denoted by $k_n: \ker n_A \to A$.

DEFINITION 2.3. (1) $A \in$ AbE is called a <u>torsion free group</u> iff n_A is a monomorphism for all $0 \neq n \in N$.

(2) $A \in$ AbE is called a <u>torsion group</u> iff all k_n, $0 \neq n \in N$ are jointly epic, that is, for any two homomorphisms f and g with domain A, if $fk_n = gk_n$ for all $0 \neq n \in N$ then f=g.

2.4. Recall that by AbPShL and AbShL one means the categories of Abelian groups in the topos PShL and ShL of presheaves and sheaves, respectively, on a locale L with values in the category Ab of abelian groups. For any $U, V \in L$ and $A \in$ AbShL, AU will denote the component of A at U and if $V \leq U$ the restriction map $AU \to AV$ will be written as $a \to a|V$. If A is the sheaf reflection of the given presheaf B (also denoted by $A = \tilde{B}$) then we shall write $AU \doteq BU$. Also if $h: A \to B$ is a morphism in AbShL then its component at $U \in L$ is denoted by $h_U : AU \to BU$.

NOTE. AbSh2 $\simeq$ Ab for the two-element locale 2 and if X is a discrete topological space then AbShX $\simeq \mathrm{Ab}^{|X|}$. Further AbSh3 for the three-element locale is the same as AbPSh2 that is the arrow category of Ab. Further AbSh3 is also AbShS for the Sierpinski space S with points 0 and 1 and non-trivial open set $\{1\}$.

2.5. Recall that for any local lattice homomorphism $\phi: L \to M$ we get a pair of adjoint functors AbshM $\underset{\phi^*}{\overset{\phi_*}{\rightleftarrows}}$ AbShL where $(\phi_* A)U = A(\phi(U))$ for $U \in L$, and for any $V \in M$ $(\phi^* C)V = \underset{\phi(W) \geq V}{\ell t}\, CW \quad (W \in L)$. Then ϕ^* is left exact, left adjoint to ϕ_*. As a special case we get for each $U \in L$ a pair of adjoint functors R_U: AbShL $\to$ AbSh$\downarrow U$ and E_U: AbSh$\downarrow U \to$ AbShL defined by $(R_U A)W = A(W \wedge U)$ and

$$(E_U A)V \doteq \begin{cases} AV \text{ if } V \leq U \\ 0 \text{ if } V \nleq U \end{cases}$$

Then E_U is left adjoint left exact to R_U. We shall also denote $R_U A$ by $A|U$. Further R_U preserves all limits and co-limits.

2.6. Besides the obvious external Ab valued hom-functor $H = H_L$:

$\mathrm{AbShL}^{opp} \times$ AbShL $\to$ Ab, AbShL also has an internal hom-functor $[-,-]$:

$\mathrm{AbShL}^{opp} \times$ AbShL $\to$ AbShL, for which $[A,B]U = H_{\downarrow U}(A|U, B|U)$, with the restriction maps $[A,B]U \to [A,B]V$ $(V \leq U)$, given by $h = (h_W)_{W \leq U} \to h|V = (h_W)_{W \leq V}$ [10].

2.7. In (2.3) we described what we mean by torsion free and torsion groups in AbE. For the case E = ShL, we have the following:

(1) $A \in$ AbShL is a torsion free group iff each AU is torsion free in Ab.

(2) $A \in$ AbShL is a torsion group iff $A = \underset{0 \neq n \in N}{\ell t} \mathrm{Ker}\, n_A$. That is for $a \in AU$, there exists a cover $U = \bigvee_{i \in I} U_i$, and $0 \neq m_i \in Z$, such that $m_i a|U_i = 0$ for all $i \in I$.

PROPOSITION 2.8. For any $U \in L$, the functors R_U and E_U preserve torsion groups.

PROOF. Let $A \in$ AbShL which is a torsion group. Then $A = \underset{0 \neq n \in N}{\ell t} \mathrm{Ker}\, n_A$ since R_U preserves all co-limits and limits (2.5), it follows $R_U A = \underset{0 \neq n \in N}{\ell t} R_U(\mathrm{Ker}\, n_A) = \underset{0 \neq n \in N}{\ell t} \mathrm{Ker}\, n_{A|U}$, hence $A|U$ is torsion in AbSh$\downarrow U$. By a similar argument it can be shown that E_U preserves torsion groups.

3. TORSION GROUPS.

THEOREM 3.1. $A \varepsilon$ AbShL is a torsion group iff there is a cover $E = V_{i\varepsilon I} U_i$ such that $A|U_i$ is torsion in $AbSh{\downarrow}U_i$ for all $i \varepsilon I$.

PROOF. ($\to$) Clear by taking the trivial cover of E. On the other hand if all $A|U_i$ are torsion groups in $AbSh{\downarrow}U_i$, we claim A is torsion. So consider any $b \varepsilon AU$, $U \varepsilon L$. Then $U = V_{i\varepsilon I}(U \wedge U_i)$ and $b|(U \wedge U_i) \varepsilon A(U \wedge U_i) = A|U_i(U \wedge U_i)$ for all $i \varepsilon I$. all $i \varepsilon I$.

But $A|U_i$ is torsion in $AbSh{\downarrow}U_i$, and so for each $i \varepsilon I$, there is a cover

$U \wedge U_i = \bigvee_{j\varepsilon J_i} W_{ji}$ and $0 \neq n_{ji} \varepsilon N$ such that $n_{ji} b|W_{ji} = 0$, $j \varepsilon J_i$. Hence for

$b \varepsilon AU$, we can find a cover $U = \bigvee_{j\varepsilon J_i \ i\varepsilon I} W_{ji}$ such that

$n_{ji} b|W_{ji} = 0$ for all i, j, $0 \neq n_{ji} \varepsilon N$, which shows that A is a torsion group in AbShL.

COUNTEREXAMPLE 3.2. Proposition 3.1 shows that torsion is a local property. However it is not a global property as we shall see from the following counter example: Consider $L = \omega{+}1$ and $A \varepsilon$ AbShL given by

$$\begin{array}{ccccccccc} \prod_{n<\omega} Z/nZ & \to \ldots \to & Z/2Z & \chi & Z/32 & \to & Z/2Z & \to & 0\ (= Z/Z) \to 0 \\ \omega & > \ldots\ldots\ldots > & 3 & & > & & 2 & > & 1 > 0 \end{array}$$

By Proposition 3.1, A is torsion, since for the cover $\omega = V_{n<\omega} n$, the group

$A|n = \Pi_{k<n} Z/kZ$ is torsion in $AbSh{\downarrow}n$ for all $n < \omega$. But $A\omega = \Pi_{n<\omega} Z/nZ$

is not torsion in Ab, as the element $(1+nZ)_{n<\omega}$ does not have a finite order.

DEFINITION 3.3. For a given prime p, by the p-primary component of a group $A \varepsilon$ AbShL we mean the subgroup of A given by $U_{0\neq n\varepsilon N} \operatorname{Ker} p^n{}_A$. We denote the p-primary component of A by A_p. $A\varepsilon$AbShL is called a p-primary group if $A = A_p$.

DEFINITION 3.4. By the torsion subgroup B of an any group $A\varepsilon$AbShL we mean the subgroup of A given by $B = U_{0\neq n\varepsilon N} \operatorname{Ker} n_A$.

THEOREM 3.5. Every torsion group is a direct sum of its p-primary components.

PROOF. Let A be a torsion group and denote by B the presheaf $BU = t(AU)$ the torsion subgroup of AU. Then A is the sheaf reflection of [illegible]. Now $BU = t(AU) = \oplus(t(AU))_p$ where $(t(AU))_p$ denotes the p-primary component of t(AU). If $B_p \subseteq B$

is the subpresheaf $B_pU = (t(AU))_p$ then clearly $B = \oplus B_p$ in AbPShL. The Sheaf reflection being a left adjoint preserves co-limits, in particular direct sums and so

$A = \tilde{B} = (\oplus B_p)^{\sim} = \oplus \tilde{B}_p$. But $B_p = A_p$ and hence we get $A = \oplus A_p$.

DEFINTION 3.6. By the torsion type of a group A we mean the set of all prime numbers p such that $A_p \neq 0$.

PROPOSITION 3.7. If A is a torsion group and $B \supseteq A$ is an essential extension then B and A have the same torsion type.

PROOF. Since $A \subseteq B$, it follows $A_p \subseteq B_p$ and therefore $A_p \subseteq A \cap B_p$ for all p. Consider any $U \in L$, then

$$(A \cap B_p)U = AU \cap B_pU = AU \cap (\cup_{0 \neq n \in N} \mathrm{Ker}\ p_B^n)U$$

$$\doteq AU \cap (\cup_{0 \neq n \in N} \mathrm{Ker}\ p_{BU}^n)$$

$$\doteq \cup_{0 \neq n \in N} (AU \cap \mathrm{Ker}\ p_{BU}^n)$$

$$\doteq \cup_{0 \neq n \in N} (A \cap \mathrm{Ker}\ p_B^n)U$$

$$\doteq \cup_{0 \neq n \in N} (\mathrm{Ker}\ p_A^n)U = A_pU$$

Hence $A \cap B_p = A_p$ for all primes p. We now want to show that $A_p \subseteq B_p$ is an essential extension. If $0 \neq C \subseteq B_p$, then since $A \subseteq B$ is essential it follows $A \cap C \neq 0$. This means $0 \neq A \cap C \cap B_p = A_p \cap C$, thereby showing that $A_p \subseteq B_p$ is essential. Hence $B_p \neq 0$ iff $A_p \neq 0$ which means that A and B have the same torsion type.

DEFINITION 3.8. We call an $A \in$ AbShL to be a reduced group if it has no non zero injective subgroups. Recall that in the category Ab, for any torsion group B the group Hom(B,K) is reduced for all $K \in$ Ab. We shall prove the analogue of this for the Ab-valued hom-functor H and the internal hom-functor [-,-] of AbShL(2.6).

LEMMA 3.9. If $A \in$ AbShL is a torsion group then H(A,P) is reduced in Ab for all $P \in$ AbShL.

PROOF. Let $0 \neq C, \subseteq H(A,P)$ be an injective subgroup. Consider any $0 \neq \alpha \in C$, then for some $U \in L$ and $A \in AU$, $\alpha_U(a) \neq 0$. Since A is torsion and $a \in AU$ there exists a cover $U = \vee_{i \in I} U_i$ and $0 \neq n_i \in N$ such that

$n_i a|U_i = 0$ for all $i \in I$. But $\alpha_U(a) \neq 0$ implies that $\alpha_{U_k}(a|U_k) \neq 0$ for some k

I. Consider now $0 \neq n_k \in N$, then C an injective hence divisible group in Ab implies that there exists some $\beta \in C$ such that $n_k\beta = \alpha$. Therefore $n_k\beta_{U_k}(a|U_k) = \beta_{U_k}(n_k a|U_k) = \beta_{U_k}(0) = 0$, which means $\alpha_{U_k}(a|U_k) = 0$, a contradiction,

hence C = 0 which shows that AbShL(A,P)=H(A,P) is reduced in the category Ab.

PROPOSITION 3.10. If A is a torsion group in AbShL, then [A,P] is reduced in AbShL for all $P \in$ AbShL.

PROOF. Let $0 \neq B \subseteq [A,P]$ be an injective subgroup. Then for some $U \in L$, $BU \neq 0$ is an injective subgroup of $[A,P]U = H_{\downarrow U}(A|U,P|U)$. Since A is torsion, it follows $A|U$ is torsion (2.8) in AbSh↓U and so by last lemma $H_{\downarrow U}(A|U,P|U)$ is reduced in Ab. Thus BU = 0 for all $U \in L$, hence B = 0 which means [A,P] is reduced in AbShL.

REMARK. Recall that in the category Ab, the torsion subgroup of an injective group is always injective. We show in the following example that, for an arbitrary L, the torsion subgroup of an injective group need not be injective, except for some special locales which we shall discuss in the next section.

EXAMPLE 3.11. Consider the locale $L = \omega + 2$ and A AbShL given by

$$\begin{array}{ccccc} P_1 & \to \prod_{n<\omega} P_n & \to \ldots \to & P_2 \times P_1 \to P_1 & \to 0 \\ \omega + 1 & > \omega > \ldots > & & 2 > 1 > 0 & \end{array}$$

where the P_i are finite groups with increasing exponent. By one of our previous results ([1], proposition 2.3) the injective hull of A is given by the group

$$B = \prod_{n<\omega} E(P_n) \overset{1d}{\to} \prod_{n<\omega} E(P_n) \to \ldots \to E(P_2) \times E(P_1) \to E(P_1) \to 0$$

where $E(P_i)$ denotes the injective hull of group P_i in Ab. If TB is the torsion subgroup of B, then $(TB)n = Bn$ all $n < \omega$, and so $(TB)\omega = B\omega = \prod_{n<\omega} E(P_n)$, but $(TB(\omega + 1)) = T(B(\omega + 1)) = \bigoplus_{n<\omega} E(P_n)$. Hence $TB \subseteq B$ and since $A \subseteq TB$, it follows TB is not injective since B, being the injective hull of A, is the minimal injective extension of A, hence the result.

4. ESSENTIAL EXTENSIONS OF TORSION GROUPS.

If for any torsion group $A \in$ AbShL, all essential extensions of A are torsion, then we say that essential extensions in AbShL preserve torsion. The following proposition shows "essential extensions preserve torsion" is a local property.

PROPOSITION 4.1. Essential extensions preserve torsion in AbShL iff there exists a cover $E = \bigvee_{i\in I} U_i$ such that essential extensions preserve torsion in $AbSh{\downarrow}U_i$ for all $i \in I$.

PROOF. ($\to$) Clear by taking the trivial cover of E. For the converse, consider any essential extension B of the torsion group A in AbShL. Since for each $i \in I$ the functor R_{U_i}: AbShL $\to AbSh{\downarrow}U_i$, preserves essential extensions and torsion [1] it follows $B|U_i$ is an essential extension of the torsion group $A|U_i$ in $AbSh{\downarrow}U_i$.

By hypothesis, $B|U_i$ is torsion in $AbSh{\downarrow}U_i$ all $i \in I$, hence by Theorem 2.1, B is torsion in AbShL.

PROPOSITION 4.2. For any L, essenital extensions in AbShL preserve torsion iff every injective group splits into a direct sum of a torsion group and a torsion free group.

PROOF. ($\to$) Let B denote the torsion subgroup of an injective group $A \in$ AbShL. If $C \supseteq B$ is any essential extension, then by hypothesis C is a torsion group. Since A is injective we may assume that $C \subseteq A$, so C torsion implies $C \subseteq B$ and hence $C = B$. Thus B has no proper essential extensions which means that B is injective. Therefore $A = B \oplus E$ for some subgroup E of A. If TE denotes the torsion subgroup of E, then $TE \subseteq B$ and so $TE \subseteq B \cap E = 0$, hence $TE = 0$. Thus E is torsion free.

($\to$) Let P be a torsion group and H the injective hull of P. By hypothesis H =

$T \oplus F$ where T is a torsion group and F is a torsion free group. If $F \neq 0$, then since H is an essential extension of P it follows that $P \cap F = 0$, a contradiction, since P is torsion. Hence $F = 0$ which shows that H is a torsion group. Since every essential extension of P has an embedding into H, it follows all essential extensions of P are torsion, hence the result.

THEOREM 4.3. For a Boolean locale, essential extensions in AbShL preserve torsion.

PROOF. Consider an essential extension B of a torsion group A in AbShL. Let C denote the torsion subgroup of B(3.4). For any $U \in L$ consider an arbitrary element $b \in BU$. Let $W \leq U$ be the largest element in $\downarrow U$ such that $b|W \in CW$. We claim W is dense in $\downarrow U$. If not, then there exists $S \in \downarrow U$, $S \neq 0$ such that $S \wedge W = 0$. Now for any $V \leq S$, $b|V \in CV$ gives $V \leq W$ and so $V + V \wedge W \leq S \wedge W = 0$ implies $V = 0$. In particular $b|S \neq 0$. Since $B \supseteq A$ is an essential extension therefore there exists a $V \leq S$ and $m \in Z$ such that $0 \neq mb|V \in AV \subseteq CV$. Now C is the torsion subgroup of B and $0 \neq mb|B \in CV$ implies $b|V \in CV$. But then $V = 0$, a contradiction, since $0 \neq mb|V \in AV$. Hence W is dense in $\downarrow U$. Since L is Boolean we have $W = U$, thus $BU \subseteq CU$ for all $U \in L$ and so $B = C$. Hence B is torsion.

REMARK. On the other hand, one can see that if essential extensions preserve torsion in AbShL, then it does not necessarily follow that L is Boolean. Here is a counterexample:

Consider $L = 3$. If $B = \begin{matrix} B_1 \\ \downarrow h \\ B_2 \end{matrix}$ is torsion in AbSh3, then both B_1 and B_2 are torsion in Ab. By ([1], Proposition 2.3) the injective hull of B is

given by $A = \begin{matrix} E(B_2) \times E(\mathrm{Ker}\ h) \\ \downarrow \\ E(B_2) \end{matrix}$ which is torsion in AbSh3. Hence

Hence all essential extensions of B are torsion, although $L = 3$ is not Boolean. Of course the remark is a special case of the following more general result which shows that there are non-Boolean L such that essential extensions in AbShL preserve torsion.

THEOREM 4.4. For any finite L essential extensions in AbShL preserve torsion.

PROOF. Let B be any essential extension of the torsion group A. Then for an arbitrary $a \in AU$, $U \in L$. A torsion implies that there is a cover $U = U_1 \vee U_2 \vee \ldots \vee U_k$ and $0 \neq n_i \in N$ such that $n_i a|U_i = 0$ for all $i = 1,2,\ldots,k$. If $m = n_1 n_2 \ldots n_k$ then $ma|U_i = 0$ for all i and therefore $ma = 0$ and $m \neq 0$. This shows for each $U \in L$, AU is a torsion group in the category Ab. Now, if there are $V \in L$ such that BV is not a torsion group then let S be minimal such that BS is not torsion. Then $S \neq 0$ and for all $U < S$, BU is a torsion group in Ab. If $W = \bigvee_{U<S} U$, then since each BU is torsion it follows by proposition 3.1 that $B|W$ is torsion in AbSh$\downarrow$W. By the same argument as above it follows BW is torsion and hence W<S. Consider an arbitrary $b \in BS$ of infinite order. Since $B \supseteq A$ is an essenital extension, there exists $V \leq S$ and $0 \neq m \in Z$ such that $0 \neq mb|V \in AV$. Then $V \neq S$, for otherwise $0 \neq mb \in AV$ has finite order and so b will have finite order, a contradiction, since b has infinite order. Hence $V \leq W$. This implies $b|W \neq 0$. But BW is torsion and so for some $0 \neq n \in N$, $nb|W = 0$. But $0 \neq nb \in BS$ is again of infinite order and so by the same argument $0 \neq nb|W$, a contradiction. Hence BS is a

torsion group which contradicts the definition of S. This shows B is a torsion group in AbShL.

REMARK 3.4. Recall from (2.1) that the finite locales L are exactly those L in which both ACC and DCC hold. It is therefore of interest to note that there exists an L which satisfies DCC but for which essential extensions in AbShL do not preserve torsion. Here is an example which is actually the same as that considered in (3.11) for a different purpose: If A and its injective hull $B \supseteq A$ are as in 3.11, then B is not torsion because its torsion subgroup is proper.

THEOREM 4.5. If essential extensions preserve torsion in AbShL, then for all $U \varepsilon L$, the following are true:

(i) Essential extensions preserve torsion in $AbSh{\downarrow}U$.

(ii) Essential extensions preserve torsion in $AbSh{\uparrow}U$.

PROOF. Let B be any essential extension of the torsion group A in $AbSh{\downarrow}U$. Since the functor E_U: $AbSh{\downarrow}U \to AbShL$ preserves essential extensions [1] and also torsion (2.9), it follows E_UB is an essential extension of the torsion group E_UA. By hypothesis E_UB is torsion in AbShL. Therefore $R_U(E_UB) = B$ is again torsion since the functor R_U preserves torsion (2.9), hence the result.

(ii) Consider the local lattice homomorphism ϕ: $L \to {\uparrow}U$ given by $\phi(W) = W \vee U$. Then ϕ produces ϕ_*: $AbSh{\uparrow}U \to AbShL$(2.6) where $(\phi_*A)W = A(U \vee W)$, $W \varepsilon L$.

Let B be an essential extension of the torsion group A in $AbSh{\uparrow}U$. We claim that B is torsion. We first show that ϕ_* preserves torsion. Let $0 \neq a \varepsilon (\phi_*A)W = A(U \vee W)$. Since A is torsion, there is a cover $(U \vee W) = \bigvee_{i \varepsilon I} U_i$ in ${\uparrow}U$, and $0 \neq n_i \varepsilon N$ such

that $n_ia|U_i = 0$ for all $i \varepsilon I$. So we can for a cover $W = (U \vee W)$ $W = \bigvee_{i \varepsilon I} (U_i \wedge W)$

in L such that $(n_ia)|U_i \wedge W) = 0$ all $i \varepsilon I$. Hence for $0 \neq a \varepsilon (\phi_*A)W$, we can always

find a cover $W = \bigvee_{i \varepsilon I} (U_i \wedge W)$ in L, such that $0 = n_ia|(U_i \wedge W)$, and that proves ϕ_*A is torsion in AbShL.

To show that ϕ_* preserves essential extesnions take $0 \neq b$ in $(\phi_*B)W = B(W \vee U)$, $W \varepsilon L$. Since $B \supseteq A$ is essential in $AbSh{\uparrow}U$. there exists $V \leq W \vee U$ and $m \varepsilon Z$ such that $0 \neq mb|V \varepsilon AV$. But $U < V$ and $V \leq W \vee U$ implies $V = (V \wedge W) \vee U$ and therefore 0 $mb|(V \vee W) \vee U \varepsilon A((V \wedge W) \vee U)$. Thus for 0 $b \varepsilon (\phi_*B)W$, there

is $(V \wedge W) \leq W$ such that $0 \neq mb|V \wedge W \varepsilon \phi_*A) (V \wedge W)$ for some $m \varepsilon Z$. This shows ϕ_*B is an essential extension of ϕ_*A in AbShL. Finally we show that ϕ_* reflects torsion. So, let ϕ_*P be a torsion group in AbShl for some $P \varepsilon AbSh{\uparrow}U$. If $0 \neq a \varepsilon PW$, $W \varepsilon {\uparrow}U$, then $0 \neq a \varepsilon (\phi_*P)W = P(W\ U) = PW$, and so ϕ_*P being a torsion group implies, that there is a cover $W = V_{i \varepsilon I}W_i$ in L, and $0 \neq n_i \varepsilon N$ such that

$n_ia|W_i = 0$ all $i \varepsilon I$, where $a|W_i \varepsilon (\phi_*P)Wi = P(W_i \vee U)$. If we consider the cover $W = V_{i \varepsilon I} (W_i \vee U)$ in ${\uparrow}I$. then we get $0 = n_ia|(W_i \vee U)$ for all $i \varepsilon I$, which proves that P is torsion in $AbSh{\uparrow}U$. Thus, in order to prove (ii), we consider an essential extension D of the torsion group C in $AbSh{\uparrow}U$. Then by the above argument ϕ_*D is an essential extension of ϕ_*C in AbShL. But ϕ_*C is torsion since C is torsion, hence by hypothesis $\phi_* D$ is torsion. Now ϕ_* reflects torsion and that proves D is torsion in AbSh↑U. Hence the result.

REMARK 4.6. As a special case, if L = OX for some topological space X and $Y \subseteq X$ is a closed subspace then $\uparrow CY \simeq \Theta Y$, the isomorphism being given by $U \to U \cap Y$, $U \in \uparrow CY$. Hence, by the last proposition, essential extensions preserve torsion in AbShY, if they do in AbShX.

LEMMA 3.7. On the space $X = \{0\} \cup \{1/n | n = 1,2,\ldots\} \subseteq R$ there is a torsion group C with a non-torsion essential extension.

PROOF. Consider $A \in Ab^{|X|}$ by $A\{0\} = 0$, $A(n) = z(p^\infty)$ for all $n \neq 0$. Then the functor $F: Ab^{|X|} \to AbShX$ [1] produces B = FA, (FA)U = $\Pi_{x\, U} A\{x\} = \{\phi: U \to Z(p^\infty),\ \phi(0) = 0 \text{ if } 0 \in U\}$ in AbShX. Let C be the torsion subgroup of B. Assume C = B, then CX = BX and so the function $\phi \in BX$ given by $\phi(0)=0$, $\phi(1/n) = a_n$ where a_n has order p^n, n = 1,2... is in CX. This means there exists a cover $X = \cup_{i\in I} U_i$ and $0 \neq k_i \in N$ such that $k_i\phi|U_i=0$ for all $i \in I$. Since $0 \in U_j$ for some $j \in I$ and hence U_j contains infinitely many $\{1/n, n \in N\}$ thus $k_j\phi|U_j = 0$ a contradiction. Hence $\phi \notin CX$, which shows B is not a torsion group in AbShX. We now show that B is an essential extension of C. Let $0 \neq \alpha \in BU$, then $\alpha(1/n) \neq 0$ for some $1/n \in U$. If $W = \{1/n\}$, then $\alpha|W \neq 0$ is of finite order since $\alpha(1/n) \in Z(p^\infty)$, hence $0 \neq \alpha|W \in CW$. Thus B is an essential extesnion of C which is torsion, although B itself is not torsion.

THEOREM 4.8. If X is a first countable Hausdorff space, then essential extensions preserve torsion in AbShX iff X is discrete.

PROOF. (→) Suppose that X is not discrete. Then there is a point $x_o \in X$ for which $\{x_0\}$ is not open. Let the countable basic neighbourhoods of x_0 be arranged in the form $U_1 \supset U_2 \supset \ldots$ and for each $n \in N$ pick an element $x_n \in U_n - U_{n+1}$. Denote by X_0 the subspace of X consisting of the points $\{x_0, x_1, x_2, \ldots\}$. Since the sequence $\{x_k\}_{k\in N}$ converges to x_0, it follows that the space X_0 is compact in X. But X is Hausdorff and so X_0 is closed in X. For any x_n, $n \neq 0$ the subset $X_0 - \{x_n\}$ also being compact, is also closed in X. Hence $\{x_n\} = \{X - \{X_0 - \{x_n\}\}\} \cap X_0$ is open in the space X_0. It is then easy to see that the subspace X_0 consisting of $\{x_0, x_1, \ldots\}$ is homeomorphic to the space $\{0\} \cup \{1/n | n\in N\} \subseteq R$. By the above lemma essential extensions of torsion groups need not be torsion in $AbShX_0$, a contradiction to Remark 4.6 hence X is discrete.

(←) If X is discrete then $AbShX \simeq AbSh^{|X|}$ and so if $A \in AbShX$ is a torsion group then clearly each A{x}, $x\in X$ is a torsion group in Ab. So, if $B \supseteq A$ is an essential extension in AbShX, then $B|\{x\} = B\{x\} \supseteq A|\{x\} = A\{x\}$ is essential in Ab, hence each B{x} is a torsion group in Ab. Thus B is torsion in AbShX.

COROLLAY 4.9. If $X = \Pi_{\alpha\in I} X_\alpha$, where each X_α is a first countable, Hausdorff space, and essential extensions in AbShX preserve torsion, then X is discrete.

PROOF. If $X = \Pi_{\alpha\in I} X_\alpha$, then each $X_\alpha \simeq$ closed subspace of X, hence by Remark 4.6, essential extensions preserve torsion in $AbShX_\alpha$. But X_α is given to be first countable and Hausdorff, therefore by Proposition 4.8, each X_α is discrete. Suppose X_α is non-trivial for infinitely many α, then $2^\omega \simeq$ subspace of X. But 2^ω is compact, hence closed in X. Also 2^ω is first countable, Hausdorff. But it is not discrete, hence only finitely many X_α are non-trivial which implies that X is disrete.

REMARK. All finite L are spatial, and for all finite L, essential extensions in AbShL preserve torsion. Hence there are many non-discrete spaces X such that essential extensions preserve torsion in AbShX.

ACKNOWLEDGMENTS. This is a part of my doctoral dissertation submitted to the Graduate School of McMaster University for partial fulfillment of the requirements for the degree of Doctor of Philosophy. The author is grateful to her supervisor, Professor Bernhard Banaschewski, for his valuable guidance, encouragement, and criticism throught the research work. I would also like to thank the Laboratory of Statistical and Mathematical Methododlogy at the National Institutes of Health, Bethesda, MD, for providing the facilities for the preparation and revision of this paper.

REFERENCES

1. BHUTANI, K.R. Injectivity and injective hulls of abelian groups in a localic topos, Bulleting of the Australian Mathematical Society Vol. 37 (1988), 43-59.
2. BANASCHEWSKI, B. and BHUTANI, K.R. Boolean algebras in a localic topos, Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 100, Part 1 43-56, July 1986.
3. FUCHS, L. Infinite abelian groups, Academic press, Vol. 1 (1970).
4. KAPLANSKY, I. Infinite abelian groups. University of Michigan.
5. JOHNSTONE, P.T. Stone spaces, Cambridge studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1982.
6. MACLANE, S. Categories for the working Mathematician, Graduate Text in Mathematics 5, Springer-Verlag, 1971.
7. JOHNSTONE, P.T. Topos Theory. Academic Press, London (1977).
8. BANASCHEWSKI, B. Coherent Frames, Lecture notes in Mathematics, Springer Verlag, 871 1-19.(1981).
9. FREYD, P. Abelian Categories. Harper & Row Publishers, 1964.
10. EBRAHIMI, M. Algebra in a topos of sheaves. Doctoral Dissertation, McMaster University, 1980.
11. BALBES, R. and DWINGER, P. Distributive lattices. University of Missouri Press, 1975.
12. GODEMENT, R. Theorie des Faisceaux, Merman, Paris, 1958.