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OPERATIONAL CALCULUS FOR THE CONTINUOUS LEGENDRE TRANSFORM WITH APPLICATIONS

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ABSTRACT. This paper develops an operational calculus for the continuous Legendre transform introduced and studied by Butzer, Stens and Wehrens [1]. It is an extension of the work done by Churchill et al [2], [3] for the discrete case. In particular, a differentiation theorem and a convolution theorem are proved and the results are applied to the solution of some boundary value problems.

KEY WORDS AND PHRASES. Continuous Legendre Transform, Operational Calculus, Convolution, Boundary Value Problems.
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1. INTRODUCTION. For a given function f belonging to an appropriate function space, the continuous Legendre transform is defined by

$$(Tf)(\lambda) = \frac{1}{2}\int_{-1}^{1} P_\lambda(x)f(x)dx \tag{1}$$

where $P_\lambda(x)$ is the Legendre function and $\lambda \geq -\frac{1}{2}$. This transform has been introduced and studied by Butzer, Stens and Wehrens [1]. The discrete analog of the transform in (1) has been studied by Churchill [2] and Churchill and Dolph [3]. The object of this paper is to develop an operational calculus for the transform which is useful in solving partial differential equations whose underlying differential form is given by

$$D = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]. \tag{2}$$

In section 2 we present the background material needed in the sequel. In section 3, we derive the operational calculus for (1) including a convolution theorem and a table of transforms of some functions. In the last section we apply the results to solving some boundary value problems.

2. PRELIMINARIES. We recall basic properties of the transform $(Tf)(\lambda)$ (see [1]) and important contiguous relations that hold for the Legendre function.

The Legendre function $P_\lambda(x)$ is given by

$$P_\lambda(x) = {}_2F_1(-\lambda,\ \lambda+1;1;\frac{1-x}{2}) = \sum_{k=0}^{\infty} \frac{(-\lambda)_k(\lambda+1)_k}{(k!)^2}(\frac{1-x}{2})^k,\ x\epsilon(-1,1].$$

Since $P_{\lambda-1}(x) = P_{-\lambda}(x)$, it sufficies to consider the case $\lambda \geq -\frac{1}{2}$. $P_\lambda(x)$ satisfies the differential equation

$$Dy + \lambda(\lambda+1)y = 0$$

where D is as given in (2). Further, it satisfies the relations $P_\lambda(1) = 1$, $P'_\lambda(1) = \frac{\lambda(\lambda+1)}{2}$, $\lim_{x\to -1^+}(1+x)P_\lambda(x) = 0$ and $\lim_{x\to-1^+}(1+x)P'_\lambda(x) = \frac{\sin\pi\lambda}{\pi}$.

The following contiguous relations (see [4]) will be useful in the derivation of the calculus for $(Tf)(\lambda)$:

$$(2\lambda+1)xP_\lambda(x) = (\lambda+1)P_{\lambda+1}(x) + \lambda P_{\lambda-1}(x) \tag{3}$$

and

$$(1-x^2)P'_\lambda(x) = -\lambda x P_\lambda(x) + \lambda P_{\lambda-1}(x). \tag{4}$$

From (3) and (4) we obtain the relation

$$(1-x^2)P'_\lambda(x) = -\frac{\lambda(\lambda+1)}{2\lambda+1}(P_{\lambda+1}(x) - P_{\lambda-1}(x)). \tag{5}$$

The addition formula for the Legendre functions (see [4]) is given by

$$P_\lambda(\cos\alpha)P_\lambda(\cos\beta) = P_\lambda(\cos\nu) - 2\sum_{m=1}^{\infty}\frac{\Gamma(\lambda-m+1)}{\Gamma(\lambda+m+1)}P_\lambda^m(\cos\alpha)P_\lambda^m(\cos\beta)\cos m\gamma \tag{6}$$

where $P_\lambda^m(\cdot)$ is the associated Legendre function and $\cos\nu = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\gamma$ with $0 \leq \alpha,\ \beta \leq \pi,\ \alpha+\beta < \pi,\ \gamma$ real. Formula (6) will be useful in deriving the convolution theorem. Another useful relation involving the Legendre functions is

$$\int_{-1}^{1} P_\lambda(x)P_\nu(-x)dx = \frac{\sin\pi\lambda - \sin\pi\nu}{\pi(\lambda-\nu)(\lambda+\nu+1)},\ \lambda \neq \nu,\ \lambda+\nu+1 \neq 0. \tag{7}$$

The Legendre transform $(Tf)(\lambda)$ is a linear integral transform from $L_2(-1,1]$ into the space $C_0(-1,1] \cap L_2(-1,1]$. For $f\epsilon L_2(-1,1]$, it was shown in [1] that $(Tf)(\lambda) = 0(\lambda^{-\frac{1}{2}})$ as $\lambda \to \infty$ and $(Tf)(\lambda - \frac{1}{2})\epsilon C_0(-1,1] \cap L_2(-1,1]$. Further, it was shown that if $f\epsilon L_2(-1,1] \cap C(-1,1]$ and if $\sqrt{\lambda}(Tf)(\lambda-\frac{1}{2})\epsilon L_1(\mathbf{R}^+)$, then the inversion formula is given by

$$f(x) = T^{-1}((Tf)(\lambda)) = 4\int_0^\infty (Tf)(\lambda - \frac{1}{2})P_{\lambda-\frac{1}{2}}(-x)\lambda\sin\pi\lambda d\lambda. \tag{8}$$

3. BASIC OPERATIONAL PROPERTIES FOR $(Tf)(\lambda)$. In this section we shall [illegible] operational calculus for the continuous Legendre transform $(Tf)(\lambda)$ thus extending the calculus obtained by Churchill [2] and Churchill and Dolph [3] for the discrete case. We shall also derive the Legendre transform of some functions.

The first property in this direction involves the Legendre transform of the differential operator D as given in (2).

Theorem 3.1. Let f be a function such that (i) $f^{(k)} \epsilon C(-1,1] \cap L_2(-1,1]$ $k = 0,1$ (ii) $\lim_{x\to\pm 1}(1-x^2)f(x) = \lim_{x\to\pm 1}(1-x^2)f'(x) = 0$ and (iii) $(Tf)(\lambda)$ exists. Then

$$(T(Df))(\lambda) = -\lambda(\lambda+1)(Tf)(\lambda). \tag{9}$$

Proof. From (1) together with successive integration by parts, we obtain

$$\begin{aligned}
(T(Df))(\lambda) &= \frac{1}{2}\int_{-1}^{1} P_\lambda(x) Df(x)dx \\
&= \frac{1}{2}\int_{-1}^{1} P_\lambda(x)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}f(x)\right]dx \\
&= \left[\frac{1}{2}P_\lambda(x)(1-x^2)f'(x) - \frac{1}{2}P'_\lambda(x)(1-x^2)f(x)\right]_{-1^+}^{+1^-} \\
&- \frac{1}{2}\lambda(\lambda+1)\int_{-1}^{1} P_\lambda(x)f(x)dx.
\end{aligned}$$

The result follows from the facts that $P_\lambda(1) = 1$, $P'_\lambda(1) = \frac{\lambda(\lambda+1)}{2}$, $\lim_{x\to -1^+}(1+x)P_\lambda(x) = 0$ and $\lim_{x\to -1^+}(1+x)P'_\lambda(x) = \frac{\sin \pi\lambda}{\pi}$ together with the hypothesis (ii).

This basic operational property reduces a given differential equation which involves the operator D into an algebraic one or into a differential equation with one less independent variable.

Remark 3.1. (a) If, in Theorem 3.1, $D^k f = D^{k-1}(Df)$ and $f^{(k)}$ satisfy the same hypotheses, then

$$T((D^k f(x)))(\lambda) = (-1)^k \lambda^k (\lambda+1)^\lambda (Tf)(\lambda), \; k = 1,2,\ldots .$$

(b) We note that (9) can be cast into the form

$$\frac{1}{4}(Tf)(\lambda) - T((Df))(\lambda) = (\lambda + \frac{1}{2})^2 (Tf)(\lambda). \tag{10}$$

The second operational property involves the relationship between the transform of a given function f and the function $g(x) = \int_{-1}^{x} f(t)dt$.

Theorem 3.2. If f is a piecewise continuous function defined on $(-1,1)$ and $g(x) = \int_{-1}^{x} f(t)dt$ and if $(Tf)(\lambda)$ exists, then

$$(Tg)(\lambda) = -\frac{(Tf)(\lambda+1) - (Tf)(\lambda-1)}{2\lambda+1}. \tag{11}$$

Proof. Since $D(P_\lambda(x)) = -\lambda(\lambda+1)P_\lambda(x)$, it follows that

$$\begin{aligned}
(Tg)(\lambda) &= -\frac{1}{2\lambda(\lambda+1)}\int_{-1}^{1}\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_\lambda(x)\right]g(x)dx \\
&= -\frac{1}{2\lambda(\lambda+1)}(1-x^2)P'_\lambda(x)g(x)|_{-1}^{1} + \frac{1}{2\lambda(\lambda+1)}\int_{-1}^{1}(1-x^2)P'_\lambda(x)f(x)dx .
\end{aligned}$$

Since $P'_\lambda(1)$ and $g(1)$ are defined, $g(-1) = 0$ and $\lim_{x\to -1^+}(1+x)P'_\lambda(x) = \frac{\sin \pi\lambda}{\pi}$, the first term is identically zero. Thus

$$(Tg)(\lambda) = \frac{1}{2\lambda(\lambda+1)}\int_{-1}^{1}(1-x^2)P'_\lambda(x)f(x)dx.$$

The contiguous relation (5) will then imply that

$$(Tg)(\lambda) = \frac{1}{2\lambda(\lambda+1)}\int_{-1}^{1} -\frac{\lambda(\lambda+1)}{2\lambda+1}(P_{\lambda+1}(x) - P_{\lambda-1}(x))f(x)dx$$

Equivalently,

$$(Tg)(\lambda) = -\frac{(Tf)(\lambda+1) - (Tf)(\lambda-1)}{2\lambda+1}.$$

Remark 3.2. Similar difference relations to that of (11) can be obtained in the following situation.

(a) If $g(x) = xf(x)$ and if $(Tf)(\lambda)$ exists, then under appropriate conditions on f, one obtains

$$(Tg)(\lambda) = \frac{(\lambda+1)(Tf)(\lambda+1) + \lambda(Tf)(\lambda-1)}{2\lambda+1} \tag{12}$$

This will follow by applying the contiguous relation (3).

(b) If $g(x) = \int_{-1}^{x}(x-t)f(t)dt$ and if $(Tf)(\lambda)$ exists, then, again under appropriate conditions on f, the contiguous relation (5) and Theorem 3.2 yields

$$(Tg)(\lambda) = \frac{(Tf)(\lambda+2) - 2(Tf)(\lambda) + (Tf)(\lambda-2)}{(2\lambda+1)^2} \tag{13}$$

The next operational property that we will derive involves the inverse of the differential operator D. We define the inverse of D, denoted by D^{-1}, by $D^{-1}(f(x)) = g(x)$ if and only if $D(g(x)) = f(x)$. If $(Tf)(\lambda)$ is known, then we want to relate $T((D^{-1}f))(\lambda)$ to the transform of f.

If, for a given function $f(x)$, $D(g(x)) = f(x)$, then on integrating twice, we obtain

$$g(x) = \int_0^x \frac{1}{1-t^2}\int_{-1}^{t} f(\alpha)d\alpha dt + c$$

for some constant c. If $f(x)$ is in addition an even function on $(-1,1)$, then one can show by employing a continuity argument that $\lim_{x\to\pm1}(1-x^2)g(x) = \lim_{x\to\pm1}(1-x^2)g'(x) = 0$. Theorem 3.1 will then imply that

$$(Tf)(\lambda) = T((Dg))(\lambda) = -\lambda(\lambda+1)(Tg)(\lambda).$$

Equivalently,

$$(Tg)(\lambda) = -\frac{1}{\lambda(\lambda+1)}T((Dg))(\lambda) = -\frac{(Tf)(\lambda)}{\lambda(\lambda+1)}.$$

Thus

$$T(D^{-1}f)(\lambda) = -\frac{1}{\lambda(\lambda+1)}(Tf)(\lambda).$$

This last relation implies that $D^{-1}f$ is the inverse Legendre transform of $-\frac{(Tf)(\lambda)}{\lambda(\lambda+1)}$. We thus have

Theorem 3.3. If $f(x)$ is such that $f(x)$ is even on $(-1,1)$, $f\epsilon L_2(-1,1]\cap C(-1,1]$, $(Tf)(\lambda)$ exists and $\frac{(Tf)(\lambda)}{\sqrt{\lambda(\lambda+1)}}\epsilon L_1(\mathbf{R}^+)$, then

$$D^{-1}(f(x)) = T^{-1}\left(-\frac{(Tf)(\lambda)}{\lambda(\lambda+1)}\right) \tag{14}$$

where the inverse transform T^{-1} is given by (8).

We shall finally develop a convolution property for the Legendre transform. In particular, we will show

Theorem 3.4. If $f(x)$ and $g(x)$ are given functions for which $(Tf)(\lambda)$ and $(Tg)(\lambda)$ respectively exist, then their product $(Tf)(\lambda)(Tg)(\lambda)$ is the transform of the function $h(x) = f(x) * g(x)$ where $h(x)$ is given by

$$h(\cos\nu) = \frac{1}{2\pi}\int_0^\pi \int_0^\pi f(\cos\alpha)g(\cos\beta)\sin\alpha d\alpha d\theta$$

where $\cos\beta = \cos\alpha\cos\nu + \sin\alpha\sin\nu\cos\theta$ with $0 \le \alpha,\ \nu \le \pi,\ \alpha + \nu < \pi$ and θ is real. The variables $\alpha,\ \nu$ and β may be interpreted as the sides of a spherical triangle on the unit hemisphere and θ is the angle between the sides α and ν (see Figure 1).

Proof. From (1), we have

$$(Tf)(\lambda)(Tg)(\lambda) = \frac{1}{4}\int_{-1}^{1} P_\lambda(x)f(x)dx \int_{-1}^{1} P_\lambda(y)g(y)dy.$$

Set $x = \cos\alpha$ and $y = \cos\beta$. Then

$$(Tf)(\lambda)(Tg)(\lambda) = \frac{1}{4}\int_0^\pi f(\cos\alpha)\sin\alpha \int_0^\pi P_\lambda(\cos\alpha)P_\lambda(\cos\beta)g(\cos\beta)\sin\beta d\beta d\alpha.$$

The addition formula for the Legendre function (6) will yield upon an integration with respect to γ from 0 to π

$$P_\lambda(\cos\alpha)P_\lambda(\cos\beta) = \frac{1}{\pi}\int_0^\pi P_\lambda(\cos\nu)d\gamma$$

where $\cos\nu = \cos\alpha\cos\beta + \sin\alpha\sin\beta\cos\gamma$ (see figure 1).

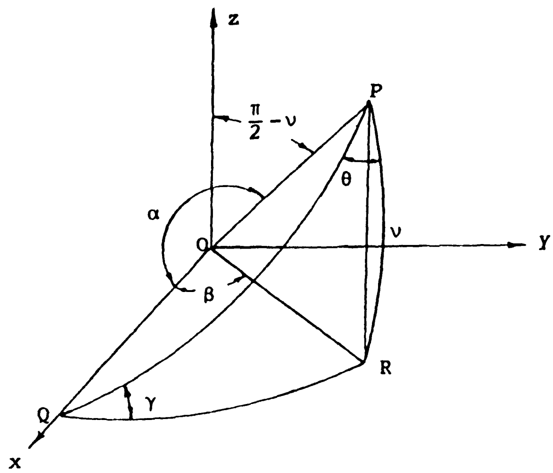

Figure 1

Thus

$$(Tf)(\lambda)(Tg)(\lambda) = \frac{1}{4\pi}\int_0^\pi f(\cos\alpha)\sin\alpha \int_0^\pi \int_0^\pi P_\lambda(\cos\nu)g(\cos\beta)\sin\beta d\gamma d\beta d\alpha.$$

In the spherical triangle PQR, we have

$$\cos\beta = \cos\alpha\cos\nu + \sin\alpha\sin\nu\cos\theta.$$

Using this relation along with the sine law and transformation of co-ordinates, the double integral can be written as:

$$\int_0^\pi \int_0^\pi P_\lambda(\cos\nu) g(\cos\beta) \sin\nu d\theta d\nu.$$

Hence,

$$(Tf)(\lambda)(Tg)(\lambda) = \frac{1}{2}\int_0^\pi P_\lambda(\cos\nu)\sin\nu \left[\frac{1}{2\pi}\int_0^\pi\int_0^\pi f(\cos\alpha)g(\cos\beta)\sin\alpha d\alpha d\theta\right] d\nu.$$

The expression in the bracket is a function of ν and we then write

$$h(\cos\nu) = \frac{1}{2\pi}\int_0^\pi\int_0^\pi f(\cos\alpha)g(\cos\beta)\sin\alpha d\alpha d\theta \tag{15}$$

This may be interpreted as a convolution product of f and g and $(Th(\cos\nu))(\lambda) = (Tf)(\lambda)(Tg)(\lambda)$. This proves Theorem 3.4.

Geometrically, the expression (15) is the mean value of $f(\cos\alpha)g(\cos\beta)$ over the unit hemisphere $x^2+y^2+z^2 = 1$, $z \geq 0$. To see this, we note that the element surface area is $dS = sin\alpha d\alpha d\theta$. This is clear if we identify the coordinate transformation in Figure 1 by

$$\begin{aligned} x &= \cos\alpha \\ y &= \sin\alpha\sin\theta \\ z &= \sin\alpha\cos\theta \end{aligned}$$

Thus (15) reads

$$h(\cos\nu) = \frac{1}{2\pi}\int_S\int f(\cos\alpha)g(\cos\beta)dS.$$

We will now evaluate the Legendre transform of some functions.

1. $f(x) = \text{constant} = k$

$$(Tf)(\lambda) = \begin{cases} k\frac{\sin\pi\lambda}{\pi\lambda(\lambda+1)} & \lambda \neq 0 \\ k & \lambda = 0 \end{cases}$$

2. $f(x) = P_n(x)$. Then by (2.5) we have, for $n = 0, 1, 2, \ldots$,

$$\begin{aligned} (Tf)(\lambda) &= \frac{1}{2}\int_{-1}^1 P_\lambda(x)P_n(x) = \frac{(-1)^n}{2}\int_{-1}^1 P_\lambda(x)P_n(-x)dx \\ &= \frac{\sin\pi(\lambda-n)}{2\pi(\lambda-n)(\lambda+n+1)}, \ \lambda \neq n, \ -(n+1). \end{aligned}$$

3. $f(x) = \log(1-x)$.

$$\begin{aligned} (Tf)(\lambda) &= \frac{1}{2}\int_{-1}^1 P_\lambda(x)\log(1-x)dx \\ &= -\frac{1}{2\lambda(\lambda+1)}\int_{-1}^1 \frac{d}{dx}\left[(l-x^2)\frac{d}{dx}P_\lambda(x)\right]\log(1-x)dx \\ &= (\log 2)\frac{\sin\pi\lambda}{\lambda(\lambda+1)} - \frac{1}{\lambda(\lambda+1)} - \frac{1}{2\lambda(\lambda+1)}\int_{-1}^1 P_\lambda(x)\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\log(1-x)\right]dx. \end{aligned}$$

Observe that $D(\log(1-x)) = \frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\log(1-x)\right] = -1$. Thus

$$(Tf)(\lambda) = (\log 2)\frac{\sin\pi\lambda}{\lambda(\lambda+1)} - \frac{1}{\lambda(\lambda+1)} - \frac{\sin\pi\lambda}{\lambda^2(\lambda+1)^2}$$

4. $f(\lambda) = \int_{-1}^x \frac{1}{1-t}dt$. By using 1 and 3 above, we obtain

$$(Tf)(\lambda) = \frac{1}{\lambda(\lambda+1)} + \frac{\sin \pi\lambda}{\lambda^2(\lambda+1)^2}.$$

5. $f(x) = (1 - 2tx + x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$, $-1 < t < 1$. From (2) above

$$(Tf)(\lambda) = \frac{sin\pi\lambda}{2\pi} \sum_{n=0}^{\infty} \frac{t^n}{(\lambda - n)(\lambda + n + 1)}.$$

We finally remark that for λ equal to a non- negative integer, the results of this section yield those obtained in [2] and [3].

4. APPLICATIONS. In this section we consider some applications of the Legendre transform. We consider problems arising in heat conduction and in potential theory.

A. Heat Conduction Problem. Consider a non-homogeneous bar with extremities at $x = \pm 1$ and is insulated at these end points. Let $u(x,t)$ be the temperature of the bar at position x at time t. The one dimensional heat equation with prescribed initial temperature is given by

$$\begin{aligned} \frac{\partial}{\partial x}\left(k\frac{\partial}{\partial x}u(x,t)\right) &= \rho c\frac{\partial u}{\partial t}(x,t) \\ u(x,0) &= g(x)\,, \qquad -1 < x < 1 \end{aligned}$$

where k, ρ and c are physical constants representing thermal conductivity, density and specific heat respectively. We assume that the thermal conductivity k is given by $k = \alpha(1 - x^2)$, α being a real constant. The above equation reads

$$\begin{aligned} \frac{\partial}{\partial x}\left((1 - x^2)\frac{\partial}{\partial x}u(x,t)\right) &= \frac{\rho c}{\alpha}\frac{\partial u}{\partial t}(x,t) \\ u(x,0) &= g(x)\,, \qquad -1 < x < 1. \end{aligned}$$

If $U(\lambda,t) = T(u(x,t))(\lambda)$ and $G(\lambda) = (Tu(x,0))(\lambda)$, then, by Theorem 3.1, we obtain upon the application of the transform

$$\begin{aligned} \frac{d}{dt}U(\lambda,t) &= -\frac{\alpha}{\rho c}\lambda(\lambda+1)U(\lambda,t) \\ U(\lambda,0) &= G(\lambda). \end{aligned}$$

The solution is given by

$$U(\lambda,t) = G(\lambda)e^{-\frac{\alpha}{\rho c}(\lambda+1)\lambda t}$$

Now $u(x,t)$ can be obtained by either employing the inversion formula (8) or the convolution theorem. By employing the inversion formula and under the assumption that $u(x,t)\epsilon C(-1,1] \cap L_2(-1,1)$ and $\sqrt{\lambda}\, U(\lambda - \frac{1}{2}, t)\epsilon L_1(\mathbf{R}^+)$, one obtains

$$u(x,t) = 4\int_0^\infty G(\lambda - \frac{1}{2})e^{-\frac{\alpha}{\rho c}(\lambda^2 - \frac{1}{4})t}P_\lambda(-x)\lambda \sin \pi\lambda d\lambda$$

On the other hand the convolution property (Theorem 3.4) will yield

$$u(\cos\nu, t) = \frac{1}{2\pi}\int_0^\pi \int_0^\pi g(\cos\alpha)f(\cos\beta)\sin\alpha d\alpha d\theta$$

where α, β, θ are as in Figure 1 and $\cos\beta = \cos\alpha\cos\nu + \sin\alpha\sin\nu\cos\theta$ and f is the inverse transform of $e^{-\frac{\alpha}{\rho c}(\lambda^2 - \frac{1}{4})t}$. That is, by (8)

$$f(x) = e^{\frac{\alpha}{4\rho c}t}4\int_0^\infty e^{-\frac{\alpha}{\rho c}(\lambda - \frac{1}{2})^2}P_\lambda(-x)\lambda \sin \pi\lambda d\lambda.$$

B. Dirichlet Problem for the Unit Sphere (see[2]) Consider the problem of determining the potential $v(r,\cos\theta)$ in the interior of a unit sphere with a prescribed potential $f(\cos\theta)$ on $r = 1,\ 0 \le \theta \le \pi$. The Laplace equation defining this potential is

$$\nabla^2 v = \frac{1}{r}(rv)_{rr} + \frac{1}{r^2\sin\theta}(\sin\theta v_\theta)_\theta = 0.$$

If $x = \cos\theta$, then the equation reduces to

$$\begin{aligned} r(rv)_{rr} + \left((1-x^2)v_x\right)_x &= 0 \\ v(1,x) &= f(x)\,, \qquad -1 \le x \le 1. \end{aligned}$$

If $V(r,\lambda)$ and $F(\lambda)$ denote respectively the Legendre transform of $v(r,x)$ and $f(x)$, then, upon applying the transform to the underlying equation, we obtain

$$\begin{aligned} r\frac{d^2}{dr^2}(rV(r,\lambda)) - \lambda(\lambda+1))V(r,\lambda) &= 0, \\ V(1,\lambda) &= F(\lambda). \end{aligned}$$

The solution of this equation is given by

$$V(r,\lambda) = c_1 r^\lambda + c_2 r^{-(\lambda+1)}.$$

In order to apply the inversion formula (8) we need to have $v(r,x)\varepsilon L_2(-1,1] \cap C(-1,1]$ and $\sqrt{\lambda}\ V(r,\lambda)\varepsilon L_1(\mathbb{R}^+)$. This will imply that $c_2 = 0$ and $v(1,\lambda) = F(\lambda)$ will imply that $c_1 = F(\lambda)$. Hence the solution is given by

$$v(r,\lambda) = F(\lambda)r^\lambda$$

and

$$v(r,x) = r\int_0^\infty F(\lambda - \frac{1}{2})r^{\lambda-\frac{1}{2}}P_\lambda(-x)\lambda\sin\pi\lambda d\lambda.$$

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