PSEUDO-ANALYTIC FINITE PARTITION APPROACH TO TEMPERATURE DISTRIBUTION PROBLEM IN HUMAN LIMBS

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ABSTRACT.   The main object of this paper is to introduce the Pseudo Analytic Finite Partition method and to illustrate its use in solving physiological heat distribution problems pertaining to limbs and similar body organs.   A two dimensional circular region resembling the cross section of a human or animal limb is considered.   The biological properties are assumed to vary along the radial direction.   The theoretical model incorporates the effect of blood mass flow and metabolic heat generation.   The region is divided into annular sub-regions and Ritz variational finite element method is applied along the radial direction, while for the angular direction, Fourier Series has been used due to uniformity in each annular part.

1. INTRODUCTION.

A human body maintains its body core temperature at a uniform temperature under the normal atmospheric conditions.   In order to maintain this core temperature, parameters like rate of blood mass flow, rate of metabolic heat generation and, thermal conductivity vary in response to changes in atmospheric conditions.   However, in extreme parts of a human body, the core temperature is not uniform at low atmospheric temperatures, where the core temperature of limbs varies extensively as we move away from the body core.   This may be because the arterial blood has cooled down while travelling towards the extremities.   The heat flow in in-vivo tissues is given by W. Perl [1] as given below.

$$\rho \bar{c} \frac{\partial u}{\partial t} = \text{Div}(K \text{ grad } u) + m_b c_b (u_b - u) + s \tag{1.1}$$

where $, \rho, \bar{c}$, K and S are respectively the density, specific heat, thermal conductivity and, rate of metabolic heat generation in tissues. m_b and c_b are the mass blood flow rate and specific heat of the blood respectively. W. Perl [1] derived and used this equation to study simple problems of heat flow in tissue medium. Chao, Eisley and Yang [2] and Chao and Yang [3] also studied temperature distributions in infinite tissue mediums. Cooper and Trezek [4] obtained a solution for a cylindrical symmetry with all the parameters as constant. Saxena [5,6], Saxena and Arya [7,8], and Saxena and Bindra [9,10] used this model to study temperature distributions in skin and subcutaneous tissues using analytical and numerical techniques. This study was performed for a one dimensional steady state case. Later Arya [11], Arya and Saxena [12] and Saxena and Bindra [13] investigated this problem for a two dimensional steady state case in skin and subcutaneous tissues. Here a cylindrical limb having circular cross section with layers of tissues with different properties is considered. The outer boundary is assumed to be exposed to the environment and heat loss takes place due to convection, radiation and, evaporation. The innermost solid cross section is assumed to be at a known variable temperature. This case may occur when one side of the limb contains major blood vessels and thus heated constantly by the blood comming out of main trunk.

2. MATHEMATICAL FORMULATION.

Equation (1.1) in polar steady state form can be written as:

$$\frac{\partial}{\partial r}(Kr\frac{\partial u}{\partial r}) + \frac{K}{r}\frac{\partial^2 u}{\partial r^2} + m_b c_b \; r(u_b - u) + rS = 0 \; . \tag{2.1}$$

The region is divided into N layers with inner and outer radii equal to a_o and a_N respectively. (see Fig.1).
The boundary and initial conditions imposed are

$$\left| -K \frac{\partial u}{\partial r} \right|_{r=a_N} = h(u-u_a) + LE \tag{2.2}$$

and $\qquad u(a_o, \theta) = f(\theta) = u_o \tag{2.3}$

where $f(\theta)$ is known. Here h is the heat transfer coefficient and, L and E are respectively the latent heat and rate of sweat evaporation. S is assigned temperature dependent values given by

$\qquad$ S=s $(u_b - u)/u_b$, where s is constant.

Thus equation (2.1) along with boundary conditions (2.2) and (2.3) is reduced to descretized polar variational form as given below:

$$I_i = \frac{1}{2} \int_{a_{i-1}}^{\bar{a}_i} [\bar{r} \; K_i (\frac{\partial v^{(i)}}{\partial r})^2 - \frac{K_i}{\bar{r}}(\frac{\partial^2 v^{(i)}}{\partial \bar{\theta}^2})^2 + \bar{M}_i \bar{r} \; v^{(i)2}] \; d\bar{r} + \delta_i, \; i=1(1)N \tag{2.4}$$

where $\delta_N = [\frac{h}{2}(v^{(N)} - v_a)^2 - LE \; v^{(N)}/u_b] \; \bar{}_{r=a_N}$

$\qquad \delta_i = 0, \; i=1(1)N-1$

$\bar{a}_i = a_i/a_o$, $a_i(i=1(1)N)$ is the external radius of ith annular region as shown in Fig.

(1), K_i, $\bar{M}_i$ and $v^{(i)}$ are values of K, $\bar{M}$ and, v in the ith region,

$$\bar{M} = a_o^2 (m_b c_b + s/u_b), \quad \bar{r} = r/a_o, \quad v = (u_b-u)/u_b, \quad v_a = (u_b-u_a)/u_b,$$

The following linear shape functions have been taken for each region:

$$v^{(i)} = \frac{a_i v_{i-1} - a_{i-1} v_i}{\bar{a}_i - \bar{a}_{i-1}} + \frac{v_i - v_{i-1}}{\bar{a}_i - \bar{a}_{i-1}} \bar{r}. \qquad (2.5)$$

Evaluating integrals (2.4) and assembling these we get

$$I = \sum_{i-1}^{N} I_i . \qquad (2.6)$$

Now I is extremized with respect to v_i and the following equations are obtained:

$$\sum_{j=0}^{N} (E_j^i v_j + F_j^i \frac{\partial^2 v_j}{\partial \theta^2}) = H_i. \qquad (2.7)$$

Where E_j^i, F_j^i and H_i are constants depending upon physical and physiological

parameters.

Now Fourier Series is applied to eliminate the θ variable from the equation (2.7).
We take

$$v_o = A_{oo} + \sum_{n=1}^{\infty} (A_{no} \text{Cos} n\theta + B_{no} \text{Sin} n\theta) \qquad (2.8)$$

and

$$v_i = A_{oi} + \sum_{n=1}^{\infty} (A_{ni} \text{Cos} n\theta + B_{ni} \text{Sin} n\theta) . \qquad (2.9)$$

Here the coefficients A_{oo}, A_{no} and B_{no} are known due to boundary condition (2.3). All
the coefficients A_{oi}, A_{ni} and B_{ni} (i=1,2,...,N) are unknown. Accordingly, the
following system of linear equations is obtained:

$$\bar{E}^o \bar{A}_o = \bar{G}^o$$

$$\bar{E}^1 \bar{A}_n = \bar{G}^1$$

$$\bar{E}^2 \bar{B}_n = \bar{G}^2 \qquad (2.10)$$

where $\bar{E}^v = [E_{ij}^v]$ (i=1(1) N, j = 1(1) N) are square matrices of order N.

$\bar{A}_o = [A_{oi}]$, $\bar{A}_n = [A_{ni}]$, $\bar{B}_n = [B_{ni}]$ and, $\bar{G}^v = [G_i^v]$ where $i=1(1)N$, are vectors. E_{ij}^v, A_{oi}, A_{ni}, B_{ni} and G_i^v are constants and they depend on the parameters mentioned earlier. Here only a special case in which the annular cross-section of the limb has been divided into two layers i.e. N=2 is considered. These two layers have different biological properties. The outermost layer is supposed to be made of mainly dead tissues. Hence it does not have any metabolic heat generation or blood flow. These quantities have been prescribed in the inner annular part. The core of the limb is assumed to have unsymmetric temperature. Parabolic variation of temperature along the circular boundary has been taken, so

$$V_o = F(\theta) = \frac{(\theta-\pi)(\theta-2\pi)}{2\pi^2} w_o + \frac{2\theta(\theta-2\pi)}{(-2\pi^2)} w_1 + \frac{\theta(\theta-\pi)}{2\pi^2} w_o$$

where $w_o = (u_b - u_\alpha)/u_b$, $w_1 = (u_b - u_\beta)/u_b$, u_α and u_β are temperatures at $\theta=0$ and $\theta=\pi$ respectively. Using $F(\theta)$ in terms of Fourier series, the constants A_{oo}, A_{no}, and B_{no} are determined and then substituted in the system of equations (2.10). These equations are solved to find the values of A_{oi}, A_{ni}, and B_{ni} which in turn are substituted in expressions (2.9) to obtain v_1. Using (2.5) and (2.9), the temperature profiles are obtained for each sub-region.

3. NUMERICAL RESULTS

The following values of physical and physiological parameters and constants have been taken:

$K_1 = 0.06$ Cal/cm-min. deg. C, $K_2 = 0.03$ Cal/cm-min. deg. C.,

$a_o = 2.5$ cm, $a_1 = 5$ cm, $a_2 = 7.5$ cm, $L = 579$ Cal/gm,

$h = 0.009$ Cal/cm^2-min. deg. C, $u_b = 37^{\circ}$C,

$M_1 = (m_b c_b)_1 = 0.003$ Cal/cm^3-min. deg. C, $M_2 = (m_b c_b)_2 = 0$

$(S)_1 = s_1 = 0.0357$ Cal/cm^3-min, $(S)_2 = s_2 = 0$,

$u_\alpha = 30^{\circ}$C, $u_\beta = 34^{\circ}$C, $u_a = 15^{\circ}$, $E = 0$,

Graphs have been plotted between u and θ and the temperature profiles have been shown in fig. 2, 3 and, 4. Fig. 2 is a geometrical representation of the boundary conditions and figures 3 and 4 give temperature variation in direction around the two annular partitions. These two curves are significantly different from the curve in Fig. 2. There is a slow rise in the temperature at $\theta=0$. This rise becomes sharper later on and obviously the temperature takes on its maximum value at $\theta=\pi$. The linear variation of temperature with respect to r is easily seen by comparing the figures 3 and 4.

The number of computations involved in this method are less as compared to those in variational finite element method for two dimensional case.
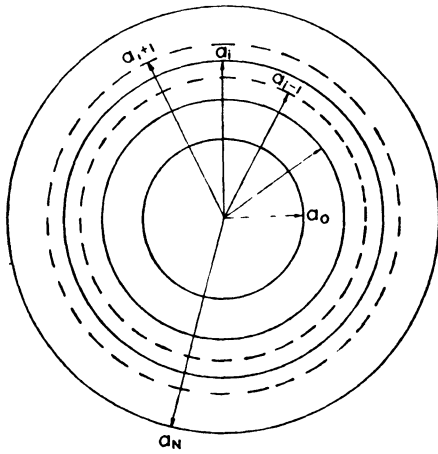
Fig. 1. ANNULAR CROSS-SECTION OF A HUMAN
LIMB DIVIDED INTO N LAYERS. SHADED
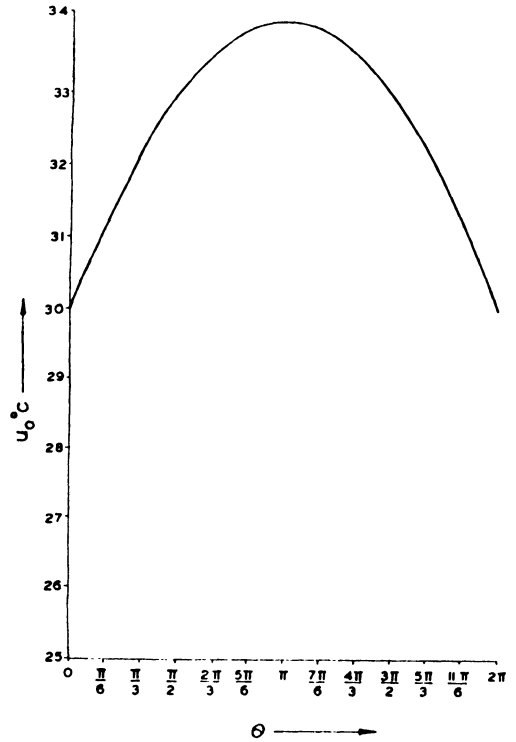PORTION IS SOLID CIRCULAR SECTION.

Fig.2. GRAPH SHOWING THE
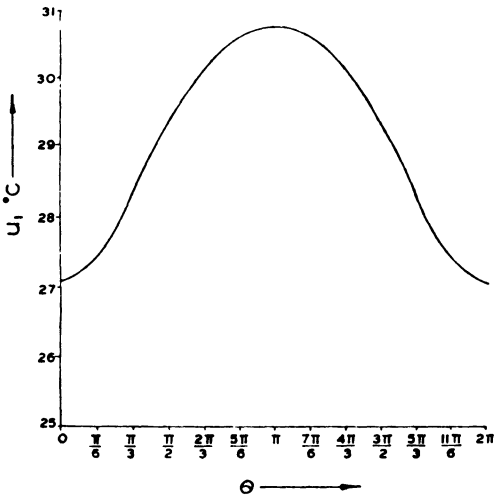GEOMETRICAL REPRESENTATION OF
THE BOUNDARY CONDITIONS.

Fig. 3. GRAPH BETWEEN U_1 AND Θ, SHOWING THE
TEMPERATURE VARIATIONS AROUND THE
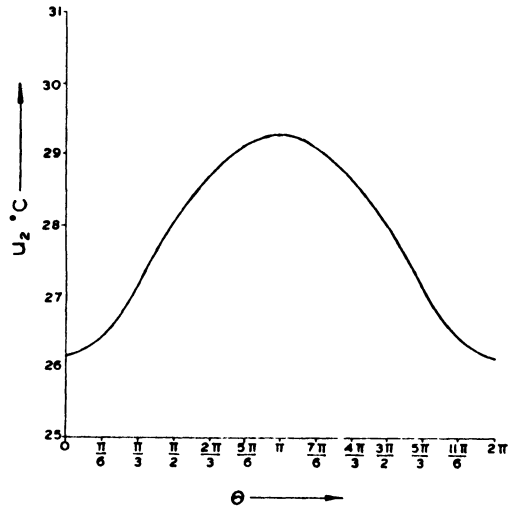INNER ANNULAR PARTITION.

Fig. 4. GRAPH BETWEEN U_2 AND Θ,
SHOWING THE TEMPERATURE VARIATIONS
AROUND THE OUTER ANNULAR PARTITION.

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