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CONSERVATION LAWS FOR INCOMPRESSIBLE FLUIDS

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ABSTRACT. By means of a direct approach, a complete set of conservation laws for incompressible fluids is determined. The problem is solved in the material (Lagrangian) description and the results are eventually rewritten in the spatial (Eulerian) formulation. A new infinite family of conservation laws is determined, besides those for linear momentum, angular momentum, energy and helicity.

KEY WORDS AND PHRASES. Conservation laws, incompressible fluids, Lagrangian description, complete classification.
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1. INTRODUCTION.

The determination of conservation laws, and of the corresponding invariants, for the 3-dimensional Euler equations can be of fundamental importance in studying existence and uniqueness of solutions [1-3]. In this scheme, Serre examined Euler's equations for the motion for an incompressible inviscid fluid under the assumption that the conserved quantities are independent of the space-time variables. He has shown [1] that the conserved quantites are in fact the linear momentum, energy, and the so-called helicity [4], only. Later, on adopting a Hamiltonian formulation, Olver has derived the conservation law of angular momentum by considering the generators of symmetry transformations [5].

Recently, an analysis of inviscid compressible fluid flows based on infinitesimal generators of symmetry transformations and adjoint variables [6,7] has led to the discovery of an additional conservation law [8] describing, in a sense, the center-of-mass theorem. Taking this result as a motivation and looking for a more exhaustive approach, the problem was dealt with in the material description [9], where the Euler equations allow for a variational formulation and thus Noether's theorem permits (cf. [10]) the determination of a complete set of conservation laws. Really, new conservation laws depending on arbitrary functions, besides the center-of-mass theorem, have been determined, but the procedure is not completely exhaustive, because of some technical limitations that have been imposed from the very beginning, in order to render the required calculations more manageable.

In this paper we consider the incompressible fluid model and look for the full set of independent conservation laws, without any restrictive assumption on the form of such laws. In this case no variational formulation, and thus none of the various forms of Noether's theorem, is available. That is why we adopt a direct approach and seek solutions to the vanishing of a 4-dimensional divergence. We find it convenient to solve the problem in the material description and, eventually, to rewrite the results in the spatial (Eulerian) one.

As a result of our approach, the *complete* set of conserved vectors is given and, as a by-product, a new infinite class of conservation laws is determined explicitly. This class extends to incompressible fluids a remarkable result discovered very recently for compressible fluids [9]. Incidentally, the procedure elaborated in this paper is likely to work for more involved situations.

2. PRELIMINARIES.

To set up a general framework for conservation laws in fluid dynamics we consider a system described by n functions ϕ_β, $\beta = 1,2,...,n$, of the (space-time) variables X_Σ, $\Sigma = 1,2,...,m$. The functions ϕ_β satisfy the second order system of differential equations

$$F_\alpha(X_\Lambda, \phi_\beta, \phi_{\beta,\Sigma}, \phi_{\beta,\Sigma\Lambda}) = 0, \qquad \alpha = 1,2,...,n, \tag{2.1}$$

where a comma followed by a letter, Σ say, denotes the partial derivative with respect to X_Σ. The functions F_α are supposed to be of class C^1 with respect to their argument.

Let D_Σ denote the (total) derivative with respect to X_Σ. A *conservation law* for the system (2.1) is a second order differential equation

$$D_\Sigma I_\Sigma \doteq 0 \tag{2.2}$$

for suitable functions I_Σ of the form

$$I_\Sigma = I_\Sigma(X_\Lambda, \phi_\beta, \phi_{\beta,\Sigma}).$$

The symbol $\doteq$ is a reminder that equality is required to hold in connection with the solutions ϕ_β to (2.1) only. Operatively, determining a conservation law amounts to finding a m-tuple of functions I_Σ such that (2.2) is satisfied identically by the solutions to (2.1).

Trivial conservation laws arise when the I_Σ's vanish for all solutions to (2.1) or when (2.2) holds for all functions ϕ_β regardless of whether they solve the system (2.1) [10]. Such trivial conservation laws do not provide any information about the properties of the solutions and then two conservation laws are regarded as equivalent whenever they differ by a trivial conservation law. Non-equivalent conservation laws are said to be independent. Accordingly we are only interested in the determination of independent and non-trivial conservation laws.

3. GOVERNING EQUATIONS.

Let $\mathbf{v}$ be the velocity field of the fluid and p the pressure. Moreover let latin indices run over $1,2,3$ and denote Cartesian components. So x_i is the i-th component of the position vector $\mathbf{x}$ in the three-dimensional Euclidean space $\mathcal{E}^3$. The motion of an inviscid, incompressible fluid is governed by the system of differential equations [11]

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0, \tag{3.1}$$

$$\frac{\partial v_j}{\partial x_j} = 0, \tag{3.2}$$

in the unknown functions v_i, p. Equations (3.1)-(3.2) reflect the Eulerian description of motion where v_i and p are viewed as $\mathcal{C}^1$ functions of $\mathbf{x} \in \mathcal{E}^3$ and time $t \in \mathcal{R}$.

As it often happens, here it is convenient to account also for the motion of a fluid particle, usually denoted by a function $\mathbf{x}(t)$. To avoid any confusion about the meaning of the symbol $\mathbf{x}$ (independent variable or function of time) one might adopt the Lagrangian description where the particles are labelled by the position $\mathbf{X}$ they occupy in a reference configuration $\mathcal{F} \in \mathcal{E}^3$. In that case the independent variables are $\mathbf{X} \in \mathcal{F}$ and $t \in \mathcal{R}$ and the unknown functions are the motion $\mathbf{x}(\mathbf{X}, t)$ and the pressure $p(\mathbf{X}, t)$. Accordingly, adopting the Lagrangian description in the standard way [11] makes (3.1)-(3.2) in a second order differential system.

For the present purposes the Lagrangian description as such has another drawback, besides that of increasing the order of the system, because it makes less immediate to contrast our results with those obtained by Olver [5] within the Eulerian description. That is why eventually the results determined via the material formulation will be given the corresponding Eulerian form.

For convenience in calculations we denote by upper case indices the Cartesian components of $\mathbf{X}$. Moreover, in connection with the function $\mathbf{x}(\mathbf{X}, t)$ and its inverse $\mathbf{X}(\mathbf{x}, t)$ we let

$$x_{iH} = \frac{\partial x_i}{\partial X_H}, \qquad x_{it} = \frac{\partial x_i}{\partial t}, \qquad X_{Hi} = \frac{\partial X_H}{\partial x_i}.$$

Similarly,

$$p_H = \frac{\partial p}{\partial X_H}, \qquad p_t = \frac{\partial p}{\partial t}.$$

Accordingly, the system (3.1)-(3.2) may be written in the form

$$x_{itt} + \frac{J}{\rho_0} p_H X_{Hi} = 0, \tag{3.3}$$

$$x_{itH} X_{Hi} = 0, \tag{3.4}$$

where the condition of mass conservation , $\rho = \rho_0/J$, has been used with $J = \det(x_{iH})$. Really $J = 1$ but, to simplify the comparison with the compressible case, we prefer to write just J.

For later convenience we write now some identities that will be freely used without further reference:

$$J X_{Aa} = \frac{\partial J}{\partial x_{aA}} = \frac{1}{2} \epsilon_{ahk} \epsilon_{AHK} x_{hH} x_{kK}, \qquad D_H(J X_{Hh}) = 0.$$

4. GENERAL FORM OF THE CONSERVATION LAWS.

Back to the general framework of section 2, we identify the unknown fields ϕ_β with x_i, $i = 1, 2, 3$, and p. The independent variables are X_H, $H = 1, 2, 3$, and t. Then we search for conservation laws (2.2) in the form

$$D_t I(X_H, t, x_i, p, x_{it}, x_{iH}) + D_K I_K(X_H, t, x_i, p, x_{it}, x_{iH}) \doteq 0 \tag{4.1}$$

where I is the conserved density and I_K is the associated flux. As a consequence, the helicity integral [1,4,5] is ruled out, since it involves the curl of the velocity field. In view of the previous discussion on equivalence, it is also to be observed that the density is only defined up to the divergence of a spatial vector; this remark will lead to a simplification in the subsequent calculations.

The explicit form of (4.1) reads

$$\begin{aligned}\frac{\partial I}{\partial t}+\frac{\partial I}{\partial x_i}x_{it}+\frac{\partial I}{\partial p}p_t+\frac{\partial I}{\partial x_{it}}x_{itt}+\frac{\partial I}{\partial x_{iH}}x_{iHt}\\ +\frac{\partial I_K}{\partial X_K}+\frac{\partial I_K}{\partial x_i}x_{iK}-\frac{\rho_0}{J}\frac{\partial I_K}{\partial p}x_{iH}x_{itt}+\frac{\partial I_K}{\partial x_{it}}x_{itK}+\frac{\partial I_K}{\partial x_{iH}}x_{iHK}\doteq 0\end{aligned} \tag{4.2}$$

where (3.3) has been taken into account. Here p_t, x_{itt}, and x_{iHK} are regarded as arbitrary quantities while x_{itH} satisfies the constraint equation (3.4). Hence (4.2) holds identically if and only if

$$\frac{\partial I}{\partial p}=0 \tag{4.3}$$

$$\frac{\partial I}{\partial x_{it}}-\frac{\rho_0}{J}\frac{\partial I_K}{\partial p}x_{iK}=0 \tag{4.4}$$

$$\frac{\partial I_K}{\partial x_{iH}}+\frac{\partial I_H}{\partial x_{iK}}=0 \tag{4.5}$$

$$\frac{\partial I}{\partial t}+\frac{\partial I}{\partial x_i}x_{it}+\frac{\partial I_K}{\partial X_K}+\frac{\partial I_K}{\partial x_i}x_{iK}=0 \tag{4.6}$$

$$\left(\frac{\partial I}{\partial x_{iK}}+\frac{\partial I_K}{\partial x_{it}}\right)x_{itK}\doteq 0. \tag{4.7}$$

Once we account for the constraint (3.4), the condition (4.7) yields [10]

$$\frac{\partial I}{\partial x_{iK}}+\frac{\partial I_K}{\partial x_{it}}-2J\lambda X_{Ki}=0 \tag{4.8}$$

where $2J\lambda$ is the Lagrange multiplier and λ is allowed to depend on X_H, t, x_i, p, x_{it}.

The solution to (4.5) may be represented as

$$I_K=\beta_l\epsilon_{lrs}\epsilon_{KPQ}x_{rP}x_{sQ}+\omega_{jHK}x_{jH}+\alpha_K \tag{4.9}$$

where β_l, ω_{jHK}, and α_K are functions of X_H, t, x_i, p, x_{it} ; $\omega_{jHK}=-\omega_{jKH}$. Then (4.4) and (4.8) yield respectively

$$\frac{\partial I}{\partial x_{it}}=\frac{\rho_0}{J}\left(\frac{\partial\beta_l}{\partial p}\epsilon_{lrs}\epsilon_{KPQ}x_{rP}x_{sQ}+\frac{\partial\omega_{jHK}}{\partial p}x_{jH}+\frac{\partial\alpha_K}{\partial p}\right)x_{iK} \tag{4.10}$$

$$\frac{\partial I}{\partial x_{iK}}=2J\left(\lambda\delta_{il}-\frac{\partial\beta_l}{\partial x_{it}}J\right)X_{Kl}-\frac{\partial\omega_{jHK}}{\partial x_{it}}x_{jH}-\frac{\partial\alpha_K}{\partial x_{it}}. \tag{4.11}$$

where δ_{il} denotes as usual the Kronecker symbol. Eqs.(4.3), (4.10), and (4.11) allow the determination of the explicit dependence of I on p, x_{it}, and x_{iK}. For example, since I is independent of p in view of (4.3), it follows in particular from (4.10) that β_l may be represented as

$$\beta_l=p\hat{B}_l+\hat{b}_l$$

with $\hat{B}_l$ and $\hat{b}_l$ as functions of X_H, t, x_i, x_{it}. Then, after substitution into (4.11), we can use the arbitrariness in λ to set $\lambda\delta_{il}=\partial\hat{B}_l/\partial x_{it}$, so as to eliminate dependence on p in the contribution proportional to X_{Kl}. Back to more general considerations, when we impose the standard integrability conditions on (4.3), (4.10), and (4.11) we find further restrictions on the arbitrary functions at hand so that we arrive at

$$\beta_l = p(Bx_{lt} + B_l) + bx_{lt} + b_l \tag{4.12}$$

$$\omega_{jKH} = C_{iG}\epsilon_{ijl}\epsilon_{GKH}x_{lt} + C_{jKH} \tag{4.13}$$

$$\alpha_H = pJa_H - \frac{1}{2}\rho J a_H x_{it} x_{it} + \sigma_H . \tag{4.14}$$

with B, B_l, b, b_l, C_{iG}, C_{jKH}, a_H, and σ_H depending on X_H, t, and x_i; $C_{jKH} = -C_{jHK}$. An obvious integration provides

$$I = 2\rho_0\left(\frac{1}{2}Bx_{it}x_{it} + B_i x_{it} + d\right) + \rho_0 a_H x_{iH} x_{it} - C_{iK} J X_{Ki}, \tag{4.15}$$

with $d = d(X_H, t, x_i)$, whereas the expression for I_H becomes

$$\begin{aligned} I_H =& [p(Bx_{lt} + B_l) + bx_{lt} + b_l]2JX_{Hl} \\ &+ (C_{iG}\epsilon_{ijl}\epsilon_{GKH}x_{lt} + C_{jKH})x_{jK} + pJa_H - \frac{1}{2}\rho_0 a_H x_{lt}x_{lt} + \sigma_H . \end{aligned} \tag{4.16}$$

Further information on I and I_H follows from the analysis of (4.6). Substitution for I and I_H from (4.15) and (4.16) into (4.6) yields

$$\begin{aligned} & pJ\left[\left(\frac{\partial B}{\partial X_H}x_{lt} + \frac{\partial B_l}{\partial X_H}\right)2X_{Hl} + \left(\frac{\partial B}{\partial x_i}x_{it} + \frac{\partial B_i}{\partial x_i}\right)2 + \frac{\partial a_H}{\partial X_H} + \frac{\partial a_H}{\partial x_i}x_{iH}\right] \\ &+ \frac{1}{2}\rho_0\left[2\frac{\partial B}{\partial x_i}x_{it} - \left(\frac{\partial a_H}{\partial X_H} + \frac{\partial a_H}{\partial x_i}x_{iH}\right)\right]x_{ht}x_{ht} \\ &+ \rho_0\left(\frac{\partial B}{\partial t}\delta_{ih} + 2\frac{\partial B_h}{\partial x_i} + \frac{\partial a_H}{\partial x_i}x_{hH}\right)x_{it}x_{ht} \\ &+ \left[\left(-\frac{1}{2}\frac{\partial C_{rS}}{\partial x_i}x_{it} + \frac{\partial b}{\partial X_S}x_{rt}\right)x_{bB} + \frac{\partial C_{rS}}{\partial x_i}x_{bt}x_{iB}\right]\epsilon_{rjb}\epsilon_{SRB}x_{jR} \\ &+ \left(\rho_0\frac{\partial a_H}{\partial t}\delta_{ij} + \frac{\partial C_{rS}}{\partial X_K}\epsilon_{rji}\epsilon_{SHK}\right)x_{it}x_{jH} + \left(\frac{\partial C_{jKH}}{\partial X_H} + \frac{\partial \sigma_K}{\partial x_j}\right)x_{jK} \\ &+ \left[\left(-\frac{1}{2}\frac{\partial C_{rS}}{\partial t} + \frac{\partial b_r}{\partial X_S}\right)\epsilon_{rjb}\epsilon_{SRB} + \frac{\partial C_{jRB}}{\partial x_b}\right]x_{jR}x_{bB} \\ &+ \left[2\rho_0\left(\frac{\partial B_i}{\partial t} + \frac{\partial d}{\partial x_i}\right) + 2J\frac{\partial b}{\partial x_i}\right]x_{it} + 2\rho_0\frac{\partial d}{\partial t} + \frac{\partial \sigma_H}{\partial X_H} + 2J\frac{\partial b_i}{\partial x_i} = 0. \end{aligned} \tag{4.17}$$

The vanishing of the overall contribution of the terms linear in p and terms of second and higher-order in the components x_{it} leads to

$$\begin{aligned} & B = \text{constant}, \\ & B_i = \omega_{ik}(t)x_k + m_i(t), \qquad \omega_{ik} = -\omega_{ki} \\ & a_H = a_H(X_K, t), \qquad \frac{\partial a_H}{\partial X_K} = 0. \end{aligned} \tag{4.18}$$

Then the vanishing of the terms in the products $x_{it}x_{jR}$ provides

$$\frac{\partial a_H}{\partial t} = 0, \qquad b = b(x_i, t), \qquad C_{iK} = C_{iK}(t).$$

Further, the vanishing of the coefficient of x_{it} yields

$$\frac{\partial}{\partial x_i}(2\rho_0 d + 2Jb) = -2\rho_0\frac{\partial B_i}{\partial t}. \tag{4.20}$$

Taking the $\partial/\partial x_j$ derivative of (4.20) and requiring the symmetry in i and j of the resulting expression we find, in view of (4.18),

$$\omega_{ik} = \text{const.}$$

Equation (4.20) can now be solved to give

$$2\rho_0 d = -2Jb - 2\rho_0 \dot{m}_i x_i + \alpha(\mathbf{X}, t). \tag{4.21}$$

We now observe that b and α may always be represented as

$$b = \frac{\partial \nu_i(\mathbf{x}, t)}{\partial x_i}, \qquad \alpha = \frac{\mu_H(\mathbf{X}, t)}{\partial X_H};$$

then (4.21) can be given the form

$$2\rho_0 d = D_H\left(-2\nu_i J X_{Hi} - 2\rho \dot{m}_i x_h x_h J X_{Hi} + \mu_H\right)$$

thus showing that the contribution of d to the conserved density (4.15) is represented in the form of a divergence and consequently may be disregarded, independently of the explicit form of ν, μ, and $\dot{m}_i$, that is of b and α. Similarly, we have

$$-C_{iK} J X_{Ki} = D_H\left(-\frac{1}{2} C_{iK} \epsilon_{ijh} \epsilon_{KAH} x_{jA} x_h\right),$$

where the condition $C_{iK} = C_{iK}(t)$ has been taken into account, whence it follows that the contribution of C_{iK} to the conserved density can be disregarded as well.

As a consequence, the explicit determination of the unknowns b, α, C_{iK}, C_{jKH}, σ_H, and b_l is not required, since we are allowed to consider an equivalent conservation law where the expression of the density does not involve d and C_{iK}; at most the preceeding unknowns can influence the expression of the flux, and no essential information is lost if we set them equal to zero.

In conclusion, we can write the set of conservation laws through the 4-tuple (I, I_H) as

$$I = 2\rho_0\left(\frac{1}{2} B x_{ht} x_{ht} + x_{ht}\omega_{hk} x_k + m_h(t) x_{ht}\right) + \rho_0 a_H(\mathbf{X}) x_{it} x_{iH},$$

$$I_H = p[B x_{lt} + \omega_{lk} x_k + m_l(t)] 2 J X_{Hl} + a_H(\mathbf{X})(p - \frac{1}{2}\rho x_{lt} x_{lt}) J,$$

a_H being a divergence-free vector.

5. EXPLICIT FORM OF THE CONSERVATION LAWS.

The direct approach of the previous section has provided an exhaustive set of independent conservation laws for incompressible fluid motions. To discuss the physical interpretation of the results we observe that the arbitrariness of m_h, ω_{hk}, and B leads to the conservation laws for the linear momentum (in a generalized form), the angular momentum, and the energy, respectively. Indeed, the law for linear momentum derived here, where each conserved component - say $\rho_0 x_{ht}$ - is multiplied by an arbitrary function of the time $m_h(t)$, constitutes an extension of the usual formulation which corresponds to m_h being constant. As a direct calculation can show, the greater generality is due to the incompressibility constraint; this interpretation is further enforced by the observation that a similar result also holds for incompressible viscous fluids [8].

As to a_H, we have

$$I = \rho_0 a_H v_i x_{iH}, \qquad I_H = a_H(p - \frac{1}{2}\rho v^2) J. \tag{5.1}$$

The law (5.1) mirrors the constraint of mass conservation for the coordinate transformation in the reference configuration $\mathcal{F}$, as follows from a comparison with the results obtained in [9].

Each of the above conservation laws has a counterpart holding for compressible fluids, with the only exception of the conservation of linear momentum which requires a constant m_h [9]. In this connection it is rather unexpected that no center-of-mass theorem has been found, although it has been shown to hold for compressible fluids [8,9] as a continuum counterpart of a relation derived by Hill [12] for a system of N particles. In the present case this result is a consequence of the conservation law of linear momentum. In fact, it suffices to set $m_h(t) = \bar{m}_h t/2$, with $\bar{m}_h$ constant, and to let the remaining arbitrary functions and constants vanish. Then the expression of the conserved density is $I = \rho_0 \bar{m}_h x_{ht} t$ and may be trivially modified by addition of a divergence term of the form $D_H(-\bar{m}_i x_h x_h \rho J X_{Hi}/2) = -\rho_0 \bar{m}_h x_h$. The resulting expression of the conserved density turns out to be given by

$$I = \bar{m}_h \rho_0 (x_{ht} t - x_h)$$

thus yielding the motion of the center-of-mass [9].

Now we are ready for the final step toward the formulation of conservation laws in the Eulerian description. To this aim we observe that the densities in the Eulerian and the Lagrangian descriptions are related through multiplication by the factor J, as follows from the change of variable theorem within multiple integrals [13]. We also make use of the correspondence between material and spatial descriptions of vector fields, $Y_H \leftrightarrow y_h$, namely [13]

$$Y_H = J y_h X_{Hh},$$

and we observe that $D_H Y_H = J D_h y_h$ [13], where D_h denotes the total derivative with respect to x_h. Then, on examining separately the various contributions to the general form of the conservation laws in the Lagrangian formulation, we are led to the following independent contributions in the Eulerian description:

$$D_t(\rho m_h v_h) + D_h(\rho m_j v_j v_h + m_h p - \dot{m}_j x_j v_h) \doteq 0, \tag{5.2}$$

$$D_t(\rho \omega_{hk} v_h x_k) + D_h(\rho \omega_{jk} v_j x_k v_h + \omega_{hk} p x_k) \doteq 0, \qquad \omega_{hk} = -\omega_{kh}, \tag{5.3}$$

$$D_t(\frac{1}{2}\rho v^2) + D_h(\frac{1}{2}\rho v^2 v_h + p v_h) \doteq 0, \tag{5.4}$$

where constant factors have not been considered and the contributions to the conserved flux without analogue in in the expression for I_H are due to the fact that we consider fixed regions in the physical space. It has already been pointed out that eq.(5.2) generalizes the usual conservation law of linear momentum, m_h being an arbitrary function of time. Equations (5.3) and (5.4) are the usual conservation laws of angular momentum and energy. In this connection the center-of-mass theorem takes the form

$$D_t[\rho(x_k - v_k t)] - D_h[\rho(x_k - v_k t)v_h - tp\delta_{kh}] \doteq 0, \tag{5.5}$$

and may be regarded as equivalent to the conservation law (5.2).

To find the analogue of the conservation law determined by (5.1) it is appropriate to introduce the Eulerian counterpart of Ja_H, say a_i, which is given by $a_i = a_H x_{iH}$. Accordingly, we find $I = \rho a_i v_i$ and $I_H = (p - \frac{1}{2}v^2) J a_l X_{Hl}$ and hence the Eulerian form of the conservation law reads

$$D_t(\rho a_i v_i) + D_h[\rho a_i v_i v_h + (p - \frac{1}{2}v^2)a_h] \doteq 0. \tag{5.6}$$

The conserved density is just the projection of the linear momentum on the field a_i. The vector a_i is divergence free, since a_H is, and satisfies the further condition

$$\frac{\partial a_i}{\partial t} + \frac{\partial a_i}{\partial x_h} v_h - a_h \frac{\partial v_i}{\partial x_h} = 0 \tag{5.7}$$

which is the Eulerian correspondent of the condition that a_H is independent of t. In practical terms , however, it is easily seen that within the *usual* Eulerian framework eq.(5.7) admits only the vanishing solution. This shows that the conservation law (5.6) can only be arrived at within the Lagrangian formulation and explains why Olver [5] could not find it.

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REFERENCES

1. SERRE, D. Les invariants du premier ordre de l'equation d'Euler en dimension 3, *C. R. Acad. Sc. Paris* **289** (1979) 267-270.
2. STRAUSS, W. A. Nonlinear invariant wave equations, in Invariant Wave Equations, G. Velo and A. S. Wightman Eds., Springer-Verlag, New York, 1978, 197-249.
3. EBIN, D. G. Integrability of perfect fluid motion, *Commun. Pure Appl. Math.* **36** (1983) 37-54.
4. MOFFATT, H. K. The degree of knottedness of tangled vortex lines, *J. Fluid Mech.* **35** (1969) 117-129.
5. OLVER, P. J. Olver A nonlinear Hamiltonian structure for the Euler equations, *J. Math. Anal. Appl.* **89** (1982) 233-250.
6. CAVIGLIA, G. Symmetry transformations, isovectors and conservation laws, *J. Math. Phys.* **27** (1986) 972-978.
7. BENATI, M. and CAVIGLIA G. Conservation laws for 3-dimensional compressible Euler equations, *Int. J. Engng. Sci.* **25** (1987) 1-7.
8. CAVIGLIA, G. Composite variational principles and the determination of conservation laws, *J. Math. Phys.*, in print.
9. CAVIGLIA, G. and MORRO, A. Noether-type conservation laws for perfect fluid motions, *J. Math. Phys.* **28** (1987) 1056-1060.
10. OLVER, P. J. Applications of Lie Goups to Differential Equations, Springer-Verlag, New York, 1986.
11. GURTIN, M. E. An Introduction to Continuum Mechanics, Academic Press, New York, 1981.
12. HILL, E. L. Hamilton's principle and the conservation theorems of mathematical physics, *Rev. Mod. Phys.* **23** (1951) 253-260.
13. MARSDEN, J. E. and HUGHES, J. R. Mathematical Foundations of Elasticity, Englewood Cliffs, N. J., 1983.