Internat. J. Math. & Math. Sci.
VOL. 12 NO. 3 (1989) 441-446

FURTHER RESULTS ON PRIMES IN SMALL INTERVALS

GEORGE GIORDANO

**Department of Mathematics
Physics and Computer Science
Ryerson Polytechnical Institute
Toronto, Ontario, Canada M5B 2K3**

(Received August 27, 1987)

ABSTRACT. In this paper we will deal with upper and lower bounds for $\pi(x + y) - \pi(x)$. In fact, given q with $0 < q \leq 1$, for sufficiently large integers m,n such that $m \geq n \geq qm > 2$ we show that $\pi(m + n) - \pi(m) < \ln(n)\pi(n)/\ln(m + 1)$. Moreover, explicit bounds are obtained and a wider range is given under the assumption of the Riemann hypothesis. Let m,n be positive integers with $m > 2657$. Let $1 \leq \theta < 2$ and $m \geq n \geq m^{1/\theta}$. If the Riemann hypothesis holds, then $\pi(m + n) - \pi(m) < n/\ln(m + 1) + \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi$. (Here $\pi(x) =$ the number of primes $\leq x$.)

KEY WORDS AND PHRASES. Primes. Small intervals. $\pi(x + y) \leq \pi(x) + \pi(y)$.
1980 AMS Subject Classification Code. 10H15, 10J15.

1. INTRODUCTION.

There are several accounts dealing with the validity of the conjecture that for $x > 1$ and $y > 1$,

$$\pi(x + y) \leq \pi(x) + \pi(y). \tag{1.1}$$

For example [1], [2], [3] deal with (1.1), whereas in [4] there is a discussion of the conjecture of the following form:

$$\pi(x + y) < \pi(x) + \pi(y) + cy/\ln^2(y). \tag{1.2}$$

(Here we let $x \geq y \geq 1$ and $c > 0$.) In fact, one of the two authors of [4] believes that (1.2) is true, whereas the other one does not.

What is interesting to this author is a paper written by Hensley and Richards [5]; they proved that if the prime k-tuple conjecture is true then (1.1) is false. Furthermore, assuming that the k-tuple conjecture is true they have shown that $\exists \mid c \mid > 0$ such that for sufficiently large y and infinitely many x we must have $\pi(x + y) - \pi(x) - \pi(y) > cy/\ln^2(y)$.

By using sophisticated techniques H.L. Montgomery and R.C. Vaughan [6] proved that if $M > 0$ and $N > 1$ are integers then $\pi(M + N) - \pi(M) \leq 2N/\ln(N)$. Now D.R. Heath-Brown and H. Iwaniec [7] show that if $\theta > 11/20$ and $x \geq x(\theta)$ then $\pi(x) - \pi(x - y) > y/(212 \ln(x))$ in the range $x^\theta \leq y \leq x/2$. The methods used in this paper are elementary and give a different range of validity. The proofs of this paper use the following definitions and results.

$\pi(x)$ = the number of primes $\leq x$

$$Li(x) = \int_2^x dt/\ln(t) \qquad \text{for } x \geq 2$$

$$Ls(m) = \sum_{k=2}^{m} 1/\ln(k) \qquad \text{for any integer } m \geq 2$$

$$\pi(x) = Li(x) + O(xe^{-a\sqrt{\ln(x)}}) \qquad \text{for } x \geq 2,\ a > 0 \tag{1.3}$$

$$Li(x) = x(1 + \sum_{k=1}^{n-1}(k!/\ln^k(x)))/\ln(x) + O(x/\ln^{n+1}(x)) \quad \text{for } x \geq 2 \tag{1.4}$$

$$\pi(m) = Ls(m) + O(me^{-c\sqrt{\ln(m)}}) \quad \text{for integer } m \geq 2,\ c > 0 \tag{1.5}$$

$$|\ Li(m) - Ls(m)\ | < C \qquad \text{for some constant } C \tag{1.6}$$

If the Riemann hypothesis holds, then (1.7) is true

$$|\ \pi(x) - Li(x)\ | < \sqrt{x}\ \ln(x)\ /\ 8\pi \qquad \text{for } x \geq 2657 \tag{1.7}$$

$$x(1 + 1/(2\ln(x)))\ /\ \ln(x) < \pi(x) \qquad \text{for } 59 \leq x \tag{1.8}$$

$$\pi(x) < x(1 + 3/(2\ln(x)))\ /\ \ln(x) \qquad \text{for } 1 < x \tag{1.9}$$

Now (1.3), (1.4) can be found in Ayoub [8], whereas (1.5), (1.6) are found in T. Estermann [9]. Furthermore, the paper written by L. Schoenfeld [10] gives us (1.7). Finally (1.8), (1.9) were proven by J.B. Rosser and L. Schoenfeld [11].

2. THEOREMS, COROLLARIES AND THEIR PROOFS.

THEOREM 1. If $0 < d \leq 1$ and x,y are sufficiently large with $x \geq y \geq dx > 2$, then $\pi(x + y) - \pi(x) - \ln(y)\pi(y)/\ln(x + y) < O(y/\ln^{n+1}(y))$ for any natural number $n \geq 2$.

PROOF. We have from (1.3) and (1.4) the following:

$$\pi(x) = x/\ln(x) + x/\ln^2(x) + \cdots + (n-1)!x/\ln^n(x) + O(x/\ln^{n+1}(x)). \tag{2.1}$$

Now it is obvious that

$$\pi(x+y) - \pi(x) = x/\ln(x+y) - x/\ln(x) + \sum_{k=1}^{n-1}\left[k!x/\ln^{k+1}(x+y) - k!x/\ln^{k+1}(x)\right]$$
$$+ y\left[1 + \sum_{k=1}^{n-1}(k!/\ln^k(x+y))\right] / \ln(x+y) + O\left[(x+y)/\ln^{n+1}(x+y)\right]. \tag{2.2}$$

Given that $x \geq 2$, $y > 0$ then for $0 \leq k \leq n-1$ we have

$$k!x\ /\ \ln^{k+1}(x+y) < k!x\ /\ \ln^{k+1}(x).$$

Hence (2.2) is replaced by

$$\pi(x+y) - \pi(x) < y\left[1 + \sum_{k=1}^{n-1}(k!/\ln^k(x+y))\right] / \ln(x+y) + O\left[(x+y)/\ln^{n+1}(x+y)\right] \tag{2.3}$$

For $k \geq 1$, we observe that $\ln^k(x+y) \geq \ln^k(2y) > \ln^k(y)$. Replacing $\ln^k(x+y)$, (2.3) now becomes

$$\pi(x+y) - \pi(x) < y\left[1 + \sum_{k=1}^{n-1}(k!/\ln^k(y))\right] / \ln(x+y) + O\left[(x+y)/\ln^{n+1}(y)\right]. \tag{2.4}$$

Multiplying the first term on the right hand side of (2.4) by ln(y)/ln(y) and using (2.1) we have replaced (2.4) by the following:

$$\pi(x+y) - \pi(x) - \ln(y)\pi(y)/\ln(x+y) < O\left[(x+y)/\ln^{n+1}(y)\right]. \tag{2.5}$$

It is obvious $\exists$ a constant $M > 0$ such that for $x + y$ sufficiently large the left hand side of (2.5) is strictly less than

$$M(x+y)/\ln^{n+1}(y) \tag{2.6}$$

Since $x \geq y \geq dx > 2$ for $0 < d \leq 1$ then

$$M(x+y)/\ln^{n+1}(y) < M(y/d + y)/\ln^{n+1}(y) < M'(y/\ln^{n+1}(y)). \tag{2.7}$$

Hence by using (2.7) we conclude that

$$\pi(x+y) - \pi(x) - \ln(y)\pi(y)/\ln(x+y) < O(y/\ln^{n+1}(y)).$$

THEOREM 2. Let $0 < q \leq 1$. If m, n are sufficiently large positive integers satisfying $m \geq n \geq qm > 2$, then $\pi(m+n) - \pi(m) < n/\ln(m+1) + Bne^{-a\sqrt{\ln(2n)}}$ for $B, a > 0$.

PROOF. By using (1.5) we see that

$$\pi(m+n) - \pi(m) = \sum_{k=m+1}^{m+n} (1/\ln(k)) + O\left[(m+n)e^{-a\sqrt{\ln(m+n)}}\right]. \tag{2.8}$$

It is obvious that we can replace (2.8) by

$$\pi(m+n) - \pi(m) - n/\ln(m+1) < O\left[(m+n)e^{-a\sqrt{\ln(m+n)}}\right]. \tag{2.9}$$

Now $\exists$ a constant $M > 0$ such that for $m + n$ sufficiently large that the left hand side of (2.9) is strictly less than

$$M(m+n)e^{-a\sqrt{\ln(m+n)}}.$$

Since $m \geq n \geq qm > 2$ and $0 < q \leq 1$ then

$$M(m + n)e^{-a\sqrt{\ln(m+n)}} < M(n/q + n)e^{-a\sqrt{\ln(2n)}} = Bne^{-a\sqrt{\ln(2n)}}.$$

Hence $\pi(m + n) - \pi(m) < n / \ln(m + 1) + Bne^{-a\sqrt{\ln(2n)}}$.

COROLLARY 1. Let $0 < q \leq 1$. If m,n are sufficiently large positive integers satisfying $m \geq n \geq qm > 2$, then $\pi(m + n) - \pi(m) < \ln(n)\pi(n)/\ln(m + 1)$.

PROOF. By using the result of Theorem 2 with a slight modification we have

$$\pi(m + n) - \pi(m) < n\ln(n)/(\ln(n)\ln(m + 1)) + Bne^{-a\sqrt{\ln(2n)}}. \tag{2.10}$$

We rearrange the terms in (2.1) so that one can give an upper bound to replace n/ln(n). With $M > 0$, we now incorporate an upper bound of n/ln(n) into (2.10) to establish that

$$\pi(m+n) - \pi(m) < \ln(n)\left[\pi(n) - \sum_{k=2}^{t-1} ((k-1)!n/\ln^k(n)) + Mn/\ln^t(n)\right]/\ln(m+1) + Bne^{-a\sqrt{\ln(2n)}}.$$

Hence for n sufficiently large we have

$$\pi(m + n) - \pi(m) < \ln(n)\pi(n)/\ln(m + 1).$$

THEOREM 3. Let $0 < q \leq 1$. If m,n are sufficiently large positive integers satisfying $m \geq n \geq qm > 2$, then $\pi(m + n) - \pi(m) > n/\ln(m + n) - Ane^{-a\sqrt{\ln(2n)}}$ for $a > 0$ and $A > 0$. constant M we have

$$\pi(m + n) - \pi(m) > \sum_{k=m+1}^{m+n} (1/\ln(k)) - M(m+n)e^{-a\sqrt{\ln(m+n)}} - Mme^{-a\sqrt{\ln(m)}}. \tag{2.11}$$

With a slight modification in (2.11) and using another constant $M' > 0$ we see that

$$\pi(m + n) - \pi(m) > n/\ln(m + n) - M'(m + n)e^{-a\sqrt{\ln(m+n)}}. \tag{2.12}$$

By rearranging the terms in (2.12) this will now become

$$M'(m + n)e^{-a\sqrt{\ln(m+n)}} > n/\ln(m+n) + \pi(m) - \pi(m + n). \tag{2.13}$$

Since $m \geq n \geq qm > 2$ and $0 < q \leq 1$ then

$$M'(m + n)e^{-a\sqrt{\ln(m+n)}} < M'(n/q + n)e^{-a\sqrt{\ln(2n)}} = Ane^{-a\sqrt{\ln(2n)}}.$$

Hence $\pi(m + n) - \pi(m) > n/\ln(m + n) - Ane^{-a\sqrt{\ln(2n)}}$.

COROLLARY 2. Let $0 < q \leq 1$, $\varepsilon > 0$. If m,n are sufficiently large positive integers satisfying $m \geq n \geq qm > 2$, then $\pi(m + n) - \pi(m) > \ln(n)(\pi(n) - (1 + \varepsilon)\, n/\ln^2(n))/\ln(m + n)$.

PROOF. By using the results of Theorem 3 with a slight modification we have

$$\pi(m + n) - \pi(m) > n\ln(n)/(\ln(n)\ln(m + n)) - Ane^{-a\sqrt{\ln(2n)}}. \tag{2.14}$$

Using an argument similar to that found in Corollary 1, we rearrange the terms in (2.1) so that one can give a lower bound to replace $n/\ln(n)$. With $D > 0$, we now incorporate a lower bound of $n/\ln(n)$ into (2.14) to establish the following

$$\pi(m+n)-\pi(m) > \ln(n)\left[\pi(n)-\sum_{k=2}^{t-1}((k-1)!n/\ln^k(n)) - Dn/\ln^t(n)\right] / \ln(m + n) - Ane^{-a\sqrt{\ln(2n)}}.$$

Hence for sufficiently large n

$$\pi(m + n) - \pi(m) > \ln(n)(\pi(n) - (1 + \varepsilon)n/\ln^2(n))/\ln(m + n).$$

THEOREM 4. Let $1 \leq \theta < 2$. Let m,n be positive integers with $m > 2657$ and $m \geq n \geq m^{1/\theta}$. If the Riemann hypothesis holds, then $\pi(m + n) - \pi(m) < n/\ln(m + 1) + \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi$.

PROOF. By using the upper and lower bounds of (1.7) we have

$$\pi(m + n) - \pi(m) < Li(m + n) - Li(m) + (\sqrt{m+n}\,\ln(m + n) + \sqrt{m}\,\ln(m))/8\pi. \tag{2.15}$$

Noting that $\sqrt{m+n}\,\ln(m + n) > \sqrt{m}\,\ln(m)$ and using (1.6), then (2.15) will now become

$$\pi(m + n) - \pi(m) < \sum_{k=m+1}^{m+n}(1 / \ln(k)) + \sqrt{m + n}\,\ln(m + n) / 4\pi. \tag{2.16}$$

It is obvious that we can replace (2.16) by

$$\pi(m + n) - \pi(m) < n / \ln(m + 1) + \sqrt{m + n}\,\ln(m + n)/4\pi.$$

Given that $m \geq n \geq m^{1/\theta}$ for $1 \leq \theta < 2$ we may now conclude

$$\pi(m + n) - \pi(m) < n/\ln(m + 1) + \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi.$$

COROLLARY 3. Let $1 \leq \theta < 2$. Let m,n be positive integers with $m > 2657$, $n > 59$, and $m \geq n \geq m^{1/\theta}$. If the Riemann hypothesis holds, then $\pi(m+n)-\pi(m) < \ln(n)\left[\pi(n)-n/(2\ln^2(n))\right]/\ln(m+1)+\sqrt{n^\theta+n}\ln(n^\theta+n)/4\pi$.

PROOF. By using the result of Theorem 4 with a slight modification we have

$$\pi(m + n) - \pi(m) < n\ln(n)/(\ln(m + 1)\ln(n)) + \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi. \tag{2.17}$$

By rearranging (1.8) and incorporating it into (2.17) we achieve the following:

$$\pi(m + n) - \pi(m) < \ln(n)\left[\pi(n) - n/(2\ln^2(n))\right] / \ln(m + 1) + \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi.$$

THEOREM 5. Let $1 \le \theta < 2$. Let m,n be positive integers with $m > 2657$ and $m \ge n \ge m^{1/\theta}$. If the Riemann hypothesis holds then $\pi(m + n) - \pi(m) > n/\ln(m + n) - \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi$.

PROOF. By using the upper and lower bounds of (1.7) we have

$$\pi(m + n) - \pi(m) > \mathrm{Li}(m + n) - \mathrm{Li}(m) - (\sqrt{m + n}\,\ln(m + n) + \sqrt{m}\,\ln(m))/8\pi. \tag{2.18}$$

Noting that $\sqrt{m + n}\,\ln(m + n) > \sqrt{m}\,\ln(m)$ and using (1.6), then (2.18) will now become

$$\pi(m + n) - \pi(m) > \sum_{k=m+1}^{n+m} (1/\ln(k)) - \sqrt{m + n}\,\ln(m + n)/4\pi. \tag{2.19}$$

It is obvious that we can replace (2.19) by

$$\pi(m + n) - \pi(m) > n/\ln(m + n) - \sqrt{m + n}\,\ln(m + n)/4\pi.$$

Given that $m \ge n \ge m^{1/\theta}$ for $1 \le \theta < 2$ we may conclude that

$$\pi(m + n) - \pi(m) > n/\ln(m + n) - \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi.$$

COROLLARY 4. Let $1 \le \theta < 2$. Let m,n be positive integers with $m > 2657$ and $m \ge n \ge m^{1/\theta}$. If the Riemann hypothesis holds, then

$$\pi(m + n) - \pi(m) > \ln(n)(\pi(n) - 3n/(2\ln^2(n)))/\ln(m + n) - \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi.$$

PROOF. By using the result of Theorem 5 with a slight modification we have

$$\pi(m + n) - \pi(m) > n\ln(n)/(\ln(m + n)\ln(n)) - \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi. \tag{2.20}$$

By rearranging (1.9) and incorporating into (2.20) we achieve the following

$$\pi(m + n) - \pi(m) > \ln(n)(\pi(n) - 3n/(2\ln^2(n)))/\ln(m + n) - \sqrt{n^\theta + n}\,\ln(n^\theta + n)/4\pi.$$

3. FINAL COMMENTS.

I feel that Theorem 1 and the Corollaries 1 and 3 are relevant to the disagreement between Erdős and Richards in their paper [4] dealing about whether the following conjecture is true.

$$\pi(x + y) - \pi(x) - \pi(y) < cy / \ln^2(y). \tag{3.1}$$

Of course, Theorem 1 states that (3.1) is true provided that for $0 < d \le 1$, x and y are sufficiently large and $x \ge y \ge dx > 2$. Under similar restrictions, Corollary 1 also states that (3.1) is true. Moreover, if we assume the conditions that are given in the Corollary 3 then we can give explicit bounds for which (3.1) is correct.

As for the mysterious person who told P. Erdős [12] that the "correct" conjecture should be $\pi(x + y) \le \pi(x) + 2\pi(y/2)$, I claim to have made some progress in this direction. From Rosser, Schoenfeld and Yohe [13] we have $\pi(2x) - \pi(x) < \pi(x)$. If $m \ge n$ then $\ln(n)\,\pi(n)/\ln(m + 1) < \pi(n) < 2\pi(n/2)$. Hence with the restrictions found in the Corollary 1 we have $\pi(m + n) \le \pi(m) + 2\pi(n/2)$.

4. ACKNOWLEDGEMENTS.

' am deeply indebted to Professor J. Repka for his suggestions which led to a better presentation of the manuscript.

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