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# EXCHANGE PF-RINGS AND ALMOST PP-RINGS

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ABSTRACT. Let R be a commutative ring with unity. In this paper, we prove that R is an almost PP-PM-ring if and only if R is an exchange PF-ring. Let X be a completely regular Hausdorff space, and let $\beta X$ be the Stone Čech compactification of X. Then we prove that the ring C(X) of all continuous real valued functions on X is an almost PP-ring if and only if X is an F-space that has an open basis of clopen sets. Finally, we deduce that the ring C(X) is an almost PP-ring if and only if C(X) is a U-ring, i.e. for each $f \in C(X)$, there exists a unit $u \in C(X)$ such that $f = u|f|$.

KEY WORDS AND PHRASES. PF-ring, PP-ring, PM-ring, almost PP-ring, pure ideal, exchange ring, idempotents, Stone-Cech compactification, Boolean space and the ring of all continuous real valued functions over a space X, C(X).
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## 1. INTRODUCTION.

All rings considered in this paper are commutative with unity. Recall that R is called a PF-ring if every principal ideal aR is a flat R-module, and it is called a PP-ring if every principal ideal aR is a projective R-module. An ideal I of a ring R is called pure if for each $x \in I$, there exists $y \in I$ such that $xy = x$. It is well-known that R is a PF-ring if and only if for $a \in R$, annihilator ideal, $\text{ann}_R(a)$, is pure, see Al-Ezeh [1]. Also it is well-known that R is a PP-ring if for each $a \in R$, $\text{ann}_R(a)$ is generated by an idempotent. In an earlier paper we introduced almost PP-rings as a generalization of PP-rings. A ring R is called an almost PP-ring if for each $a \in R$, $\text{ann}_R(a)$ is generated by idempotents of R. In fact, one can easily show that R is an almost PP-ring if and only if for each $a \in R$ and $b \in \text{ann}_R(a)$, there exists an idempotent e in $\text{ann}_R(a)$ such that $be = b$.

A ring R is called an exchange ring if every element in R can be written as the sum of a unit and an idempotent. Exchange rings have been studied extensively, see for example Monk [2] and Johnstone [3]. Our aim in this paper is to study the relationship between exchange PF-rings and almost PP-rings. To carry out our study we need two more definitions. A ring R is called a PM-ring if every proper prime ideal of R is contained in a unique maximal ideal of R. it is well-known that the ring of all continuous real valued functions over a completely regular Hausdorff space X, C(X), is a PM-ring, see Gillman and Jerison [4]. A compact Hausdorff and totally disconnected space is called a Boolean (or Stone) space.

## 2. MAIN RESULTS.

First, we state a theorem that was proved by Johnstone [3].

THEOREM 2.1 A ring R is an exchange ring if and only if it is a PM-ring and the space of maximal ideals of R, Max(R), is a Boolean space.

THEOREM 2.2 Let R be an exchange PF-ring. Then it is an almost PP. PM-ring.

PROOF. Let R be an exchange PF-ring. Let $a \in R$, and let $b \in \operatorname{ann}_R(a)$. Since R is a PF-ring, there exists $c \in \operatorname{ann}_R(a)$ such that bc = b. Because R is an exchange ring, c=e+u, where $e^2=e$ and u is a unit in R. Hence $cu^{-1} = eu^{-1} + 1$, and so $1 - e = cu^{-1}(1 - e)$. Since ac = 0, a(1 - e) = 0. But bc = b, so b(1 - e) = ub since c = e + u. Therefore $bu^{-1}(1 - e) = b$. Consequently, $b(1 - e) = bcu^{-1}(1 - e) = bu^{-1}(1 - e) = b$. Since $1 - e \in \operatorname{ann}_R(a)$, R is an almost PP-ring. By Theorem 1, R is a PM-ring. Hence R is an almost PP-PM-ring.

Now we want to establish the converse of theorem 2.2. Clearly, every almost PP-ring is a PF-ring. So, by theorem 2.1, it is enough to show that the space of maximal ideals of R, Max(R), is a Boolean space. De Marco and Orsatti [5] proved that if R is a PM-ring, then Max(R) is a compact Hausdorff space. So it is left to show that for an almost PP-PM-ring R, Max(R) is totally separated. That is for any two distinct maximal ideals M and $M_1$ there exists a clopen set in Max(R) containing M but not $M_1$.

THEOREM 2.3 Let R be an almost PP-PM-ring. Then R is an exchange PF-ring.

PROOF. By the above arguement, R is a PF-PM-ring. Moreover, Max(R) is a compact Hausdorff space. Let $M_1, M_2 \in \mathrm{Max}(R)$ and $M_1 \neq M_2$. Since R is a PM-ring, there exist $a \notin M_1$ and $b \notin M_2$ such that ab = 0, see Contessa [6]. Because R is an almost PP-ring, there exists an idempotent e $\operatorname{ann}_R(b)$ such that ea=a. Therefore $e \notin M_1$ and $e \in M_2$. Since e is an idempotent, $U=D(e)=\{M \in \mathrm{Max}(R): e \notin M\}$ is a clopen set in Max(R) containing $M_1$ but not $M_2$. So, by theorem 2.1, R is an exchange PF-ring.

For a completely regular Hausdorff space X, the ring of all continuous real valued functions, C(X), is a PM-ring, see Gillman and Jerison [4]. Moreover, Max(C(X)), is homeomorphic to $\beta X$, the Stone-Čech compatification of X. Therefore C(X) is an almost PP-ring if and only if R is an exchange PF-ring. Consequently, C(X) is an almost PP-ring if and only if it is a PF-ring and $\beta X$ is a Boolean space. Al-Ezeh et al [7], proved that C(X) is a PF-ring if and only if X is an F-space, where X is called an F-space if every finitely generated ideal is principal. It is well-known that X is an F-space if and only if any two nonempty disjoint cozero sets are completely separated. Therefore, the ring C(X) is an almost PP-ring if and only if X is an F-space and $\beta X$ is a Boolean space. In fact, $\beta X$ is a Boolean space if and only if X has an open basis of clopen sets. Thus the ring C(X) is an almost PP-ring if and only if X is an F-space that has an open basis of clopen sets.

Finally, Gillman and Henriksen [8] defined the ring C(X) to be a U-ring if for every $f \in C(X)$, there exists a unit $u \in C(X)$ such that $f = u|f|$. In the same paper they proved that the ring C(X) is a U-ring if and only if X is an F-space and $\beta X$ is a Boolean space. So we get the following theorem.

THEOREM 2.4 The ring C(X) is an almost PP-ring if and only if it is a U-ring.

We end this paper by giving some examples illustrating the relationships discussed above.

EXAMPLES.

1) Let N be the set of positive integers with the discrete topology. Let $\beta N$ be its Stone-Čech compactification. The space $\beta N \setminus N$ is a compact F-space, see Gillman and Jerisen [4]. Moreover, $\beta N \setminus N$ is totally disconnected. Hence, the space $\beta N \setminus N$ is Boolean. So the ring $C(\beta N \setminus N)$ is an almost PP-ring. However, it is not a PP-ring because the space $\beta N \setminus N$ is not basically disconnected, see Brookshear [9].

2) Let $R^+$ be set of nonnegative reals endowed with the usual topology. The space $\beta R^+ \setminus R^+$ is a compact, connected F-space, see Gillman and Henriksen [8]. Thus, the ring C(X) has no nontrivial idempotents. So, if it were an almost PP-ring, it would be an integral domain which is not the case because it has plenty of zero divisors. Consequently, $C(\beta R^+ \setminus R^+)$ is a PF-rings that is not an almost PP-ring.

3) The ring of integers is an almost PP-ring that is not a PM-ring, and so not an exchange ring.

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