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SUBLINEAR FUNCTIONALS ERGODICITY AND FINITE INVARIANT MEASURES

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ABSTRACT. By introducing a sublinear functional involving infinite matrices, we establish its connection with ergodicity and measure preserving transformation. Further, we characterize the existence of a finite invariant measure by means of a condition involving the above sublinear functional.

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1. INTRODUCTION AND DEFINITIONS.

Let ℓ_∞ be the set of all real bounded sequence $\{x_n\}$, normed by $||x|| = \sup_{n\geq 0}|x_n|$

Linear functional ϕ on ℓ_∞ are called Banach limit [1] satisfying the conditions,

i) $\phi(x_n) \geq 0$, if $x_n \geq 0$, $n = 0,1,2 \dots$.

ii) $\phi(x_{n+1}) = \phi(x_n)$

iii) $\underline{\lim}_{n\to\infty} x_n \leq \phi(x_n) \leq \overline{\lim}_{n\to\infty} x_n$.

If there is a number for all Banach limits ϕ, the sequence $x = \{x_n\}$ is called almost convergent and we write; $F - \lim_{n\to\infty} x_n = s$. It is shown by Lorentz [2] that a sequence $\{x_n\}$ is almost convergent with F-limit s, if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=i}^{i+n-1} x_k = s . \tag{1.1}$$

uniformly in i.

Let, $\mathcal{A} = (a_{n,k}^{(i)})$ be a sequence of real or complex matrices for each $i = 0,1,2\ldots$ such that $a_{n,k}^{(i)} = 0$, if any n,k,i, is a negative integer. The sequence $\{x_n\}$ is called $\mathcal{A}$ summable to s if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} x_k = s \tag{1.2}$$

uniformly in i and in this case we write:

$$\mathcal{A} - \lim_{n\to\infty} x_n = s, \quad \text{or} \quad x_n \to s(\mathcal{A}).$$

In the case $a_{n,k}^{(i)} = 1/n+1$ $(i \le k \le i+n)$ and 0 otherwise, $(\mathcal{A})$ reduces to the method (F). If $\mathcal{A} = A = a_{n,k}$, then we obtain the usual summability method (A). It is significant to note that there does not exist any regular method (A) equivalent to method (F) (See Lorentz [2]. Theorem 11 and 12). In the case $a_{n,k}^{(i)} = \frac{1}{n+1} \sum_{r=i}^{n+i} a_{r,k}$, then $(\mathcal{A})$ reduces to the almost summability method introduced by King [3].

The method $(\mathcal{A})$ is called conservative, if $x \to s \Rightarrow x \to s^1$ $(\mathcal{A})$, regular, if $s = s^1$. The following characterization of regular matrices is due to Stieglitz [4]. The method $(\mathcal{A})$ is called regular if and only if the following conditions hold:

$$\sum_{k=0}^{\infty} |a_{n,k}^{(i)}| < \infty, \text{ for all } n \text{ and } i \ge 0, \tag{1.3}$$

and there exists an integer m such that

$$\sup_{i\ge0,\ n\ge m} \sum_{k=0}^{\infty} |a_{n,k}^{(i)}| < \infty \tag{1.4}$$

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} = 1, \text{ uniformly in } i, \tag{1.5}$$

$$\lim_{n\to\infty} a_{n,k}^{(i)} = 0, \text{ for fixed } k, \text{ uniformly in } i. \tag{1.6}$$

We write

$$||\mathcal{A}|| = \sup_{i\ge0,\ n\ge0} \sum_{k=0}^{\infty} |a_{n,k}^{(i)}|$$

The matrix $\mathcal{A}$ is called translative, if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} |d_{n,k}^{(i)}| = 0 \tag{1.7}$$

uniformly in i, where

$$d_{n,k}^{(i)} = (a_{n,k-1}^{(i)} - a_{n,k}^{(i)}) \tag{1.8}$$

The matrix $\mathcal{A}$ is called positive, if

$$a_{n,k}^{(i)} \ge 0, \quad \forall\ n,k,i \tag{1.9}$$

For real λ we write,

$$\lambda^+ = \max(\lambda,0), \quad \lambda^- = \max(-\lambda,0).$$

The matrix $\mathcal{A}$ is called almost positive, if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}^{-(i)} = 0, \text{ uniformly in } i. \tag{1.10}$$

Let (x, F, m) be a finite measure space and let, $T; X \to X$ be a measurable transformation. (This is assumed throughout). The measure m is called <u>null invariant</u>, if $m(A) = 0 \Longleftrightarrow m(T^{-1}A) = 0$, $A \in F$. It is <u>conservative</u>, if $A \cap T^{-n}A = \phi \Rightarrow m(A) = 0$, for all n, and $A \in F$. A measure μ is called <u>equivalent to measure m</u>, if $m(A) \Longleftrightarrow \mu(A) = 0$, for $A \in F$. The transformation T is called <u>measure preserving or invariant</u>, if $m(A) = m(T^{-1}A)$, $A \in F$. It is called <u>ergodic</u> if, $T^{-1}A = A \Rightarrow m(A)=0$ or $m(X/A) = 0$. The set $A \in F$ is called invariant, if $A = T^{-1}A$. It is called <u>wandering</u>, if $A, T^{-1}A, T^{-2}A \ldots A \in F$, are mutually disjoint. It is called <u>weakly wandering</u>, if there is an increasing sequence of positive integers $\{r_k, k = 1,2,3 \ldots\}$ such that $A, T^{-r_1}A, T^{-r_2}A \ldots$ are mutually disjoint. A measure q is called m-continuous if, for $\varepsilon > 0$ there exists a $\delta > 0$ such that $m(A) < \delta \Rightarrow q(A) < \varepsilon$. A sequence of measures $\{q_n\}$ is called <u>uniformly m continuous</u> if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $m(A) < \delta \Rightarrow q_n(A) < \varepsilon$ for all n.

Write:

$$t(x) = \overline{\lim_{n\to\infty}} \sup_i \frac{1}{n} \sum_{k=i}^{i+n-1} x_k, \quad x_k \in \ell_\infty \tag{1.11}$$

Let $\{l_\infty, t\}$ denote the set of linear functionals ϕ, such that $\phi(x) \leq t(x)$. It is known (see Sucheston [5] Das and Misra [6]) that $\{l_\infty, t\}$ is the set of all Banach limits on l_∞ and $\phi \in \{l_\infty, t\}$ is unique if and only if $\phi(x) = -\phi(-x)$ and this happens when

$$\frac{1}{n} \sum_{k=i}^{i+n-1} x_k \to \text{a limit}$$

as $n \to \infty$, uniformly in i. Lorentz [2] calls all such sequences as almost convergent sequences. Let A be real and such that $||A|| < \infty$. Then we define, $R: l_\infty \to l_\infty$ by

$$R(x) = \overline{\lim_{n\to\infty}} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} x_k . \tag{1.12}$$

Since, for all $x \in l_\infty$

$$|R(x)| \leq ||x|| \; ||A|| < \infty ,$$

R is finite valued. It is easy to see that it is a sublinear functional on l_∞. By Hahn-Banach theorem there exists a linear functional ϕ on l_∞ such that

$$-R(-x) \leq \phi(x) \leq R(x), \quad x \in l_\infty \tag{1.13}$$

Let $\{l_\infty, R\}$ be the set of all linear functional ϕ satisfying (1.13). It is easily seen that ϕ is unique if and only if

$$-R(-x) = R(x) \tag{1.14}$$

and this happens if and only if

$$\sum_{k=0}^{\infty} a_{n,k}^{(i)} x_k \to \text{a limit}$$

as $n \to \infty$, uniformly in i.

We now state a lemma.

LEMMA 1. Let $x \in l_\infty$, then

(a) $\underline{\lim}\, x_n \leq R(x) \leq \overline{\lim}\, x_n$

if and only if A is real, regular and almost positive.

(b) $-t(-x) \le -R(-x) \le R(x) \le t(x)$

if and only if A is regular, almost positive and translative.

(c) If x is almost convergent to s, then

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} x_k = s, \text{ uniformly in } i .$$

2. ERGODICITY.

In this section, we establish that the ergodicity and invariance can be established in terms of summability of a particular sequence and thus generalizes a result of (Sucheston [5], Theorem 3) involving almost convergence.

We now examine the following conditions:

(I) For some $\phi \in \{1_\infty, R\}$ $\phi[m(T^{-n}B \cap c)] = m(B) \cap m(c)$, $n = 0,1,2 \dots$.

(II) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} m(T^{-n}B \cap C) = m(B)\, m(C)$

uniformly in i, $\forall$ $B, C \in F$.

(III) T is ergodic and measure preserving.

THEOREM 1. Let (X, F, m) be a finite measure space and let $||A|| < \infty$. Then

(a) (II)=> (I)

(b) (i) (I) => T is ergodic

(ii) If A is translative. Then

(I) => (III)

(c) If A is regular, almost positive, and translative, then

(I) <=> (II) <=> (III)

We need the following lemma for the proof of the theorem.

LEMMA 2. Let $||A|| < \infty$, $\phi \in \{1_\infty, R\}$, s: $1_\infty \to 1_\infty$ be the shift operator i.e.

$$s(x_n) = x_{n+1} , \quad s^2(x_n) = x_{n+2}.$$

Then

(a) $|\phi(SX) - \phi(x)| \le ||x|| \overline{\lim_{n\to\infty}} . \sup_i \sum_{k=0}^{\infty} |d_{n,k}^{(i)}|$

where $d_{n,k}^{(i)}$ is defined by (1.8).

Let A be translative, then for $x \in 1_\infty$.

(b) (i) $R(SX - x) = R(x-SX) = 0$

(ii) $\phi(Sx) = \phi(x)$

(c) $R(Sx) = R(x)$

Let, further

$$\lim_{n\to\infty} a_{n,k}^{(i)} = 0, \text{ fixed } k, \text{ uniformly in } i. \tag{2.1}$$

Then

(d) $R(\sum_{j=0}^{p} s^{r_j} x) = p.R(X)$

Where $r_0 = 1, r_1, r_2 \dots r_p$ is a sequence of fixed positive integers.

PROOF: Since

$$R(Sx-x) = \overline{\lim_{n\to\infty}} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} (Sx_k - x_k)$$

$$= \overline{\lim_{n\to\infty}} \sup_i \sum_{k=0}^{\infty} d_{n,k}^{(i)} x_k .$$

It follows that

$$|R(Sx-x)| \leq ||x|| \overline{\lim_{n\to\infty}} \sup_i \sum_{k=0}^{\infty} |d_{n,k}^{(i)}| . \tag{2.2}$$

Now as ϕ is linear, we obtain

$$\phi(sx) - \phi(x) = \phi(sx-x) \leq R(sx-x) . \tag{2.3}$$

Changing the role of sx and x in (2.2) and (2.3) we obtain (a). When A is translative (b) (i), (ii) follows from (2.2), and changing the role of sx and x in (2.2) (b) (ii) follows from (a). Since, R is sublinear,

$$R(Sx) = R(Sx-x+x) \leq R(Sx-x) + R(x) = R(x)$$

by b (i). Changing the role of Sx and x, we obtain $R(x) \leq R(Sx)$. So (c) follows.

Lastly $R(S^{r_1}x + S^{r_2}x) = R(S^{r_1}x - x + S^{r_2}x - x + 2x) \leq R(S^{r_1}x-x) + R(S^{r_2}x-x) + 2R(x)$.
i.e.

$$R(S^{r_1}x + S^{r_2}x) - 2R(x) \leq R(S^{r_1}x - x) + R(S^{r_2}x - x) \tag{2.3}$$

But,

$$\begin{aligned} R(S^{r_1}x - x) &= \overline{\lim_{n\to\infty}} \sup_i \sum_{k=0}^{\infty} a_{n,k}^{(i)} (x_{k+r_1} - x_k) \\ &= \overline{\lim_{n\to\infty}} \sup_i \sum_{k=0}^{\infty} (a_{n,k-r_1}^{(i)} - a_{n,k}^{(i)}) x_k \\ &= \overline{\lim_{n\to\infty}} \sup_i [\sum_{k=r_1}^{\infty} (a_{n,k-r_1}^{(i)} - a_{n,k}^{(i)}) x_k - \sum_{k=0}^{n-1} a_{n,k}^{(i)} x_k] \\ &= \overline{\lim_{n\to\infty}} \sup_i [\sum_{k=r_1}^{\infty} (a_{n,k-r_1}^{(i)} - a_{n,k}^{(i)} x_k] \qquad \text{by (2.1)} \\ &= \overline{\lim_{n\to\infty}} \sup_i \sum_{k=r_1}^{\infty} x_k \sum_{j=0}^{r_1-1} (a_{n,k-j-1}^{(i)} - a_{n,k-j}^{(i)}) \\ &\leq ||x|| \sum_{j=0}^{r_1-1} \overline{\lim_{n\to\infty}} \sup_i \sum_{k=r_1}^{\infty} |d_{n,k-j}^{(i)}| \\ &= 0 \quad (\because A \text{ is translative}) \end{aligned}$$

Similarly

$$R(S^{r_2}x - x) \leq 0 .$$

Hence,

$$R(s^{r_1}x + S^{r_2}x) \leq 2R(x), \quad x \in l_\infty .$$

Again, since

$$\begin{aligned} 2R(x) &= R(2x - S^{r_1}x - S^{r_2}x + S^{r_1}x + S^{r_2}x) \\ &\leq R(x - S^{r_1}x) + R(S - S^{r_2}x) + R(S^{r_1}x + S^{r_2}x) . \end{aligned}$$

Proceeding as above, we have

$$2R(x) \leq R(S^{r_1}x + S^{r_2}x) , \quad x \in l_\infty .$$

Hence,

$$R(S^{r_1}x + S^{r_2}x) = 2 R(x) , \quad x \in l_\infty . \tag{2.4}$$

(d) follows by repeated application of (2.4).

PROOF OF THEOREM 1.

(a) Let (II) hold. Then

$$-R\ [-m(T^{-n}B \cap C)] = R\ [m(T^{-n}B \cap C)]$$

Since

$$-R(x) \leq Q(x) \leq R(x)\ , \quad x \in l_\infty \tag{2.5}$$

It follows that

$$\phi\ [\ m(T^{-n}B \cap C)] = m(B) \cdot m(C)\ ,\ n= 0,1,2....$$

This proves (II) => (I).

(b) Take, $T^{-1}B = B$, $C = x/B = B^{-1}$in (I).

Hence it follows that

$$0 = \phi(0) = m(B)\ .\ m(B^1)$$

either $m(B) = 0$ or $m(B^1) = 0$.

i.e. T is ergodic.

Writing, $C = X$ in (I), we obtain

$$\phi\ [m(T^{-n}B)] = m(B)\ .\ m(X) \tag{2.6}$$

Replacing B by $T^{-1}B$ in (2.6), we obtain

$$\phi\ [m(T^{-n-1}B)] = m(T^{-1}B)\ .\ m(X) \tag{2.7}$$

If further, A is translative, by Lemma 2 (b)

$$\phi\ [m(T^{-n-1}B)] = \phi\ [m(T^{-1}B)]\quad .$$

Again, since $0 < m(X) < \infty$, it follows from (2.6) and (2.7) that

$$m(T^{-1}B) = m(B)$$

Hence, (I) => (III) .

(c) In veiw of (a) and (b), it is enough to show that (III) => (II). Take any fixed $B \in F$ such that $m(B) > 0$. Define, for $\phi \in \{l_\infty,R\}$ and $C \in F$

$$q_n(c) = \frac{m(T^{-n}B \cap C)}{m(B)}\quad ,\ n = 0,1,2\ ...$$

$$q(c) = \phi(q_n(c)) \tag{2.8}$$

We now show that q is an invariant measure and m = q.

Since, A is almost positive

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a^{-(i)}_{n,k}\ x_k = 0,\ \text{uniformly in } i\ , \tag{2.9}$$

for $x_k \in l_\infty$.

Write

$$R^+(x) = \overline{\lim_{n\to\infty}}\ \sup_i \sum_{k=0}^{\infty} a^{+\ (i)}_{n,k}\ x_k\ .$$

So, by (2.9)

$$R(x) = R^+(x)$$

Since, $x \geq 0 \Rightarrow R^+(x) \geq 0$

$$x \geq 0 \quad \Rightarrow \quad R(x) \geq 0\quad . \tag{2.10}$$

Again, since m is a measure, $q_n(c) \geq 0$. So it follows from (2.10) that $R(q_n(c)) \geq 0$.

Since R is sublinear, we have

$$-R\,[-q_n(c)] \geq 0.$$

Now, it follows from (2.5) that $q(c) \geq 0$, $C \in F$. Let $B_i \in F$ be a countable sequence of disjoint sets. Then

$$q(\overset{\infty}{\underset{i=1}{\cup}} B_i) = \phi\,[q_n(\overset{\infty}{\underset{i=1}{\cup}} B_i)]$$

$$= \phi\,[\sum_{i=1}^{\infty} q_n(B_i)] \quad (\because m \text{ is a measure})$$

$$= \sum_{i=1}^{\infty} \phi\,[q_n(B_i)] \quad (\phi \text{ is continuous linear functional})$$

So, q is countably additive and hence it is a measure.

Next,

$$q(T^{-1}C) = \phi\,[\frac{m(T^{-n}B \cap T^{-1}C)}{m(B)}]$$

$$= \phi\,[\frac{m(T^{-n+1}B \cap C)}{m(B)}] \quad (\because T \text{ is a measure preserving})$$

$$= \phi\,[q_{n-1}(C)]$$

Since ϕ is shift invariant by Lemma 2 (b),

$$= \phi\,[q_n(C)]$$

$$= q(C)\ , \quad C \in F.$$

This proves that q is an invariant measure. Sice T is ergodic, the invariant sets are of measure 0 or 1. Since m and q are invariant measures, an invariant measure is determined by the value it takes on invariant sets (See Sucheston [8], Theroem 2), it follows that $q = m$.

Now, we have

$$q(C) = m(C) = \phi\,[\frac{m(T^{-n}B \cap C)}{m(B)}] \quad \text{i.e.} \quad \phi\,[m(T^{-n}B \cap C)] = m(B)\,.\,m(C)\ .$$

Hence, ϕ is unique on $\{T^{-n}B \cap C\}$, $n = 0,1,2 \ldots$. But, $\phi \in \{l_\infty, R\}$ has unique value λ if and only if

$$R(x) = -R(-x) = \lambda\ .$$

Hence, it follows that

$$R[m(T^{-n}B \cap C)] = -R[-m(T^{-n}B \cap C)] = m(B)\,.\,m(C)\ .$$

i.e. (II) holds and hence proves (c) completely.

3. EQUIVALENT MEASURES.

Many necessary and sufficient conditions have been determined for the existence of equivalent invariant measures (see Sucheston [7], [8], Mrs. Dowker [9], Calderon [10], and Hajian and Kakutani [11]). In the pointwise ergodic theorem of Birkhoff [12], it was necessary to take invariant measure, but Halmos [13] has shown that even if a measure is null invariant and conservative, an equivalent measure need not exist. Sucheston [7], [8] has used Banach limit technique to prove the existence of invariant measures . We now generalize some of the theorems of Sucheston [5] involving almost convergence and some results of Mrs. Dowker on (C,1) convergence and establish the existence of invariant measure by using linear functional $\phi \in \{l_\infty, R\}$.

We now prove

THEOREM 2. Let A be a real matrix such that $||A|| < \infty$ and let A be almost positive and translative. Let (x, F, m) be a finite measure space and T be a measurable transformation. Then, the following condition are equivalent.

(I) There exists an equivalent finite invariant measure.

(II) For some $\phi \in \{l_\infty, R\}$ and all $B \in F$

$$m(B) > 0 \Rightarrow \phi\,[m(T^{-n}B)] > 0$$

(III) $m(B) > 0 \Rightarrow R\,[m(T^{-n}B)] > 0$

PROOF. (I) => (II). Suppose that p is an invariant measure which is equivalent to m. Suppose that (II) fails to hold. Then there exists a $B \in F$ such that $m(B) > 0$ and

$$\phi[m(T^{-n}B)] = 0 .$$

But, since $-R(-x) \le \phi(x) \le R(x)$, $x \in l_\infty$ and by Lemma 1

$$\underline{\lim}_{n\to\infty} x_n \le -R(-x) \le R(x) \le \overline{\lim}_{n\to\infty} x_n .$$

it follows that for all $B \in F$

$$0 = \phi\,[m(T^{-n}B)] \ge \underline{\lim}_{n\to\infty} m(T^{-n}B) .$$

But, since $\underline{\lim}_{n\to\infty} m(T^{-n}B) \ge 0$, it follows that

$$\underline{\lim}_{n\to\infty} m(T^{-n}B) = 0.$$

Hence, there exists a sub sequence $\{x_k\}$ such that

$$\underline{\lim}_{k\to\infty} m(T^{-n_k}B) = 0 .$$

Since p is equivalent to m, we obtain

$$p(B) > 0 \text{ and } \lim_{k\to\infty} p(T^{-n_k}B) = 0.$$

Since p is invariant, we have

$$p(T^{-n_k}B) = p(B) .$$

Hence $p(B) = 0$. This is a contradiction and this proves the fact that (I) => (II).

<u>(II)=> (III)</u> . Let II hold . Since, $\phi\,[m(T^{-n}B)] \le R\,[m(T^{-n}B)]$ it follows that

$$\phi\,[m(T^{-n}B)] > 0 \Rightarrow R\,[m(T^{-n}B)] > 0 .$$

<u>(III) => (I)</u> . Suppose (III) holds and (I) fails. Since Condition (I) is equivalent to non-existence of weakly wandering set (See Sucheston [7], Theorem 6) it follows that there exists positive integers $r_0 = 1, r_1, r_2 \ldots$ and a set $B \in F$ with $m(B) > 0$ such that

$$B,\ T^{-r_1}B,\ T^{-r_2}B,\ \ldots,\ T^{-r_k}B \ldots$$

are mutually disjoint. Since,

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{n,k}^{(i)} = 1, \text{ uniformly in } i, \text{ it follows that}$$

$$R[m(T^{-k}X)] = R[m(X)] = m(X)\,R(1) = m(X).$$

Again

$$m(X) = R\,[m(T^{-n}X)] \ge R\,[m(\bigcup_{j=0}^{s} T^{-r_j}B)]$$

$$m(X) = \overline{\lim_{n\to\infty}} \sup_{i} \sum_{k=0}^{\infty} a_{n,k}^{(i)} m\left[\bigcup_{j=0}^{s} T^{-rj-k}B\right]$$

$$= \overline{\lim_{n\to\infty}} \sup_{i} \sum_{k=0}^{\infty} a_{n,k}^{(i)} \sum_{j=0}^{s} S^{rj} X_k .$$

Where $X_k = m(T^{-k} B)$, S is a shift operator. Then by Lemma 2(d),

$$m(X) \geq S . R(X) \tag{3.1}$$

Then it follows from (3.1) that

$$m(X) \geq s . R \quad m(T^{-k}B) . \tag{3.2}$$

Since $m(T^{-k} S) > 0$ by hypothesis and s is an arbitrary positive integer. This contradicts (3.2). This proves (III) => (I).

In the next theorem we give yet another characterization of existence of invariant measure in terms of the sublinear functional $R(x)$.

THEOREM 3. Let A satisfy the condition of Theorem 2. Let (x, F, m) be the finite measure space. Then there exists an invariant measrure equivalent to measure m on X, if and only if,

(i) m is null-preserving.

(ii) T is conservative.

(iii) $\sum_{n=0}^{\infty} a_{n,k}^{(i)} m(T^{-k}B)$ converges uniformly in i, for every $B \in F$.

Again, whenever it has equivalent invariant measure, then the map $q : F \to R$ defined by $q(B) = R\,[m(T^{-n}B)]$ is an invariant measure equivalent to m and agrees with m on invariant sets.

PROOF: NECESSITY .

Let us assume that m admits an invariant equivalent measure μ . Then μ is m continuous (See Halmos [9] p. 125).

Write for $\phi \in \{l_\infty, R\}$

$$q(B) = \phi\,[m(T^{-n} B)]$$

We want to show

(a) q is a measure

(b) q is a m continuous

(c) q is invariant.

As in the proof of Theorem 1, we can show that

$$q(B) \geq 0, \text{ for all } B \in F .$$

It is easy to show that

$$B,C \in F , \quad B \subset C \Rightarrow q(B) \leq q(C) .$$

Since ϕ is linear, it also follows that q is finitely additive. Since μ is m-continuous, for given $\varepsilon > 0$, $\delta > 0$. Such that

$$m(T^{-n}B) < \varepsilon \text{ when } \mu(B) = \mu(T^{-n}B) < \delta , \text{ and } m(T^{-n}B) < \varepsilon \Rightarrow q(B) < \varepsilon.$$

So q is m-continuous. The countably additivity of m and m-continuity of q (See Halmos [9] p. 39).

Next,

$$q(T^{-1}B) - q(B) = \phi\,[m(T^{-n-1}B)] - \phi\,[m(T^{-n}B)]$$

$$= \phi\,[m(T^{-n-1}B) - m(T^{-n}B)] \quad (\phi \text{ is linear}).$$

$$\leq R\,[m(T^{-n-1}B) - m(T^{-n}B)]$$

$$= \overline{\lim_{n}}\ \sup_{i}\ \sum_{k=0}^{\infty} a_{n,k}^{(i)}\ [m(T^{-k-1}B) - m(T^{-k}B)]\ .$$

Since $a_{n\ -1}^{(i)} = 0,\ \forall\ n$ and i

$$= \overline{\lim_{n\to\infty}}\ \sup_{i}\ \sum_{k=0}^{\infty} [a_{n,k-1}^{(i)} - a_{n,k}^{(i)}]\ m(T^{-k}B)$$

$$\leq\ m(X)\ \overline{\lim_{n\to\infty}}\ \sup_{i}\ \sum_{k=0}^{\infty} |d_{n,k}^{(i)}|\ .$$

Since A is translative,

$$\to\ 0 \text{ as } n \to \infty\ , \text{ uniformly in } i.$$

Hence,

$$q(T^{-1}B) \leq q(B).$$

Changing the role of $T^{-1}B$ and B, we obtain

$$q(B) \leq q(T^{-1}B)\ .$$

Hence,

$$q(T^{-1}B) = q(B)$$

i.e. q is invariant under T .

Now if $T^{-1}B = B$. Then,

$$q(B) = \phi[m(T^{-1}B)] = \phi[m(B)]\ ,$$

$$= m(B).\phi(1) = m(B)\ .$$

So $q = m$ on invariant sets. Hence (Sucheston [8], Theorem 2) $q = m$ on F. Thus $q(B) = \phi[m(T^{-n}B)]$ is unique. But $\phi \in \{l_\infty,R\}$ is unique if and only if $R(x) = -R(-x) = q(B)$ and this happens if and only if

$$\lim_{n\to\infty}\ \sum_{k=0}^{\infty}\ a_{n,k}^{(i)}\ m(T^{-k}B) = q(B)$$

uniformly in i.

Now since, $q(T^{-1}B) = q(B)$, $B \in F$ and $q = m$ on F, we have $m(T^{-1}B) = m(B)$, $B \in F$ so $m(B) = 0 \Rightarrow m(T^{-1}B) = 0$

i.e. m is null-preserving .

Again (See Sucheston [7], Theorem 6) existence of invariant measure is equivalent to non-existence of weakly wandering sets and non-existence of weakly wandering sets is the same as conservativeness of T.

SUFFICIENCY:

Let (i), (ii) and (iii) hold. Define

$$q(B) = \lim_{n\to\infty}\ \sum_{k=0}^{\infty}\ a_{n,k}^{(i)}\ m(T^{-k}B)$$

Then it can be proved as before that q ia an invariant measure. So only we have to prove q is equivalent to m. Since T is null preserving,

$$m(B) = 0\ \Rightarrow\ m(T^{-1}B) = 0.$$

Then

$$q(B) = \lim_{n\to\infty}\ \sum_{k=0}^{\infty}\ a_{n,k}^{(i)}\ m(T^{-k}B) = 0$$

uniformly in i.

Conversely, let $q(B) = 0$.

Write;

$$A^* = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^{-i}B. \quad \text{Then}$$

$$\begin{aligned} q(A^*) &= q\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} T^{-i}B\right) \\ &\le q\left(\bigcup_{i=1}^{\infty} T^{-i}B\right) \\ &\le \sum_{i=1}^{\infty} q(T^{-i}B) \quad (q \text{ is a measure}) \\ &= \sum_{i=n}^{\infty} q(B) \quad (q \text{ is invariant}) \\ &= 0 \end{aligned}$$

Since, q and m agrees on invariant sets, we have $m(A^*) = 0$. Since, T is conservative by recurrence theorem $m(B/A^*) = 0 \Rightarrow m(B) = 0$.

Hence q is equivalent to m.

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