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BOUNDED ANALYTIC FUNCTIONS AND THE LITTLE BLOCH SPACE

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ABSTRACT. The radial limits of the weighted derivative of an bounded analytic function is considered.

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1. INTRODUCTION.

Let D denote the unit disc of the complex plane C and let H^∞ denote the space of bounded analytic functions on D. An analytic function on D is called a Bloch function if $\sup_{z\in D} |f'(z)|\,(1-|z|^2) < \infty$. The space $\mathcal{B}$ of Bloch functions is a Banach space with norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in D} |f'(z)|\,(1-|z|^2).$$

A Bloch function is in the little Bloch space $\mathcal{B}_0$, if $f'(z)\,(1-|z|^2) \to 0$ as $|z| \to 1-$. An immediate consequence of Schwartz lemma (see for example [2], Lemma 1.2) is that $H^\infty \subset \mathcal{B}$, however it is well known (see section 3 for an explicit example) that $H^\infty \not\subset \mathcal{B}_0$. The main result of this paper is to show that, if $f \in H^\infty$ then

$$f'(r\,e^{i\theta})\,(1-r^2) \to 0 \text{ for almost all } \theta \text{ as } r \to 1-.$$

2. APPROXIMATE IDENTITY.

In this section we establish an approximate identity akin to the Poisson Kernel.

LEMMA 1. Let $0 < r < 1$, $t \in \mathbb{R}$ and

$$\varphi(r,t) = \frac{(1-r^2)^3}{(1+r^2)} \frac{1}{(1 - 2r \cos t + r^2)^2}$$

Then $\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r,t)dt = 1.$

PROOF: Let $P_r(t) = \frac{1-r^2}{1 - 2r \cos t + r^2} = 1 + 2 \sum_1^{\infty} r^n \cos nt$ be the Poisson kernel. As usual let L^2 be the Lebesgue 2-space on $[0,2\pi]$. Then,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r,t)dt = \frac{(1-r^2)}{(1+r^2)} \|P_r(t)\|^2_{L^2}$$

$$= \frac{(1-r^2)}{(1+r^2)} \langle 1 + 2 \sum_1^{\infty} r^n \cos nt,\ 1 + 2 \sum_1^{\infty} r^n \cos nt \rangle$$

$$= \frac{1-r^2}{1+r^2} (1 + 2 \sum_1^{\infty} r^{2n}) = 1.$$

LEMMA 2. Let μ be a complex measure on $[-\pi, \pi]$ and suppose the derivative $D\mu(\theta)$ exists for some point θ. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r, \theta-t)\, d\mu(t) \to D\mu(\theta) \text{ as } r \to 1-.$$

PROOF: The usual approximate identity proof with the Poisson kernel $P_r(\theta-t)$ (see for example [1] page 4) works for $\varphi(r, \theta-t)$ as well.

Without loss of generality we may assume that $\theta = 0$. Let $A = D\mu(0)$ and

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r, \theta-t)\, d\mu(t).$$

then

$$u(re^{i\theta})-A = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r,t)\, [d\mu(t) - Adt]$$

$$= \frac{1}{2\pi} [\varphi(r,t)\, [\mu(t) - At]_{-\pi}^{\pi}$$

$$- \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mu(t) - At) \frac{\partial\varphi}{\partial t}\, dt$$

where $\mu(t)$ is the function of Bounded Variation associated with μ. Note that the first term tends to 0 as $r \to 1-$. Fix $\delta > 0$ and let $\delta \leq |t| \leq \pi$. Then

$$\left|\frac{\partial\varphi}{\partial t}(r,t)\right| \leq \frac{(1-r^2)^3}{16r^2}\,\frac{1}{\sin^6\delta/2}.$$

Hence for each $\delta > 0$,

$$u(re^{i\theta}) - A - I_\delta \to 0$$

as $r \to 1-$, where

$$I_\delta = -\frac{1}{2\pi}\int_{-\delta}^{\delta}(\mu(t) - At)\,\frac{\partial\varphi}{\partial t}\,dt = \frac{1}{\pi}\int_0^{\delta}\left[\frac{\mu(t) - \mu(-t)}{2t} - A\right] t\left(\frac{-\partial\varphi}{\partial t}\right) dt$$

Given $\epsilon > 0$, chose $\delta > 0$ such that

$$\left|\frac{\mu(t) - \mu(-t)}{2t} - A\right| < \frac{\epsilon}{2} \quad \text{for} \quad 0 < t \leq \delta.$$

Then $|I_\delta| \leq \frac{\epsilon}{2\pi}\int_0^{\pi} t\,\left|-\frac{\partial\varphi}{\partial t}\right| dt.$

But then $\frac{\partial}{\partial t}\varphi(r,t) = -\frac{(1-r^2)^3}{(1+r^2)}\,\frac{4r\sin t}{(1 - 2r\cos t + r^2)^3}.$

Hence $|I_\delta| \leq \frac{\epsilon}{2\pi}\int_0^{\pi} t\left(-\frac{\partial\varphi}{\partial t}(r,t)\right)dt$

$$= -\frac{\epsilon}{2\pi}\left[t\,\varphi(r,t)\right]_0^{\pi} + \frac{\epsilon}{2\pi}\int_0^{\pi}\varphi(r,t)dt$$

$$\leq \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi}\varphi(r,t)dt = \epsilon.$$

Now we are ready to prove the main result of this paper. We may recall that if $f \in H^\infty$ then the radial limits, $\lim_{r\to1-} f(re^{i\theta}) = f(e^{i\theta})$ exists for almost all θ and that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-r^2)}{1 - 2r\cos(\theta-t) + r^2}\,f(e^{it})dt.$$

Taking derivatives with respect to r, we get

$$e^{i\theta} f'(re^{i\theta}) = I_1(re^{i\theta}) - I_2(re^{i\theta}),$$

where $I_1(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{-2r}{1 - 2r\cos(\theta-t) + r^2} f(e^{it})dt$ and $I_2(re^{i\theta}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)(r - \cos(\theta-t))}{(1 - 2r\cos(\theta-t) + r^2)^2} f(e^{it})dt$. Note that $(1-r)\, I_1(re^{i\theta}) \to - f(e^{i\theta})$ for almost all θ as $r \to 1-$. Also

$$I_2(1-r) = \frac{1}{\pi}(1-r)\int_{-\pi}^{\pi} \frac{(1-r^2)(-\cos(\theta-t) + 1)}{(1 - 2r\cos(\theta-t) + r^2)^2} f(e^{it})dt$$

$$- \frac{(1-r)^2}{\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)}{(1 - 2r\cos(\theta-t) + r^2)^2} f(e^{it})dt.$$

The first term in $I_2(1-r)$ is dominated by $(1-r)\,\|f\|_\infty$ and hence tends to zero as $r \to 1-$. However the second term of $I_2(1-r)$ is

$$= - \frac{2(1+r^2)}{(1+r)^2}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(r, \theta-t)\, f(e^{it})dt \to - f(e^{i\theta})$$

for almost all θ as $r \to 1-$ by Lemma 2.

The Proof is complete.

3. A BLASCHKE PRODUCT NOT IN $\mathcal{B}_0$.

In this section, for completeness sake we give an explicit example of an bounded analytic function which is not in $\mathcal{B}_0$.

First we state an elementary lemma, whose proof we omit.

LEMMA 3. Let $0 < r < 1$. Then

(a) If $0 < x < r < 1$, then $-\ln(1-x) < \frac{x}{1-r}$

(b) $\frac{1-x}{1-xr}$ decreases on $-\infty < x < \frac{1}{r}$

(c) $\frac{1+x}{1-xr}$ increases on $-\infty < x < \frac{1}{r}$.

LEMMA 4. Let $\alpha_n = 1 - \frac{1}{2^n}$ and $2\beta_n = \alpha_n + \alpha_{n+1}$ with $n \geq 1$.

Then $\varliminf_{m\to\infty} \prod_{n=1}^{\infty} \left|\frac{\beta_m - \alpha_n}{1 - \overline{\alpha}_n \beta_m}\right| \geq c > 0$ for some c.

PROOF: Fix $m \geq 1$. We first show that

$$\sum_{m+1}^{\infty} - \ln \left|\left[\frac{\alpha_n - \beta_m}{1 - \overline{\alpha}_n \beta_m}\right]\right| \leq 32.$$

For,

$$\sum_{m+1}^{\infty} - \ln \left|\left[\frac{\alpha_n - \beta_m}{1 - \overline{\alpha}_n \beta_m}\right]\right| = \sum_{m+1}^{\infty} - \ln \left(1 - \frac{(1-\alpha_n)(1 + \beta_m)}{1 - \alpha_n \beta_m}\right)$$

$$\leq \left[\frac{1 - \alpha_{m+1} \beta_m}{\alpha_{m+1} - \beta_m}\right] \sum_{m+1}^{\infty} \frac{(1 - \alpha_n)(1 + \beta_m)}{1 - \beta_m}.$$ (use Lemma 3 (a) and (b)).

Then

$$\sum_{m+1}^{\infty} - \ln \left|\left[\frac{\alpha_n - \beta_m}{1 - \overline{\alpha}_n \beta_m}\right]\right| \leq \frac{(1 - |\alpha_m|^2)}{\frac{1}{2}(\alpha_{m+1} - \alpha_m)} \frac{2}{(1-\beta_m)} \sum_{m+1}^{\infty} \frac{1}{2^n} \leq 32.$$

Now applying Lemma 3(c) to $\frac{1+x}{1-x\beta_m}$, we have $\frac{(1-\beta_m)(1+\alpha_n)}{1 - \alpha_n\beta_m} \leq \frac{(1-\beta_m)(1+\alpha_m)}{1 - \alpha_m\beta_m}$ for $1 \leq n \leq m$.

Thus,

$$\sum_{n=1}^{m} - \ln \left|\frac{(\alpha_n - \beta_m}{1 - \alpha_n\beta_m}\right| = \sum_{n=1}^{m} - \ln \left[1 - \frac{(1-\beta_m)(1+\alpha_n)}{1 - \alpha_n\beta_m}\right]$$

$$\leq \left[\frac{1 - \alpha_m\beta_m}{\beta_m - \alpha_m}\right] \sum_{n=1}^{m} \frac{(1-\beta_m)(1+\alpha_n)}{1 - \alpha_n\beta_m}$$ by Lemma 3 (a),(c).

Hence;

$$\sum_{n=1}^{m} - \ln \left|\left[\frac{\alpha_n - \beta_m}{1 - \alpha_n\beta_m}\right]\right| \leq \left[\frac{1 - \alpha_m\beta_m}{\beta_m - \alpha_m}\right] (1-\beta_m) \sum_{n=1}^{m} \frac{2}{1-\alpha_n}$$

$$\leq \frac{1 - \alpha_m^2}{\frac{1}{2}(\alpha_{m+1}-\alpha_m)} (1-\beta_m) \sum_{1}^{m} 2^{n+1} \leq 64.$$

Now Lemma 3 follows.

COROLLARY 1. Let b be the Blaschke product with zeros $\{\alpha_n : n \geq 1\}$. Then

$$b'(r)(1-r)\ 0 \quad \text{as} \quad r \to 1-.$$

PROOF: Let $c > \overline{\lim_{r\to 1-}} |b'(r)|\,(1-r)$.

Then for sufficiently large n,

$$|b(\beta_n)| \leq c \int_{\alpha_n}^{\beta_n} \frac{1}{1-r}\,dr \leq c \int_{\alpha_n}^{\alpha_{n+1}} \frac{1}{1-r}\,dr = c \ln 2.$$

Hence by Lemma 4,

$$\overline{\lim_{r\to 1-}} |b'(r)|\,(1-r) \neq 0.$$

REFERENCES

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2. Garnett, John B., Bounded Analytic Functions, Academic Press, 1981.