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RESEARCH NOTES

QUASI-BOUNDED SETS

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ABSTRACT. It is proved in [1] & [2] that a set bounded in an inductive limit $E = indlimE_n$ of Fréchet spaces is also bounded in some E_n iff E is fast complete. In the case of arbitrary locally convex spaces E_n every bounded set in a fast complete $indlimE_n$ is quasi-bounded in some E_n, though it may not be bounded or even contained in any E_n. Every bounded set is quasi-bounded. In a Fréchet space every quasi-bounded set is also bounded.

KEY WORDS AND PHRASES. Locally convex space, regular inductive limit, bounded and quasi-bounded set.

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Let L be a vector space and $B \subset L$. The absolutely convex hull of B is denoted by $abcoB$. The linear hull of B, with the topology generated by the gauge of $abcoB$, is denoted by E_B. The set B is called Banach disk if it is absolutely convex, E_B is a Banach space, and B is closed in E_B. A locally convex space F is called fast complete if every set bounded in F is contained in a Banach disk. For $A \subset F$, the closure of A in F is denoted by $c\ell_F A$.

Let $E_1 \subset E_2 \subset \cdots$ be a sequence of Hausdorff locally convex spaces with the identity maps id: $E_n \to E_{n+1}, n = 1, 2, \cdots$, continuous and the inductive limit $E = indlimE_n$ Hausdorff. Then E is called regular if every set bounded in E is also bounded in some E_n. It is shown in [1] & [2] that if all spaces E_n are Fréchet then E is regular iff E is fast complete. This result can not be extended to inductive limits of arbitrary locally convex spaces.

We introduce the notion of a quasi-bounded set and show that if E is fast complete then every set bounded in E is quasi-bounded in some E_n, though it may not be bounded or even contained in any E_n.

DEFINITION. Let F be a Hausdorff locally convex space. A set B, not necessarily contained in F, is alled quasi-bounded (further we write q-bounded) in F if:

(a) E_B is Hausdorff,

(b) for any 0-neighborhood U in F, the set $c\ell_{E_B}(U \cap E_B)$ absorbs B.

PROPOSITION 1. In the above definition the property (b) could be replaced by:

(bb) for any 0-neighborhood U in F, the set $c\ell_{E_B}(U \cap B)$ absorbs B.

PROOF. Clearly $(bb) \Longrightarrow (b)$. Let a set B satisfy (b) and U be a 0–neighborhood in F. Without the loss of generality we may assume both B and U to be absolutely convex and $B \subset c\ell_{E_B}(U \cap E_B)$.

Take $b \in B$, $\beta > 0$. Then $(b+\beta B)\cap(U\cap E_B) = (b+\beta B)\cap U \neq \emptyset$ or $b \in U+\beta B$ which implies $B \subset U+\beta B$. Put $b = u+\beta v$, where $u \in U$, $v \in B$. Then $u = b-\beta v \in B+\beta B = (1+\beta)B$ and $u \in U\cap(1+\beta)B \subset (1+\beta)U\cap(1+\beta)B = (1+\beta)(U\cap B)$. Hence $b = u+\beta v \in (1+\beta)(U\cap B)+\beta B$ and $B \subset \bigcap\{(1+\beta)(U\cap B)+\beta B;\ \beta>0\} = c\ell_{E_B}(U\cap B)$.

PROPOSITION 2. Let $Q(F)$ be the family of all q-bounded sets in a Hausdorff locally convex space F. Then:

1. B bounded in $F \Longrightarrow B \in Q(F)$,
2. $B \in Q(F) \Longrightarrow abcoB \in Q(F)$,
3. $B \in Q(F) \Longrightarrow c\ell_{E_B}B \in Q(F)$,
4. $B \in Q(F)\&A \subset B \Longrightarrow A \in Q(F)$,
5. $B \in Q(F)\&B \subset F \Longrightarrow c\ell_F B \in Q(F)$,
6. $B \in Q(F)\&A$ a set bounded in $F \Longrightarrow A\cup B \in Q(F)\&A+B \in Q(F)$,
7. $B \in Q(F)\&A$ the closed unit ball in the completion of $E_B \Longrightarrow A \in Q(F)$.

PROOF. The statements 1,2, and 3, are obvious.

4. Since $A \subset B$, the topology of E_A is finer than that of E_B and E_A is Hausdorff.

Take a 0–neighborhood U in F. We may assume that all A, B, and U, are absolutely convex and $B \subset c\ell_{E_B}(U\cap B)$. Take $\alpha > 1$ and assume there exists $x \in A \setminus \alpha c\ell_{E_A}(U\cap A)$. Then $x \notin \alpha(U\cap A)$, $\frac{1}{\alpha}x \notin U$ and $\frac{1}{\alpha}x \notin U\cap B$. On the other hand, $\frac{1}{\alpha}x \in A \subset B \subset c\ell_{E_B}(U\cap B)$. Hence there exists a real $f \in E'_B$ such that $f(x) = \alpha$ and $U\cap B \subset f^{-1}[-1,1]$. But then also $c\ell_{E_B}(U\cap B) \subset f^{-1}[-1,1]$. Since $x \in A \subset B \subset c\ell_{E_B}(U\cap B)$, we have $f(x) \in [-1,1]$, a contradiction with $f(x) = \alpha$.

5. Let $B \in Q(F)$, $B \subset F$, $B = abcoB$, and $D = c\ell_F B$. By statement 4, it is sufficient to prove $D \in Q(F)$. Take $x \in E_D$, $x \neq 0$. Since E_B is Hausdorff, there exists $\beta > 0$ such that $x \notin 2\beta B$. Take a real $f \in F'$ for which $f(x) = 2$ and $\beta B \subset f^{-1}[-1,1]$. Then also $\beta D \subset f^{-1}[-1,1]$ and $x \notin \beta B$ which implies that E_D is Hausdorff.

To prove (b) take an absolutely convex 0–neighborhood U in F for which $B \subset c\ell_{E_B}(U\cap E_B)$. Since the toplogy of E_B is finer than that of E_D, we have $c\ell_{E_B}(U\cap E_B) \subset c\ell_{E_D}(U\cap E_B) \subset c\ell_{E_D}(U\cap E_D)$. For $x \in D$ and $\beta > 0$, there exists $y \in B$ such that $x-y \in \beta U$ and $x = x-y+y \in \beta(U\cap E_D)+B \subset \beta c\ell_{E_D}(U\cap E_D)+c\ell_{E_B}(U\cap E_B) \subset \beta c\ell_{E_D}(U\cap E_D)+c\ell_{E_D}(U\cap E_D) = (1+\beta)c\ell_{E_D}(U\cap E_D)$. Hence $D \subset \bigcap\{(1+\beta)c\ell_{E_D}(U\cap E_D);\beta>0\} = c\ell_{E_D}(U\cap E_D)$.

6. The set B is contained in the completion of the normed space $E_{B\cap F}$ whose topology is stronger than that of $E_{A+(B\cap F)}$. Hence both sets A and B are contained in the completion of $E_{A+(B\cap F)}$, i.e., $A\cup B$ and $A+B$ make sense as subsets of a vector space.

Next assume both A and B to be absolutely convex. To prove that E_{A+B} is Hausdorff, take $x_0 \in E_{A+B}$, $x_0 \neq 0$. If $x_0 \in E_B$, then $x_0 \notin \beta B$ for some $\beta > 0$. If $x_0 \notin E_B$, then $x_0 \notin \beta B$ for the same $\beta > 0$. Let a real $f \in F'$ be such that $f(x_0) = \beta$ and $B \in f^{-1}[-1,1]$. Put $U = f^{-1}[-1,1]$. Since U absorbs A, we have $A \subset \alpha U$ for some $\alpha > 0$. If $\lambda \in \left(0, \frac{\beta}{1+\alpha}\right)$ and $x \in \lambda(A+B)$, then $| f(x) | \leq \lambda\alpha + \lambda < \beta$ while $f(x_0) = \beta$. Hence $x_0 \notin \lambda(A+B)$.

The space $E_{A\cup B}$ is also Hausdorff since $id : E_{A\cup B} \to E_{A+B}$ is continuous.

Let U be a 0−neighborhood in F. Take $\alpha, \beta < 0$ such that $A \subset \alpha U$ and $B \subset \beta c\ell_{E_B}(U \cap E_B)$. Then $A \subset \alpha U \cap E_A \subset \alpha c\ell_{E_A}(U \cap E_A)$ and $A+B \subset \alpha c\ell_{E_A}(U\cap E_A)+\beta c\ell_{E_B}(U\cap E_B) \subset \alpha c\ell_{E_{A+B}}(U \cap E_{A+B}) + \beta c\ell_{E_{A+B}}(U \cap E_{A+B}) = (\alpha+\beta)c\ell_{E_{A+B}}(U \cap E_{A+B})$.

Similarly, $A \cup B \subset max(\alpha,\beta) \cdot c\ell_{E_{A\cup B}}(U \cap E_{A\cup B})$.

7. The Banach space E_A is Hausdorff. Take an absolutely convex 0−neighborhood U in F and assume $B \subset c\ell_{E_B}(U \cap E_B)$. Let $a \in A$. There exists a sequence $\{b_m\} \subset B$ which is Cauchy in E_B and converges to a in E_A. For every m there exists a sequence $\{u_{m,n}\} \subset U \cap B$ such that $u_{m,n} \to b_m$ in E_B as $n \to \infty$. Choose n_m so that $u_{m,n_m} - b_m \in \frac{1}{m}B$, $m = 1,2,\cdots$ and put $a_m = u_{m,n_m}$. Then $a_m \in U \cap B$ and $a_m \to a$ in E_A as $m \to \infty$. Hence $a \in c\ell_{E_A}(U \cap B) \subset c\ell_{E_A}(U \cap A)$ and $A \subset c\ell_{E_A}(U \cap A)$.

PROPOSITION 3. Let F be a locally convex space and $B \subset F$ a Banach disk. There B is q−bounded in F.

PROOF. Take a 0−neighborhood in F. Then $B \subset \bigcup\{nU \cap E_B; n = 1,2,\cdots\}$. By the Category Argument $c\ell_{E_B}(nU \cap E_B) = nc\ell_{E_B}(U \cap E_B)$ is a 0−neighborhood in E_B for some n. Hence $c\ell_{E_B}(U \cap E_B)$ absorbs B.

EXAMPLE 1. Let F be an infinitely dimensional Banach space, B its closed unit ball, and H the vector space underlying F equipped with the finest locally convex topology. Since every set bounded in H is contained in a finite-dimensional subspace, B is not bounded in H.

On the other hand, $B \subset H$ is a Banach disk and, by Prop. 3, is q-bounded in H.

PROPOSITION 4. Let F be a Fréchet space and $B \subset F$ q−bounded in F. Then B is bounded in F.

PROOF. We may assume that B is absolutely convex and closed in F. Let $U_0 \supset U_1 \subset \cdots$ be a fundamental sequence of 0−neighborhoods in F such that each U_n is absolutely convex, closed in F, and $U_{n+1} + U_{n+1} \subset U_n$, $n = 0,1,2,\cdots$. It is sufficient to show that U_0 absorbs B.

For each n, there exists $\beta_n > 0$ such that $B \subset \beta_n c\ell_{E_B}(U_n \cap E_B)$. Put $\varepsilon_n = min(n^{-1}, \beta_n^{-1})$, $C_n = c\ell_{E_B}(U_n \cap E_B)$, $n = 1,2,\cdots$, and take $x \in B$. There exists $x_0 \in \beta_0 U_0 \cap E_B$ such that $x - x_0 \in \varepsilon_1 B \subset \varepsilon_1\beta_1 C_1 \subset C_1$. Hence there exists $x_1 \in U_1 \cap E_B$ such that $x - x_0 - x_1 \in \varepsilon_2 B \subset \varepsilon_2\beta_2 C_2 \subset C_2$, etc. By the induction, there exists $x_n \in U_n \cap E_B$ such that $x - (x_0 + x_1 + \cdots + x_n) \in \varepsilon_{n+1}B \subset \varepsilon_{n+1}\beta_{n+1}C_{n+1} \subset C_{n+1}$. The sequence $x - (x_0 + x_1 + \cdots + x_n)$, $n = 0,1,2,\cdots$, converges to 0 in E_B. Hence $x = x_0 + x_1 + \cdots \subset \beta_0 U_0 + U_1 + U_2 + \cdots \subset \beta_0 U_0 + U_0$ and $B \subset (\beta_0 + 1)U_0$.

THEOREM. Let $E_1 \subset E_2 \subset \cdots$ be a sequence of locally convex spaces, with identity maps $E_n \to E_{n+1}, n = 1,2,\cdots$, continuous and $E = \ indlimE_n$ Hausdorff. Let $B \subset E$ be a Banach disk. Then B is q−bounded in some E_m.

PROOF. Put for brevity $B_n = B \cap E_n$, $n = 1,2,\cdots$. We first prove that $B = c\ell_{E_B} B_n$ for some n. By the Category Argument there exists n such that $c\ell_{E_B} B_n$ absorbs B. Hence $B \subset \lambda c\ell_{E_B} B_n$ for some $\lambda > 0$. Take $b \in B$ and $\beta > \alpha > 1$. There is a sequence $b_k \in \lambda B_n$, $b = 1,2,\cdots$, such that $b_k \to b$ in E_B. If $b_k \notin \beta B_n$ for infinitely many indices k, then $b \notin \alpha B_n$, a contradiction. Thus we may assume $b_k \in (1 + \frac{1}{k}) B_n$ for each k. This implies $c_k = \frac{k}{k+1} b_k \in B_n$, $c_k \to b$ in E_B and $b \in c\ell_{E_B} B_n$. Since B is closed in E_B, we have $B \supset c\ell_{E_B} B_n$ and $B = c\ell_{E_B} B_n$.

Next we show that there exists $m \geq n$ such that B_m is $q-$bounded in E_m. Assume the contrary. Then for every $k \geq n$, there exists an absolutely convex $0-$neighborhood U_k in E_k such that $c\ell_{E_{B_k}}(U_k \cap B_k)$ does not absorb B_k. Since $c\ell_{E_{B_k}}(U_k \cap B_k) = E_{B_k} \cap c\ell_{E_B}(U_k \cap B_k) = E_{B_k} \cap c\ell_{E_B}(U_k \cap B)$, the set $V_k = c\ell_{E_B}(U_k \cap B)$ also does not absorb B.

Put for brevity $W_k = c\ell_{E_B} B_k$, $k \geq n$. The spaces E_{V_k} and E_{W_k}, $k \geq n$, are all Banach and the identity maps: $E_{V_k} \to E_{W_k}$, $k \geq n$ are all continuous, hence the map $id : E_{W_m} \to \bigcup\{E_{V_k}; k \geq n\}$ is closed. By [3; Cor.IV.6.5] there exists $m \geq n$ such that $id : E_{W_n} \to E_{V_m}$ is continuous. But then V_m absorbs $W_n = B$, a contradiction.

Since $m \geq n$, we also have $B = c\ell_{E_B} B_m$. By Prop. 2, #7, B is $q-$bounded in E_m.

EXAMPLE 2. Let F, B, and H, be the same as in Example 1. For each n, put $E_n = F^n \times H^N$, where $N = \{1,2,3,\cdots\}$. Then $E = \ indlim E_n = F^N$ is fast complete, the set B^N is bounded in E, but not bounded in any E_n. By Example 1, B^N is $q-$bounded in every E_n, $n \in N$.

Theorem 5 in [2] reads: Let E be an inductive limit of Fréchet spaces E_n. Then E is regular iff E is fast complete. We show that this result follows from the above theorem.

To prove it, we first observe that in any inductive limit $E = \ indlim E_n$ any set bounded in some E_n is bounded in E. Assume all spaces E_n to be Fréchet and E fast complete. Take a set B bounded in E. Since E is fast complete, we may assume that B is a Banach disk. By our Theorem there exists m such that $B = c\ell_{E_B} B_m$ and B_m is $q-$bounded in E_m. By Prop. 4, B_m is bounded in E_m. It remains to show that $B_m = B$.

Take $x_0 \in B$ and a sequence $\{x_k\} \subset B_m$, such that $x_k \to x_0$ in E_B. Since B is bounded in E, the topology of E_B is stronger than that of E and $x_k \to x_0$ in E.

The topology of E_{B_m} is inherited from the superspace E_B. Thus $\{x_k\}$ is Cauchy in E_{B_m}. Now, B_m is bounded in E_m hence the toplogy of E_{B_m} is stronger than that of E_m. This implies that $\{x_k\}$ is also Cauchy in the Fréchet space E_m and as such it converges in E_m to some $y_0 \in E_m$. But then $x_k \to y_0$ also in E.

Since E is Hausdorff, we have $x_0 = y_0 \in B \cap E_m = B_m$.

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