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WEIGHTED ADDITIVE INFORMATION MEASURES

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ABSTRACT. We determine all measurable functions $I,G,L: [0,1] \to \mathbb{R}$ satisfying the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} I(p_i q_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} G(p_i)I(q_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} L(q_j)I(p_i)$$

for $P \in \Gamma_n$, $Q \in \Gamma_m$ and for a fixed pair (n,m), $n \geqslant 3$, $m \geqslant 3$, where $G(0) = L(0) = 0$ and $G(1) = L(1) = 1$. This functional equation has interesting applications in information theory.

KEY WORDS AND PHRASES. Weighted additive information measures of sum form type, entropies of degree (α,β).
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1. INTRODUCTION.

Let $\Gamma_k = \{P = (p_1,\ldots,p_k) : p_i \geqslant 0, \sum_{i=1}^{k} p_i = 1\}$, $k \geqslant 2$.

We say that an information measure $I_k : \Gamma_k \to \mathbb{R}$, $k \geqslant 2$ is (n,m)-weighted additive $(n,m \in \mathbb{N})$ if there exist weight functions $G_k, L_k : \Gamma_k \to \mathbb{R}$, $k \geqslant 2$ such that

$$I_{nm}(P\cdot Q) = G_n(P)I_m(Q) + I_n(P)L_m(Q) \quad , P \in \Gamma_n, Q \in \Gamma_m \tag{1.1}$$

where as usual $P\cdot Q = (p_1q_1,\ldots,p_iq_j,\ldots,p_nq_m) \in \Gamma_{nm}$. If in addition I_k, G_k, L_k have the sum property with generating functions $I,G,L : [0,1] \to \mathbb{R}$, that is

$$I_k(P) = \sum_{i=1}^{k} I(p_i) \ , \ G_k(P) = \sum_{i=1}^{k} G(p_i) \ , \ L_k(P) = \sum_{i=1}^{k} L(p_i) \tag{1.2}$$

then equation (1.1) goes over into the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} I(p_iq_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} G(p_i)I(q_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} L(q_j)I(p_i) \ , \tag{1.3}$$

$P \in \Gamma_n$, $Q \in \Gamma_m$. This functional equation (1.3) is of interest since the special cases

$$G(p) = p \ , \ L(p) = p + \lambda I(p) \qquad , \ \lambda \in \mathbb{R} \tag{1.4}$$

and

$$G(p) = p^{\alpha} \ , \ L(p) = p^{\beta} \qquad , \ \alpha,\beta \in \mathbb{R} \tag{1.5}$$

play important roles in the characterization of the entropies of degree a (Losonczi, [1])

$$I_k^a(P) = \begin{cases} H_k^{(a,1)}(P) = (2^{1-a} - 1)^{-1} \sum_{i=1}^{k}(p_i^a - p_i) & a \neq 1 \\ H_k^1(P) = - \sum_{i=1}^{k} p_i \log p_i & a = 1 \end{cases} \qquad P\in\Gamma_k \tag{1.6}$$

and degree (α,β) (Sharma and Taneja, [2])

$$I_k^{(\alpha,\beta)}(P) = \begin{cases} H_k^{(\alpha,\beta)}(P) = (2^{1-\alpha} - 2^{1-\beta})^{-1} \sum_{i=1}^{k} (p_i^{\alpha} - p_i^{\beta}) & \alpha \neq \beta \\ H_k^{\alpha}(P) = - 2^{\alpha-1} \sum_{i=1}^{k} p_i^{\alpha} \log p_i & \alpha = \beta \end{cases} \qquad P\in\Gamma_k, \tag{1.7}$$

respectively. Here we follow the conventions

$$\log = \log_2 \ , \ 0\cdot\log 0 = 0 \ \text{ and } 0^a = 0 \ , \ a \in \mathbb{R}. \tag{1.8}$$

The aim of this paper is to determine all measurable triples (I,G,L) satisfying (1.3) for a fixed pair (n,m), $n \geq 3$, $m \geq 3$ where - because of the known results and the convention (1.8) - we assume

$$G(0) = L(0) = 0 \quad , \quad G(1) = L(1) = 1. \tag{1.9}$$

Thus we determine not only all measurable functions I of (I_k) (see (1.2)) but also all possible choices for G and L in (1.3). Therefore the results due to Kannappan [3-5], Losonczi [1], Sharma and Taneja [2,6] are special cases of our main result. Moreover, if we assume that I is not constant and that G and L are continuous then we can interpret our result in the form that, without loss of generality, we may assume that G and L in (1.3) are continuous, non zero multiplicative functions, that is they are non zero continuous solutions of the functional equation

$$M(p\cdot q) = M(p)\cdot M(q) \qquad p,q \in [0,1]. \tag{1.10}$$

2. MAIN RESULTS.

We make use of the following well known result (Kannappan, [5]).

LEMMA 1. Let $n \geq 3$ be a fixed integer and let $F : [0,1] \to \mathbb{R}$ be a measurable function satisfying

$$\sum_{i=1}^{n} F(p_i) = 0$$

for all $P \in \Gamma_n$. Then there exists a constant a such that

$$F(p) = a(1 - np) \quad , \quad p \in [0,1].$$

Now we are ready to prove our main result which is an extension of the results, mentioned above.

THEOREM 2. Let $I,G,L : [0,1] \to \mathbb{R}$ be measurable and let I be non constant. Then I,G,L satisfy (1.9) and (1.3) for a fixed pair (n,m), $n \geq 3$, $m \geq 3$ if, and only if they are of one of the following forms :

$$I(p) = a(p^A - p^B),$$
$$G(p) = (1 - b)p^A + bp^B \quad , \quad L(p) = bp^A + (1 - b)p^B \ , \ A \neq B \ , \tag{2.1}$$
$$I(p) = ap^A \log p \ ,$$
$$G(p) = p^A(1 + b\log p) \quad , \quad L(p) = p^A(1 - b\log p) \ , \ A \neq 1 \ , \tag{2.2}$$
$$I(p) = I(0) + (mn - m - n)I(0)p + dp\log p, \ G(p) = L(p) = p, \tag{2.3}$$
$$I(1) = 0 \quad , \ G(1) = 1 \qquad , \ L(1) = 1$$
$$I(p) = ap^A \ , \ G(p) = (1 - b)p^A \ , \ L(p) = bp^A \ , \ p \in [0,1) \tag{2.4}$$
$$I(p) = ap^A\sin(c\log p) \ , \ G(p) = p^A[\cos(c\log p) + b\sin(c\log p)] \ ,$$
$$L(p) = p^A[\cos(c\log p) - b\sin(c\log p)] \ . \tag{2.5}$$

Here A,B,a,b,c,d are constants and we follow the conventions

$$0^a \cdot \cos(\log 0) = 0 \ , \ 0^a \cdot \sin(\log 0) = 0 \ , \ a \in \mathbb{R}.$$

PROOF. Obviously, the solutions (I,G,L) given by (2.1) to (2.5) satisfy (1.9) and (1.3). To prove the converse let us introduce the function $I' : [0,1] \to \mathbb{R}$ defined by

$$I'(p) = I(p) - I(0) - (I(1) - I(0))p \ , \ p \in [0,1]. \tag{2.6}$$

It is clear that I' fulfills

$$I'(0) = I'(1) = 0. \tag{2.7}$$

We now show that the triple (I',G,L) also satisfies (1.3). To see this let us put $P = (1,0,0,\ldots,0) \in \Gamma_n$ and $Q = (1,0,0,\ldots,0) \in \Gamma_m$ into (1.3). Using (1.9) we arrive at

$$I(1) + (nm - 1)I(0) = I(1) + (m - 1)I(0) + I(1) + (n - 1)I(0)$$

or

$$I(1) - I(0) = (mn - m - n)I(0). \tag{2.8}$$

Thus I' can also be written in the form

$$I'(p) = I(p) - I(0) - (mn - m - n)I(0)p. \tag{2.9}$$

Substituting $P \in \Gamma_n$, $Q = (1,0,0,\ldots,0) \in \Gamma_m$ and $P = (1,0,0,\ldots,0) \in \Gamma_n$, $Q \in \Gamma_m$ separately into (1.3) we get

$$\sum_{i=1}^{n} G(p_i)(I(1) + (m - 1)I(0)) = (nm - n)I(0) \tag{2.10}$$

and

$$\sum_{j=1}^{m} L(q_j)(I(1) + (n - 1)I(0)) = (nm - m)I(0) \tag{2.11}$$

or, using (2.8)

$$(1 - \sum_{j=1}^{n} G(p_i))(nm - n)I(0) = 0 \tag{2.12}$$

and

$$(1 - \sum_{j=1}^{m} L(q_j))(nm - m)I(0) = 0 , \tag{2.13}$$

respectively.

After these preparations we can see immediately that I',G,L satisfy

$$\sum_{i=1}^{n}\sum_{j=1}^{m} I'(p_iq_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} G(p_i)I'(q_j) + \sum_{i=1}^{n}\sum_{j=1}^{m} L(q_j)I'(p_i) \tag{2.14}$$

for all $P \in \Gamma_n$, $Q \in \Gamma_m$. Putting I', given by (2.6), into (2.14) and using (1.3) and (2.8), we see that (2.14) is equivalent to

$$(1 - \sum_{i=1}^{n} G(p_i))(nm - n)I(0) + (1 - \sum_{j=1}^{m} L(q_j))(nm - m)I(0) = 0. \tag{2.15}$$

But (2.15) is indeed valid because of (2.12) and (2.13).

In a further step we derive a functional equation for I', G and L in which no sums will occur. Setting

$$F(p,q) = I'(p\cdot q) - G(p)I'(q) - L(q)I'(p) , \quad p,q \in [0,1] \tag{2.16}$$

we get from (2.14)

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i,q_j) = 0 , \quad P \in \Gamma_n, Q \in \Gamma_m.$$

Since by hypothesis $F : [0,1]^2 \to \mathbb{R}$ is measurable in each variable we get from Lemma 1 in Kannappan [3] (This Lemma is an application of the above Lemma 1) that F can be represented in the form

$$F(p,q) = F(p,0)(1 - mq) + F(0,q)(1 - np) - F(0,0)(1 - mq)(1 - np). \tag{2.17}$$

Thus (2.16) and (2.17) imply

$$I'(p\cdot q) = G(p)I'(q) + L(q)I'(p) , \quad p,q \in [0,1] \tag{2.18}$$

since (2.17), (1.9) and (2.7) yield $F(p,0) = F(0,q) = F(0,0) = 0$. Because of $I'(0) = G(0) = L(0) = 0$ it is enough to solve (2.18) for all $p,q \in (0,1]$. Complex-valued functional equations of this type were intensively studied by Vincze [7-9]. From these results we get the solutions of (2.18) for $p,q \in (0,1]$ (Ebanks, [10]) which have one of the following forms :

$$I'(p) = a\cdot M(p)\cdot \log p,$$

$$G(p) = M(p)(1 + b\cdot \log p)\ ,\ L(p) = M(p)(1 - b\cdot \log p)\ , \tag{2.19}$$

$$I'(p) = a(M_1(p) - M_2(p)),$$

$$G(p) = (1 - b)M_1(p) + bM_2(p)\ ,\ L(p) = bM_1(p) + (1 - b)M_2(p)\ , \tag{2.20}$$

$$I'(p) = aM(p)\sin(c\log p)\ ,\ G(p) = M(p)[\cos(c\log p) + b\sin(c\log p)]\ ,$$

$$L(p) = M(p)[\cos(c\log p) - b\sin(c\log p)]. \tag{2.21}$$

Here a,b,c are constants and $M, M_1, M_2: (0,1] \to \mathbb{R}$ are measurable multiplicative functions. Let us remark that the measurable solutions of (1.10) for $p,q \in (0,1]$ are either

$$M = 0 \quad \text{or} \tag{2.22}$$

$$M(p) = p^A\ ,\ A \in \mathbb{R} \quad \text{or} \tag{2.23}$$

$$M(1) = 1\ ,\ M(p) = 0 \text{ for } p \in (0,1). \tag{2.24}$$

Since I has the form

$$I(p) = I'(p) + I(0) + (nm - n - m)I(0)p\ ,\ p \in [0,1] \tag{2.25}$$

(see (2.6) - (2.8)) we can derive the solutions (I,G,L) of (1.9) and (1.3) from (2.19) to (2.24). Let us first consider the case that

$$\sum_{i=1}^{n} (G(p_i) - p_i) = 0 \quad \text{and} \quad \sum_{j=1}^{m} (L(q_j) - q_j) = 0 \tag{2.26}$$

for all $P \in \Gamma_n$, $Q \in \Gamma_m$. Then Lemma 1 implies that

$$G(p) = rp + s\ ,\ L(p) = r'p + s'\ ,\ p \in [0,1]$$

where r,r',s,s' are constants. Because of (1.9) we arrive at

$$G(p) = L(p) = p\ ,\ p \in [0,1].$$

This is only possible if either

$$b = 0\ ,\ M(p) = p \text{ in } (2.19)$$

or if

$$b = 1\ ,\ M_1(p) = M_2(p) = p \text{ in } (2.20).$$

In both cases we get the solution (2.3). Now we assume that

$$\sum_{i=1}^{n} (G(p_i) - p_i) \neq 0 \quad \text{for some } P \in \Gamma_n \quad \text{or}$$

$$\sum_{j=1}^{m} (L(q_j) - q_j) \neq 0 \quad \text{for some } Q \in \Gamma_m.$$

Then (2.12) and (2.13) imply that $I(0) = 0$ so that in all cases where

$$G(p) \neq p \quad \text{or} \quad L(p) \neq p$$

we get that I is equal to I' and thus I is not dependent upon n and m (see (2.25)). Using $G(1) = L(1) = 1$ and the hypothesis that I is not constant we obtain from (2.19) to (2.24) the remaining solutions (2.1), (2.2), (2.4) and (2.5). Thus the Theorem is proven.

It is clear that we can obtain from Theorem 2 some new characterization theorems for information measures. For instance, we remark that the functions G and L given by (2.4) or (2.5) cannot be continuous simultaneously. Thus we get the following extension of results in Kannappan [3,4], Sharma and Taneja [2,6].

COROLLARY 3. If in addition to the hypotheses of Theorem 2, G and L are continuous then the only solutions (I,G,L) of (1.9) and (1.3) are given by (2.1), (2.2) and (2.3).

Corollary 3 implies immediately the following characterization theorem :

Let I_k be an (n,m)-weighted additive information measure where I_k, G_k, L_k have the sum property with continuous generating functions $I,G,L : [0,1] \to \mathbb{R}$. If

$$I(0) = G(0) = L(0) = 0 \ , \ G(1) = L(1) = 1 \quad \text{and} \quad I(\tfrac{1}{2}) = \tfrac{1}{2}$$

then $I_k(P) = H_k^{(A,B)}(P)$ or $I_k(P) = H_k^C(p)$, $P \in \Gamma_k$. Here A,B,C are real constants with $A \neq B$.

Finally we give two interpretations of our result. If we put $b = 0$ into (2.1), (2.2) and (2.3) then we get - with unchanged I(p) -

$$G(p) = p^A \ , \ L(p) = p^B = p^B - p^A + p^A = \frac{1}{a} I(p) + p^A \ , \quad A \neq B \ ,$$

$$G(p) = p^A \ , \ L(p) = p^A \ , \quad A \neq 1 \ ,$$

$$G(p) = p \ , \ L(p) = p,$$

respectively. Thus we may consider Corollary 3 as a justification for the fact that in the literature only two special forms of G and L were considered, namely (1.4) and (1.5).

On the other hand, the condition $b = 0$ in (2.1) and (2.2) implies that in Corollary 3 we may assume without loss of generality that G and L are continuous, non zero multiplicative functions. This result is analogous to a result concerning recursive measures of multiplicative type (Aczél and Ng, [11]).

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