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NOTE ON QUASI-BOUNDED SETS

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ABSTRACT. It is shown that a union of two quasi-bounded sets, as well as the closure of a quasi-bounded set, may not be quasi-bounded.

KEY WORDS AND PHRASES. Locally convex space, bounded and quasi-bounded set, Banach disk.

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Let A be a set in a vector space. By *abcoA* we understand the absolutely convex hull of A and by E_A the linear hull of A equipped with the topology generated by the gauge of *abcoA*. The set A is called Banach disk if it is absolutely convex and closed in E_A, and E_A is a Banach space. If X is a locally convex space, then the closure of A in X is denoted by $c\ell_X A$.

DEFINITION. Let X be a locally convex space. A set B, not necessarily contained in X, is called quasi-bounded (we write q-bounded) in X if:

(a) there exists a vector space Y such that X is a subspace of Y and $B \subset Y$,

(b) E_B is a Hausdorff space,

(c) for any 0-neighborhood U in X, the set $c\ell_{E_B}(U \cap E_B)$ absorbs B.

The condition (c) is equivalent to:

(cc) for any 0-neighborhood $U \in X$, the set $c\ell_{E_B}(U \cap B)$ absorbs B.

PROPOSITION. Let X be a locally convex space and $B \subset X$ be a Banach disk. Then B is q-bounded in X.

PROOF. Take a 0−neighborhood U in X. Then $B \subset \cup\{nU \cap E_B; n \in N\}$. By the Category Argument $c\ell_{E_B}(U \cap E_B)$ contains a 0−neighborhood in E_B and thus it absorbs B.

Let X be an infinite dimensional Banach space and B its closed unit ball. Take a countably linearly independent subset $\{x_n; n \in N\}$ in B and denote by H, resp. K, a Hamel basis for X which contains $\{x_n; n \in N\}$, resp. $\{nx_1 - x_n; n \in N\}$. Let $\varphi : H \to K$ be a bijective map such that $\varphi(x_n) = nx_1 - x_n, n \in N$, and $\psi : X \to X$ the linear extension of φ to X. Then $\psi : X \to X$ is bijective, the space $E_B = X$ is Banach, and $\psi : E_B \to E_{\psi(B)}$ is a topological isomorphism. Hence $E_{\psi(B)}$ is also Banach and $\psi(B)$ is a Banach disk in X.

CLAIM 1. B is bounded in X and $\psi(B)$ is q-bounded in X.

PROOF. Clearly the unit ball B is bounded in X. By the Proposition, the Banach disk $\psi(B)$ is q-bounded in X.

CLAIM 2. The spaces $E_{B+\psi(B)}$ and $E_{B\cup\psi(B)}$ are not Hausdorff. Consequently, the sets $B+\psi(B)$ and $B\cup\psi(B)$ are not q-bounded in any locally convex space.

PROOF. The space $E_{B+\psi(B)}$ is not Hausdorff since $nx_1 = x_n + (nx_1 - x_n) \in B + \psi(B), n \in N$.

For any sets $C, D \subset E$, and $c \in C, d \in D$, we have $c + d = 2(\frac{1}{2}c + \frac{1}{2}d) \in 2\ abco(C \cup D)$, which implies $C \cup D \subset C + D \subset 2\ abco(C \cup D)$. Hence the identity map: $E_{C+D} \to E_{C\cup D}$ is a topological isomorphism. Since the space $E_{B+\psi(B)}$ is not Hausdorff, the space $E_{B\cup\psi(B)}$ is not Hausdorff either.

CLAIM 3. Let $A = c\ell_X\psi(B)$. Then the space E_A is not Hausdorff. Consequently, the set A is not q-bounded in any locally convex space.

PROOF. Assume that E_A is Hausdorff and take $x \in E_A, x \neq 0$. Then there exists $\alpha > 0$ such that $x \notin \alpha A$. Since αA is closed in X, there exists a 0-neighborhood U in X for which $(x+U)\cap\alpha A = \emptyset$. The set B is bounded in X, hence $\beta B \subset U$ for some $\beta > 0$. Then $(x+\beta B)\cap\alpha A = \emptyset$ and $x \notin \alpha A + \beta B$, which implies $x \notin \gamma(A + B)$, where $\gamma = min(\alpha, \beta)$. Thus E_{A+B} is also a Hausdorff space. Now, $\psi(B) + B \subset A + B$ and the topology of $E_{\psi(B)+B}$ is finer than that of E_{A+B}. Hence the space $E_{\psi(B)+B}$ is Hausdorff too, a contradiction with Claim 2.

In [1], it is stated in Propositions 2.5 and 2.6 that the union of two q-bounded sets and the closure of a q-bounded set are both q-bounded. The above example shows that it is not true. The problem is in the preservation of Property (b) in the definition of q-bounded sets. Thus a natural correction of those Propositions reads as follows:

PROPOSITION. Let A, B be q-bounded sets in a locally convex space X.

(a) If either the space E_{A+B} or the space $E_{A\cup B}$ is Hausdorff, then both are Hausdorff and both sets $A + B, A \cup B$, are q-bounded in X.

(b) If $B \subset X$ and the space E_D, where $D = c\ell_X B$, is Hausdorff, then D is q-bounded in X.

PROOF. (a) From the the proof of Claim 2, we know that the spaces E_{A+B} and $E_{A\cup B}$ are topologically isomorphic. So the first statement holds.

Take a convex 0-neighborhood U in X. There is $\lambda > 0$ such that $A \subset \lambda c\ell_{E_A}(U \cap A) \subset \lambda c\ell_{E_{A+B}}(U \cap (A + B))$ and $B \subset \lambda c\ell_{E_B}(U \cap B) \subset c\ell_{E_{A+B}}(U \cap (A + B))$. Similarly $A \cup B \subset \lambda c\ell_{E_{A\cup B}}(U \cap (A \cup B))$. Hence both sets $A + B$, $A \cup B$, are q-bounded in X.

(b) Let U and λ be the same as in (a). Since the topology of E_B is finer than that of E_D, we have $B \subset \lambda c\ell_{E_B}(U \cap E_B) \subset \lambda c\ell_{E_D}(U \cap E_B) \subset \lambda c\ell_{E_D}(U \cap E_D)$.

For $x \in D$ there exists $y \in B$ such taht $x - y \in U$. Then $x = (x - y) + y \in (U \cap E_D) + B \subset c\ell_{E_D}(U \cap E_D) + \lambda c\ell_{E_D}(U \cap E_D) = (1 + \lambda)c\ell_{E_D}(U \cap E_D)$. Hence $c\ell_{E_D}(U \cap E_D)$ absorbs D and D is q-bounded in X.

References

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