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ON ISOMORPHISMS AND HYPER-REFLEXIVITY OF CLOSED SUBSPACE LATTICES

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ABSTRACT. There are some papers, such as [1], [2] and [3], in which some properties on isomorphism of closed subspace lattices of Hilbert spaces were studied. In this short paper we will point out that the hyper-reflexivity of closed subspace lattice is invariant under isomorphism of $\xi(H_1)$ on $\xi(H_2)$. We also proved that if T is in $L(H)$ such that $0 \overline{\in} \pi(T)$ and $\mathfrak{F}$ is a hyper-reflexive subspace lattice, then $\phi_T(\mathfrak{F}) \cup \{H\}$ is hyper-reflexive where ϕ_T is a homomorphism induced by T.

KEY WORDS AND PHRASES: Reflexivity, Hyper-reflexivity, Isomorphism.
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1. INTRODUCTION.

Let H be a complex Hilbert space, $L(H)$ denotes the set of all bounded linear operators on H and let $\xi(H)$ be the set of all closed subspaces of H. For any subset $\mathcal{A}$ of $L(H)$ and any family $\mathfrak{F} \subseteq (H)$, let $\mathcal{A}$ denote the lattice of closed subspaces invariant for $\mathcal{A}$ (or the lattice of invariant projections for $\mathcal{A}$) and let Alg $\mathfrak{F}$ be the set of all operators invariant for $\mathfrak{F}$. $\mathfrak{F}$ is said to be <u>reflexive</u> if Lat Alg $\mathfrak{F} = \mathfrak{F}$. A subalgebra $\mathcal{A}$ of $L(H)$ is said to be <u>reflexive</u> if $\mathcal{A} =$ Alg Lat $\mathcal{A}$.

Let $\mathcal{A}$ be a reflexive algebra and let $T \in L(H)$. It is easy to see that

$$\text{dist }(T,\mathcal{A}) \geq \sup \{ \| P^{\perp} TP \| : P \in \text{ Lat } \mathcal{A}\}.$$

$\mathcal{A}$ is called to be hyper-reflexive (See [4]) if there exists a constant $K > 0$ such that for any $T \in L(H)$

$$\text{dist }(T,\mathcal{A}) \leq \text{K} \sup \{ \| P^{\perp} TP \| : P \in \text{ Lat } \mathcal{A}\}.$$

For any subspace lattice $\mathfrak{F} \subseteq \xi(H)$, if $\mathfrak{F}$ is reflexive and Alg $\mathfrak{F}$ is hyper-reflexive, then $\mathfrak{F}$ is said to be hyper-reflexive. Let ϕ be a lattice isomorphism of $\xi(H_1)$ onto $\xi(H_2)$ (i.e., a bijection with the property that $M = N$ iff $\phi(M) = \phi(N)$). It was proved in [1] and [2] that $\mathcal{F}$ is reflexive if and only if $\phi(\mathcal{F})$ is reflexive. For an operator $A \in L(H)$, A can give rise to a map $\phi_A : \xi(H) \to \xi(H)$ given by $\phi_A(M) = \overline{AM}$, where '$-$' denotes norm closure. In the paper [3] the author proved that if the approximate point spectrum $\pi(A)$ of A does not contain 0 and $\mathcal{F}$ is reflexive, then $\phi_A(\mathcal{F}) \cup \{H\}$ is also reflexive. Here we prove the following theorem:

LEMMA 1. Let H_1 and H_2 be two complex Hilbert spaces, and let ϕ be a lattice isomorphism of $\xi(H_1)$ onto $\xi(H_2)$. Then a subspace lattice $\mathcal{F}$ of $\xi(H_1)$ is hyper-reflexive if and only if $\phi(\mathcal{F})$ is hyper-reflexive.

THEOREM 2. Let H be a complex Hilbert space, $A \in L(H)$ and $0 \overline{\in} \pi(A)$. If the subspace lattice $\mathcal{F}$ of $\xi(H)$ is hyper-reflexive, then so is $\phi_A(\mathcal{F})U\{H\}$.

2. THE PROOF OF THE THEOREMS.

Lemma 1 may be known, we give a proof by the following theorem:

THEOREM A. ([4]) Let $\mathcal{A} \subseteq L(H)$ be a σ-weakly closed unital subalgebra of $L(H)$. Then $\mathcal{A}$ is hyper-reflexive iff every element $f \in \mathcal{A}_\perp$ has a representation

$$f = \sum_{n=1}^{\infty} f_n$$

where $\mathcal{A}_\perp = \left\{ f : f \text{ is a } \sigma\text{-weakly continuous linear functional on } L(H) \text{ and } f(\mathcal{A}) = \{0\} \right\}$, each f_n is an elementary functional in $\mathcal{A}_\perp$ and $\sum_{n=1}^{\infty} \| f_n \| < \infty$.

REMARK. A σ-weakly continuous functional f on $L(H)$ is said to be elementary if there exist $x, y \in H$ suc that $f(T) = (Tx, y)$ for any $T \in L(H)$. We write $f = (x \otimes y)$.

Let S be a conjugate linear continuous map from H_1, into H_2. It can be defined uniquely the adjoint S^* of S by the formula

$$(S^*x, y) = (Sy, x) = \overline{(x, Sy)}. \tag{2.1}$$

It is easy to check that $(S^*)^* = S$, and $(S^{-1})^* = (S^*)^{-1}$ when S has continuous inverse.

PROOF OF LEMMA 1. It is sufficient to prove the necessity. First $\phi(\mathcal{F})$ is reflexive by the reflexivity of $\mathcal{F}$. If dim $H_1 < \infty$, it is easy to prove that Alg $\phi(\mathcal{F})$ is hyper-reflexive ([5]). Now suppose that dim $H_1 = \infty$. Then there exists a bicontinuous linear or conjugate linear bijection $S: H_1 \to H_2$ such that $\phi = \phi_S$ i.e., $\phi(M) = SM$ for every $M \in \xi(H_1)$ (see [1]).
We may suppose that S is conjugate linear. For any $f \in (\text{ Alg } \phi(\mathcal{F}))_\perp$, define $g(A) = \overline{f(SAS^{-1})}$, then $g \in (\text{Alg } \mathcal{F})_\perp$ since Alg $\phi(\mathcal{F}) = S(\text{ Alg } \mathcal{F})S^{-1}$.
By theorem A, there exist $x_n, y_n \in H_1$ such that $(x_n \otimes y_n) \in (\text{ Alg } \mathcal{F})_\perp$,

$$g = \sum_{n=1}^{\infty} (x_n \otimes y_n)$$

$$\sum_{n=1}^{\infty} \| x_n \| \, \| y_n \| < \infty.$$

For any $T \in L(H_2)$

$$f(T) = \overline{g(S^{-1}TS)} = \sum_{n=1}^{\infty} \overline{(S^{-1}TSx_n, y_n)}$$

$$= \sum_{n=1}^{\infty} \overline{((S^{-1})^* y_n, TSx_n)}$$

$$= \sum_{n=1}^{\infty} (TSx_n, (S^*)^{-1} y_n).$$

Let $u_n = Sx_n$, $v_n = (S^*)^{-1} y_n$, then

$$f = \sum_{n=} (u_n, \otimes v_n)$$

and $(u_n \otimes v_n) \in (\text{Alg } \phi(\mathcal{F}))_{\perp}$, $\sum_{n=1}^{\infty} \| u_n \| \; \| v_n \| < \infty$. And therefore Alg $\phi(\mathcal{F})$ is hyperreflexive by

Theorem A. The proof is complete.

A PROOF OF THEOREM 2. Since $0 \overline{\in} \pi(A)$, $R(A)$, the range of A, is a closed subspace of H. Let $H_1 = H(A)$, then ϕ_A defines a lattice isomorphism from $\xi(H)$ onto $\xi(H_1)$. From Lemma 1 we have that $\text{Alg}_{H_1} \phi_A(\mathcal{F}) = \{T \in L(H_1)$: $\phi_A(\mathcal{F}) \subseteq \text{Lat}(T)$ is hyper-reflexive. By the definition of hyper-reflexivity, there exists $K > 0$ such that for any $T \in L(H_1)$

$$\text{dist}\,(T, \; \text{Alg}_{H_1} \phi_A(\mathcal{F}))$$

$$\leq K \sup \; \{ \| (I_{H_1} - \overline{P}_{\phi_A(M)}) T \overline{P}_{\phi_A(M)} \| : M \in \mathcal{F} \}$$

where $P_{\phi_A(M)}$ denotes the orthogonal projection from H_1 onto $\phi_A(M)$. For any $T \in \text{Alg}_{H_1}$ $(\phi_A(\mathcal{F}))$ and $S \in L(H)$, we define an operator $\widetilde{T} \in L(H)$ by formula

$$\widetilde{T}\,(x \otimes y) = Tx + Sy, \quad x \otimes y \; H = H_1 \otimes H_1^{\perp}.$$

Then $\widetilde{T} \in \text{Alg}\,(\phi_A(\mathcal{F}) \cup \{H\})$ since $T \in \text{Alg}_{H_1} \phi_A(\mathcal{F})$. Denote by E the orthogonal projection from H onto H_1, then

$$\text{dist}\,(S, \; \text{Alg}\,(\phi_A(\mathcal{F}) \cup \{H\}))$$

$$\leq \| S - T \|$$

$$\leq \| ES \,|_{H_1} - T \| + \| E^{\perp} SE \|$$

$$\leq \| ESE - T \| + \sup \; \{ P^{\perp}_{\phi_A(M)} S P_{\phi_A(M)} \| : M \in \mathcal{F} \}$$

and so

$$\text{dist}\,(S, \; \text{Alg}\,(\phi_A(\mathcal{F}) \cup \{H\}$$

$$\leq \text{dist}\,(ESE, \; \text{Alg}_{H_1} \phi_A(\mathcal{F})) + \sup \; \{ \| P^{\perp}_{\phi_A(M)} S P_{\phi_A(M)} \| : M \in \mathcal{F} \}$$

$$\leq (K+1) \sup \; \{ \| P_N^{\perp} S P_N \| : N \in \phi_A(\mathcal{F}) \cup \{H\} \}$$

which implies the hyper-reflexivity of $\text{Alg}_A(\mathcal{F}) \cup (H\}$. Together with the reflexivity of $\phi_A(\mathcal{F}) \cup \{H\}$

(see [3]), we obtain $\phi_A(\mathcal{F}) \cup \{H\}$ is hyper-reflexivity.

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