Internat. J. Math. & Math. Sci.
VOL. 15 NO. 1 (1992) 103-106



# $\Gamma$-GROUP CONGRUENCES ON REGULAR $\Gamma$-SEMIGROUPS

A. SETH
Department of Pure Mathematics, University of Calcutta
35, Ballygunge Circular Road, Calcutta - 700 019.
India.

(Received April 23, 1990 and in revised form September 16, 1990)

ABSTRACT. In this paper a $\Gamma$-group congruence on a regular $\Gamma$-semigroup is defined, some equivalent expressions for any $\Gamma$-group congruence on a regular $\Gamma$-semigroup and those for the least $\Gamma$-group congruence in particular are given.

KEY WORDS AND PHRASES. Regular $\Gamma$-semigroup, $\alpha$-idempotent, Right (left) $\Gamma$-ideal, Right (left) simple $\Gamma$-semigroup, $\Gamma$-group, Congruence, Normal family.
1980 AMS SUBJECT CLASSIFICATION CODE. 20M.

## 1. INTRODUCTION.

Let S and $\Gamma$ be two nonempty sets, S is called a $\Gamma$-semigroup if for all $a,b,c \in S$, $\alpha,\beta \in \Gamma$ (i) $a\alpha b \in S$ and (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ hold. S is called regular $\Gamma$-semigroup if for any $a \in S$ there exist $a' \in S$, $\alpha,\beta \in \Gamma$ such that $a = a\alpha a'\beta a$. We say $a'$ is $(\alpha,\beta)$-inverse of a if $a = a\alpha a'\beta a$ and $a' = a'\beta a\alpha a'$ hold and in this case we write $a' \in V_\alpha^\beta(a)$. An element e of S is called $\alpha$-idempotent if $e\alpha e = e$ holds in S. A right (left) $\Gamma$-ideal of a $\Gamma$-semigroup S is a nonempty subset I of S such that $I\Gamma S \subseteq I$ ($S\Gamma I \subseteq I$). A $\Gamma$-semigroup S is said to be left (right) simple if it has no proper left (right) $\Gamma$-ideal. For some fixed $\alpha \in \Gamma$ if we define $a \circ b = a\alpha b$ for all $a,b \in S$ then S becomes a semigroup. We denote this semigroup by $S_\alpha$. Throughout our discussion we shall use the notations and results of Sen and Saha [1-2]. For the sake of completeness let us recall the following results of Sen and Saha [1].

THEOREM 1.1. $S_\alpha$ is a group if and only if S is both left simple and right simple $\Gamma$-semigroup. (Theorem 2.1 of [1]).

COROLLARY 1.2. Let S be a $\Gamma$-semigroup. If $S_\alpha$ is a group for some $\alpha \in \Gamma$ then $S_\alpha$ is a group for all $\alpha \in \Gamma$. (Corollary 2.2 of [1]).

A $\Gamma$-semigroup S is called a $\Gamma$-group if $S_\alpha$ is a group for some (hence for all) $\alpha \in \Gamma$.

THEOREM 1.3. A regular $\Gamma$-semigroup S will be a $\Gamma$-group if and only if for all $\alpha,\beta \in \Gamma$, $e\alpha f = f\alpha e = f$ and $e\beta f = f\beta e = e$ for any two idempotents $e = e\alpha e$ and $f = f\beta f$ of S. (Theorem 3.3 of [1]).

## 2. $\Gamma$-GROUP CONGRUENCES IN A REGULAR $\Gamma$-SEMIGROUP.

An equivalence relation $\rho$ on a $\Gamma$-semigroup S is called a congruence if $(a,b) \in \rho$ implies $(c\alpha a, c\alpha b) \in \rho$ and $(a\alpha c, b\alpha c) \in \rho$ for all $a,b,c \in S$, $\alpha \in \Gamma$. A congruence $\rho$ in a regular $\Gamma$-semigroup S is called $\Gamma$-group congruence if $S/\rho$ is a $\Gamma$-group (In $S/\rho$ we define $(a\rho)\alpha(b\rho) = (a\alpha b)\rho$). Henceforth we shall assume S to be a regular $\Gamma$-semigroup and $E_\alpha$ to be its set of $\alpha$-idempotents.

A family $\{K_\alpha : \alpha \in \Gamma\}$ of subsets of S is said to be a normal family if

(i) $E_\alpha \subseteq K_\alpha$ for all $\alpha \in \Gamma$ ;

(ii) for each $a \in K_\alpha$ and $b \in K_\beta$, $a\alpha b \in K_\beta$ and $a\beta b \in K_\alpha$ ;

(iii) for each $a' \in V_\alpha^\beta(a)$ and $c \in K_\gamma$, $a\alpha c\gamma a'$ and $a\gamma c\alpha a' \in K_\beta$.

Now let $e \in E_\alpha$ and $f \in E_\beta$ and $\mu \in \Gamma$. Let $x \in V_\theta^\phi(e\mu f)$. Then $f\theta x\phi e \in E_\mu$. Thus $E_\mu \neq \phi$ for all $\mu \in \Gamma$, consequently $K_\mu \neq \phi$ for all $\mu \in \Gamma$. We further note that in an orthodox Γ-semigroup S of Sen and Saha [2] $\{E_\alpha : \alpha \in \Gamma\}$ is a normal family of S.

Let N be the collection of all normal families $K_i$ of $S(i \in \Lambda)$ where $K_i = \{K_{i\alpha} : \alpha \in \Gamma\}$. Let $U_\alpha = \bigcap_{i \in \Lambda} K_{i\alpha}$ and $U = \{U_\alpha : \alpha \in \Gamma\}$. Then obviously $E_\alpha \subseteq U_\alpha$. Also if $a \in U_\alpha$, $b \in U_\beta$, then $a \in K_{i\alpha}$ for all $i \in \Lambda$, $b \in K_{i\beta}$ for all $i \in \Lambda$. Thus $a\alpha b \in K_{i\beta}$ and $a\beta b \in K_{i\alpha_\beta}$ for all $i \in \Lambda$ implying $a\alpha b \in U_\beta$ and $a\beta b \in U_\alpha$. Similarly we can show that if $a' \in V_\alpha^\beta(a)$ and $c \in U_\alpha$ then $a\alpha c\gamma a'$, $a\gamma c\alpha a' \in U_\beta$. Thus U is a normal family of subsets of S and U is the least member in N if we define a partial order in N by $K_i \leq K_j$ iff $K_{i\alpha} \subseteq K_{j\alpha}$ for all $\alpha \in \Gamma$. We also observe that when S is orthodox Γ-semigroup, $U = \{E_\alpha : \alpha \in \Gamma\}$.

THEOREM 2.1. Let S be a regular Γ-semigroup. Then for each $K = \{K_\alpha : \alpha \in \Gamma\} \in N$, $\rho_K = \{(a,b) \in S \times S : a\alpha e = f\beta b$ for some $\alpha, \beta \in \Gamma$ and $e \in K_\alpha$, $f \in K_\beta\}$ is a Γ-group congruence in S.

PROOF. Let $a \in S$ and $a' \in V_\alpha^\beta(a)$. Then $a\alpha(a'\beta a) = (a\alpha a')\beta a$ implies $(a,a) \in \rho_K$. Next let $(a,b) \in \rho_K$. Then there exist $e \in K_\alpha$, $f \in K_\beta$ for some $\alpha, \beta \in \Gamma$ such that $a\alpha e = f\beta b$. Let $a' \in V_\gamma^\delta(a)$ and $b' \in V_\theta^\phi(b)$ such that $b\theta((b'\phi f\beta b)\gamma(a'\delta a))$ $= ((b\theta b')\phi(a\alpha e\gamma a'))\delta a$. But $b'\phi f\beta b \in K_\theta$, $a'\delta a \in K_\gamma$ and so $(b'\phi f\beta b)\gamma(a'\delta a) \in K_\theta$, and $b\theta b' \in K_\phi$, $a\alpha e\gamma a' \in K_\delta$ and so $(b\theta b')\phi(a\alpha e\gamma a') \in K_\delta$. Consequently, $(b,a) \in \rho_K$. Now let $(a,b) \in \rho_K$, $(b,c) \in \rho_K$. Then there exist $\alpha,\beta,\gamma,\delta \in \Gamma$, $e \in K_\alpha$, $f \in K_\beta$, $g \in K_\gamma$, $h \in K_\delta$ such that $a\alpha e = f\beta b$ and $b\gamma g = h\delta c$. But $a\alpha(e\gamma g) = (a\alpha e)\gamma g = (f\beta b)\gamma g = f\beta(b\gamma g)$ $= f\beta(h\delta c) = (f\beta h)\delta c$ where $e\gamma g \in K_\alpha$ and $f\beta h \in K_\delta$. Thus $(a,c) \in \rho_K$ and consequently $\rho_K$ is an equivalence relation. Let $(a,b) \in \rho_K$, $\theta \in \Gamma$, $c \in S$. Then $a\alpha e = f\beta b$ for some $\alpha,\beta \in \Gamma$ and some $e \in K_\alpha$, $f \in K_\beta$. Let $c' \in V_\gamma^\delta(c)$, $y \in V_{\gamma_1}^{\delta_1}(b\theta c)$, $x \in V_{\gamma_2}^{\delta_2}(a\theta c)$. Now $(a\theta c)\gamma(c'\delta((c\gamma_2 x\delta_2 a)\alpha e)\theta c)\gamma_1(y\delta_1(b\theta c)) = (a\theta c\gamma_2 x)\delta_2 f\beta(b\theta c\gamma_1 y)\delta_1(b\theta c)$. But $c\gamma_2 x\delta_2 a \in E_\theta \subseteq K_\theta$, so $(c\gamma_2 x\delta_2 a)\alpha e \in K_\theta$, $c'\delta((c\gamma_2 x\delta_2 a)\alpha e)\theta c \in K_\gamma$. Again $y\delta_1(b\theta c) \in E_{\gamma_1} \subseteq K_{\gamma_1}$ and consequently $(c'\delta((c\gamma_2 x\delta_2 a)\alpha e)\theta c)\gamma_1(y\delta_1 b\theta c) \in K_\gamma$. By a similar argument we can show that $(a\theta c\gamma_2 x)\delta_2 f\beta(b\theta c\gamma_1 y) \in K_{\delta_1}$. Thus $(a\theta c, b\theta c) \in \rho_K$. Also it is immediate from the foregoing by duality that $(c\theta a, c\theta b) \in \rho_K$. Thus $\rho_K$ is a congruence on S. Also as S is regular, $S/\rho_K$ is a regular Γ-semigroup. Let $e \in E_\alpha$, $f \in E_\beta$. Then $e\alpha f$, $f\alpha e \in K_\beta$, $e\beta f$, $f\beta e \in K_\alpha$. Now $(e\alpha f)\beta f = (e\alpha f)\beta f$ shows that $(e\alpha f, f) \in \rho_K$ and $(f\alpha e)\beta f = (f\alpha e)\beta f$ implies that $(f\alpha e, f) \in \rho_K$. Thus $(e\rho_K)\alpha(f\rho_K) = f\rho_K$ and $(f\rho_K)\alpha(e\rho_K) = f\rho_K$. Similarly we can show $(e\rho_K)\beta(f\rho_K) = e\rho_K$ and $(f\rho_K)\beta(e\rho_K) = e\rho_K$. So it follows from Theorem 1.3 that $S/\rho_K$ is a Γ-group. Thus $\rho_K$ is a Γ-group congruence on S.

For any normal family $K = \{K_\alpha : \alpha \in \Gamma\}$ of S, the closure KW of K is the family defined by $KW = \{(KW)_\gamma : \gamma \in \Gamma\}$ where $(KW)_\gamma = \{x \in S : e\alpha x \in K_\gamma$ for some $\alpha \in \Gamma$ and $e \in K_\alpha\}$. We call K closed if $K = KW$.

THEOREM 2.2. For each $K \in N$, $\rho_K = \{(a,b) \in S \times S : a\gamma b' \in (\ \ )_\delta$ for some $b' \in V_\gamma^\delta(b)\}$.

PROOF. Let $(a,b) \in \rho_K$. Then $f\beta a = b\alpha e$ for some $\alpha,\beta \in \Gamma$ and $e \in K_\alpha$, $f \in K_\beta$. Then $f\beta(a\gamma b') = b\alpha e\gamma b' \in K_\delta$ for some $b' \in V_\gamma^\delta(b)$. Consequently $a\gamma b' \in (KW)_\delta$. Conversely, let $a\gamma b' \in (KW)_\delta$ for some $b' \in V_\gamma^\delta(b)$. Then $e\alpha a\gamma b' \in K_\delta$ for some $\alpha \in \Gamma$ and $e \in K_\alpha$. Therefore $e\alpha a\gamma b' = f$ where $f \in K_\delta$. So $(b\theta(a'\phi e\alpha a)\gamma b')\delta a = b\theta(a'\phi f\delta a)$, for some $a' \in V_\theta^\phi(a)$ where $b\theta(a'\phi e\alpha a)\gamma b' \in K_\delta$ and $a'\phi f\delta a \in K_\theta$. Consequently $(a,b) \in \rho_K$.

For any congruence ρ on S, let $\ker\rho = \{(\ker\rho)_\alpha : \alpha \in \Gamma\}$ where $(\ker\rho)_\alpha = \{x \in S : e\rho x$ for some $e \in E_\alpha\}$.

LEMMA 2.3. For any $K \in N$, $\ker \rho_K = KW$.

PROOF. To prove $\ker \rho_K = KW$, we are to show that $(\ker \rho_K)_\alpha = (KW)_\alpha$ for all $\alpha \in \Gamma$. For this let $x \in (\ker \rho_K)_\alpha$ for some $\alpha \in \Gamma$. Then $e\rho_K x$ for some $e \in E_\alpha$ that is $e\beta f = g\gamma x$ for some $\beta,\gamma \in \Gamma$, $e \in E_\alpha$, $f \in K_\beta$, $g \in K_\gamma$. So $g\gamma x \in K_\alpha$ as $e\beta f \in K_\alpha$. Thus $x \in (KW)_\alpha$. Next let $x \in (KW)_\alpha$. Then $g\gamma x \in K_\alpha$ for some $\gamma \in \Gamma$ and $g \in K_\gamma$. Now for some $e \in E_\alpha$ $e\alpha(g\gamma x) = (e\alpha g)\gamma x$ where $g\gamma x \in K_\alpha$ and $e\alpha g \in K_\gamma$. Thus $e\rho_K x$. Consequently $x \in (\ker \rho_K)_\alpha$. So $(\ker \rho_K)_\alpha = (KW)_\alpha$ for all $\alpha \in \Gamma$.

Let $K \in N$ and suppose $a\gamma b' \in (KW)_\delta$ for some $b' \in V_\gamma^\delta(b)$. Then $e\alpha a\gamma b' \in K_\delta$ for some $\alpha \in \Gamma$ and $e \in K_\alpha$. Then for any $a' \in V_\theta^\phi(a)$, $a'\phi(e\alpha a\gamma b')\delta a \in K_\theta$ and $(a'\phi e\alpha a\gamma b'\delta a)\theta a'\phi b = (a'\phi e\alpha a)\gamma b'\delta(a\theta a')\phi b \in K_\theta$. Thus $a'\phi b \in (KW)_\theta$. Conversely, suppose $a'\phi b \in (KW)_\theta$ for some $a' \in V_\theta^\phi(a)$. Then $f\beta(a'\phi b) \in K_\theta$ for some $\beta \in \Gamma$ and $f \in K_\beta$ and $a\theta(f\beta a'\phi b)\theta a' \in K_\phi$. Therefore for some $b' \in V_\gamma^\delta(b)$, $(a\theta f\beta a'\phi b\theta a')\phi(a\gamma b') = (a\theta f\beta a')\phi b\theta(a'\phi a)\gamma b' \in K_\delta$. Therefore $a\gamma b' \in (KW)_\delta$. Thus $a\gamma b' \in (KW)_\delta$ for some (all) $b' \in V_\gamma^\delta(b)$ iff $a'\phi b \in (KW)_\theta$ for some (all) $a' \in V_\theta^\phi(a)$. Interchanging roles of a and b we see that $b\theta a' \in (KW)_\phi$ for some (all) $a' \in V_\theta^\phi(a)$ iff $b'\delta a \in (KW)_\gamma$ for some (all) $b' \in V_\gamma^\delta(b)$. Moreover, the symmetric property of $\rho_K$ shows that $a\gamma b' \in (KW)_\delta$ for some (all) $b' \in V_\gamma^\delta(b)$ iff $b\theta a' \in (KW)_\phi$ for some (all) $a' \in V_\theta^\phi(a)$. Thus we have the following.

LEMMA 2.4. For each $K \in N$, $a\rho_K b$ iff one of the following equivalent conditions hold.

(i) $a\gamma b' \in (KW)_\delta$ for some (all) $b' \in V_\gamma^\delta(b)$.

(ii) $b'\delta a \in (KW)_\gamma$ for some (all) $b' \in V_\gamma^\delta(b)$.

(iii) $a'\phi b \in (KW)_\theta$ for some (all) $a' \in V_\theta^\phi(a)$.

(iv) $b\theta a' \in (KW)_\phi$ for some (all) $a' \in V_\theta^\phi(a)$.

Let $\overline{N}$ denote the collection of all closed families in N, then $\overline{N} \subseteq N$.

THEOREM 2.5. The mapping $K \to \rho_K = \{(a,b) \in S{\times}S : a\gamma b' \in K_\delta \text{ for some } b' \in V_\gamma^\delta(b)\}$ is a one to one order preserving mapping of $\overline{N}$ onto the set of $\Gamma$-group congruences on S.

PROOF. Let $\rho$ be a $\Gamma$-group congruence on S. Let us denote $\ker \rho$ by K and $(\ker \rho)_\alpha$ by $K_\alpha$. Then $K_\alpha = \{x \in S : x\rho e \text{ when } e \in E_\alpha\}$. Then $E_\alpha \subseteq K_\alpha$. Let $a \in K_\alpha$, $b \in K_\beta$ then $a\rho e$ and $b\rho f$ where $e \in E_\alpha$ and $f \in E_\beta$. Now $(a\alpha b)\rho = (a\rho)\alpha(b\rho) = (e\rho)\alpha(f\rho) = f\rho$. Thus $a\alpha b\rho f$, where $f \in E_\beta$. Thus $a\alpha b \in K_\beta$. Similarly $a\beta b \in K_\alpha$. Next let $a' \in V_\alpha^\beta(a)$ and $c \in K_\gamma$. Then $c\rho g$ where $g \in E_\gamma$. Then $(a\alpha c\gamma a')\rho = (a\rho)\alpha(c\rho)\gamma(a'\rho) = (a\rho)\alpha((g\rho)\gamma(a'\rho)) = (a\rho)\alpha(a'\rho) = (a\alpha a')\rho$. Thus $a\alpha c\gamma a'\rho a\alpha a'$ where $a\alpha a' \in E_\beta$. Hence $a\alpha c\gamma a' \in K_\beta$. Similarly $a\gamma c\alpha a' \in K_\beta$. Therefore K is a normal family of subsets of S. Next $(KW)_\gamma = \{x \in S : e\alpha x \in K_\gamma \text{ where } e \in K_\alpha \text{ for some } \alpha \in \Gamma\}$. Then $K_\gamma \subseteq (KW)_\gamma$. To show $(KW)_\gamma \subseteq K_\gamma$, let $x \in (KW)_\gamma$. Then $e\alpha x \in K_\gamma$ for some $\alpha \in \Gamma$ and $e \in K_\alpha$. Consequently $(e\alpha x)\rho = g\rho$ where $g \in E_\gamma$ or, $(e\rho)\alpha(x\rho) = g\rho$ or, $x\rho = g\rho$ or, $x \in K_\gamma$. Thus $(KW)_\gamma \subseteq K_\gamma$. Therefore $K = KW$ and so $K = \ker \rho \in \overline{N}$. Thus if $\rho$ is a $\Gamma$-group congruence, then $\ker \rho = K \in \overline{N}$. We shall now prove that $\rho_K = \rho$. If $(a,b) \in \rho_K$, then $a\gamma b' \in K_\delta$ for some $b' \in V_\gamma^\delta(b)$. Thus $a\gamma b'\ \rho\ h$ for some $h \in E_\delta$ and $a\rho = (a\rho)\gamma((b'\delta b)\rho) = (h\rho)\delta(b\rho) = b\rho$. Thus $\rho_K \subseteq \rho$. Conversely, if $(a,b) \in \rho$ and $b' \in V_\gamma^\delta(b)$, then $a\gamma b'\rho\, b\gamma b' \in E_\delta$ and so $(a,b) \in \rho_K$. Therefore $\rho = \rho_K$. Thus from above and by lemma 2.3 for any $K \in \overline{N}$, $K \to \rho_K$ is a one-to-one mapping from $\overline{N}$ onto the set of all $\Gamma$-group congruences on S. Also it is easy to see that $K \to \rho_K$ is an order preserving mapping.

Let $\tau$ be a $\Gamma$-group congruence on S, by the proof of Theorem 2.5 $\tau = \rho_K$, where $K = \ker \tau \in \overline{N}$. Thus each $\Gamma$-group congruence is of the form $\rho_K$ for some $K \in \overline{N} \subseteq N$.

Thus by lemma 2.3 we have,

THEOREM 2.6. The least Γ-group congruence $\sigma$ on S is given by $\sigma = \rho_U$ and $\ker\sigma = UW$.

THEOREM 2.7. For any Γ-group congruence $\rho_K$ with K in N, on a regular Γ-semigroup, the following are equivalent.

(i) $a\rho_K b$.

(ii) $a\mu x\gamma b' \in K_\delta$ for some $x \in K_\mu$ $(\mu \in \Gamma)$ and some (all) $b' \in V_\gamma^\delta(b)$.

(iii) $a'\phi x\mu b \in K_\theta$ for some $x \in K_\mu$ $(\mu \in \Gamma)$ and some (all) $a' \in V_\theta^\phi(a)$.

(iv) $b\mu x\theta a' \in K_\phi$ for some $x \in K_\mu$ $(\mu \in \Gamma)$ and some (all) $a' \in V_\theta^\phi(a)$.

(v) $b'\delta x\mu a \in K_\gamma$ for some $x \in K_\mu$ $(\mu \in \Gamma)$ and some (all) $b' \in V_\gamma^\delta(b)$.

(vi) $a\alpha e = f\beta b$ for some $\alpha,\beta \in \Gamma$ and some $e \in K_\alpha$, $f \in K_\beta$.

(vii) $e\alpha a = b\beta f$ for some $\alpha,\beta \in \Gamma$ and some $e \in K_\alpha$, $f \in K_\beta$.

(viii) $K_\beta\beta a\alpha K_\alpha \cap K_\beta\beta b\alpha K_\alpha \neq \phi$ for some $\alpha,\beta \in \Gamma$.

PROOF. (ii) => (iii) Suppose $a\mu x\gamma b' \in K_\delta$ for some $x \in K_\mu$ and $b' \in V_\gamma^\delta(b)$. Then for any $a' \in V_\theta^\phi(a)$, $a'\phi(a\mu x\gamma b')\delta b = (a'\phi a)\mu(x\gamma(b'\delta b)) \in K_\theta$ as $a'\phi a \in K_\theta$ and $x\gamma b'\delta b \in K_\mu$.
(iii) => (vi) Let $a'\phi x\mu b \in K_\theta$ for $a' \in V_\theta^\phi(a)$ and $x \in K_\mu$.
Then $a\theta(a'\phi x\mu b) = (a\theta a'\phi x)\mu b$ which is (vi) as $a'\phi x\mu b \in K_\theta$ and $a\theta a'\phi x \in K_\mu$.
(vi) => (viii) Let $a\alpha e = f\beta b$ for some $\alpha,\beta \in \Gamma$ and $e \in K_\alpha$, $f \in K_\beta$. Then we have $f\beta a\alpha e\alpha e = f\beta f\beta b\alpha e$ implying $K_\beta\beta a\alpha K_\alpha \cap K_\beta\beta b\alpha K_\alpha \neq \phi$.
(viii) => (ii) Let $K_\beta\beta a\alpha K_\alpha \cap K_\beta\beta b\alpha K_\alpha \neq \phi$. Then $x\beta a\alpha y = x_1\beta b\alpha y_1$ for some $x,x_1 \in K_\beta$, $y,y_1 \in K_\alpha$. If $a' \in V_\theta^\phi(a)$, $b' \in V_\gamma^\delta(b)$, then $a'\phi x\beta a \in K_\theta$ and $(a'\phi x\beta a)\alpha y \in K_\theta$ and we have, $a\theta(a'\phi x\beta a\alpha y)\gamma b' = (a\theta a')\phi(x\beta a\alpha y)\gamma b' = (a\theta a')\phi(x_1\beta b\alpha y_1)\gamma b' = (a\theta a')\phi x_1\beta(b\alpha y_1\gamma b') \in K_\delta$ as $b\alpha y_1\gamma b' \in K_\delta$, $x_1\beta(b\alpha y_1\gamma b') \in K_\delta$ and $a\theta a' \in K_\phi$.
Thus (ii), (iii), (vi) and (viii) are equivalent.
Interchanging the roles of a and b we see that (iv), (v), (vii) and (viii) are equivalent. Also (i) and (vi) are equivalent by Theorem 2.1. Thus all the conditions (i) - (viii) are equivalent.

COROLLARY 2.8. Let $\sigma$ denote the least Γ-group congruence on a regular Γ-semigroup S. Then the following are equivalent.

(i) $a\sigma b$.

(ii) $a\mu x\gamma b' \in U_\delta$ for some $x \in U_\mu (\mu \in \Gamma)$ and some (all) $b' \in V_\gamma^\delta(b)$.

(iii) $a'\phi x\mu b \in U_\theta$ for some $x \in U_\mu (\mu \in \Gamma)$ and some (all) $a' \in V_\theta^\phi(a)$.

(iv) $b\mu x\theta a' \in U_\phi$ for some $x \in U_\mu (\mu \in \Gamma)$ and some (all) $a' \in V_\theta^\phi(a)$.

(v) $b'\delta x\mu a \in U_\gamma$ for some $x \in U_\mu (\mu \in \Gamma)$ and some (all) $b' \in V_\gamma^\delta(b)$.

(vi) $a\alpha e = f\beta b$ for some $\alpha,\beta \in \Gamma$ and $e \in U_\alpha$, $f \in U_\beta$.

(vii) $e\alpha a = b\beta f$ for some $\alpha,\beta \in \Gamma$ and $e \in U_\alpha$, $f \in U_\beta$.

(viii) $U_\beta\beta a\alpha U_\alpha \cap U_\beta\beta b\alpha U_\alpha \neq \phi$ for some $\alpha,\beta \in \Gamma$.

**ACKNOWLEDGEMENT.** I express my earnest gratitude to Dr. M.K. Sen, Department of Pure Mathematics, University of Calcutta, for his guidance and valuable suggestions. I also thank C.S.I.R. for financial assistance during the preparation of this paper. I am also grateful to the learned referee for his valuable suggestions for the improvement of this paper.

REFERENCES

1. SEN, M.K. and SAHA, N.K., On Γ-semigroup-I. Bull.Cal. Math. Soc., 78 (1986), 180-186.

2. SEN, M.K. and SAHA, N.K., Orthodox Γ-semigroup. Internat. J. Math. & Math. Sci., to appear.