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SOME RESULTS ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS AND HADAMARD PRODUCT

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Abstract. By using a certain linear operator defined by a Hadamard product or convolution, several interesting subclasses of analytic functions in the unit disc are introduced and some unifying relationships between them are established. A variety of characterization results involving a certain functional and some general functions of hypergeometric type are investigated for these classes.

Key Words and Phrases: Analytic, Hadamard product, Hypergeometric functions univalent, starlike, convex, close-to-convex, Quasi-convex, Linear operator.
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1. INTRODUCTION. Let A denote the class of the function f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. A function $f \varepsilon A$ is said to be in the class $R(\beta)$ if, for $z \varepsilon E$ and $\beta > -1$,

$$Re \frac{zf'(z)}{f(z)} > -\beta$$

Also, a function $f \varepsilon A$ is said to belong to the class $V(\beta)$ if, for $z \varepsilon E$ and $\beta > -1$,

$$Re \frac{(zf'(z))'}{f'(z)} > -\beta .$$

We note that

$$f(z) \varepsilon R(\beta) \leftrightarrow zf'(z) \varepsilon V(\beta), \tag{1.2}$$

and $\quad v(\beta) \subset R(\beta)$.

The classes $V(\beta)$ and $R(\beta)$ of analytic functions have been defined and studied in [9].

We define the following.

Let $f \varepsilon A$ and let $g \varepsilon R(\beta)$. Then $f \varepsilon T(\alpha, \beta)$ if, for $\alpha > -1$ and $z \varepsilon E$, $Re \frac{zf'(z)}{g(z)} > -\alpha$.

Also, let $f \varepsilon A$. Then $f \varepsilon T^*(\alpha, \beta)$ if, for $\alpha > -1$, $z \varepsilon E$ and $g \varepsilon V(\beta)$,

$$Re\frac{(zf'(z))'}{g'} > -\alpha \tag{1.4}$$

From (1.3) and (1.4), it is clear that

$$f \in T^*(\alpha,\beta) \leftrightarrow zf' \in T(\alpha,\beta) \tag{1.5}$$

and

$$T^*(\alpha,\beta) \subset T(\alpha,\beta)$$

Let $f_j(z)$ $(j = 1,2)$ in A be given by

$$f_j(z) = \sum_{n=0}^{\infty} a_{n+1,j} z^{n+1} \qquad (a_{ij} = 1)$$

Then the Hadamard product (or convolution) $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{n+1,1} a_{n+1,2} z^{n+1} \tag{1.6}$$

Let $a_j (j = 1,\ldots,p)$ and $\beta_j (j = 1,2,\ldots,q)$ be complex numbers with $\beta_j \neq 0,-1,-2,\ldots, j = 1,\ldots,q$.

Then the generalized hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q(z) \equiv {}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n n!} z^n \qquad (p \le q+1) \tag{1.7}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{if } n = 0 \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & \text{if } n \in N = \{1,2,3\ldots\}. \end{cases}$$

We now define the function $\phi(a,c)$ by

$$\phi(a,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, \qquad (c \neq 0,-1,-2,\ldots, z \in E) \tag{1.8}$$

so that $\phi(a,c)$ is an incomplete Beta function with

$$\phi(a,c,z) = z\,{}_2F_1(1,a;c,z).$$

Corresponding to the function $\phi(a,c)$, Carlson and Shaffer [2] defined a linear operator $L(a,c)$ on A by the convolution

$$L(a,c)f = \phi(a,c) * f \tag{1.9}$$

for $f \in A$. Clearly, $L(a,c)$ maps A onto itself, and $L(c,a)$ is an inverse of $L(a,c)$ provided that $a \neq 0,-1,-2,\ldots$

Furthermore, $L(a,a)$ is the identity operator, and

$$R(\beta) = L(1,2)V(\beta), \text{ and } V(\beta) = L(2,1)R(\beta).$$

Also

$$T(\alpha,\beta) = L(1,2)T^*(\alpha,\beta), \text{ and } T^*(\alpha,\beta) = L(2,1)T(\alpha,\beta),$$

where $\alpha > -1$ and $\beta > -1$.

We can now define the classes of analytic function with which we shall be dealing.

Definition 1.1. A function $f \in A$ is said to be in the class $R(a,c;\beta)$ if $L(a,c)f$ belongs to $R(\beta)$ for $\beta > -1$, and $f \in V(a,c;\beta)$ if, and only if, $zf' \in R(a,c;\beta)$ for $\beta > -1$.

Similarly we have:

Definition 1.2. A function $f \varepsilon A$ is said to be in the class $T(a,c;\alpha,\beta)$ if $L(a,c)f \varepsilon T(\alpha,\beta)$ for $\alpha > -1$ and $\beta > -1$. Further $f \varepsilon T^*(a,c;\alpha,\beta)$ if, and only if, $zf' \varepsilon T(a,c;\alpha,\beta)$ for $\alpha > -1$.

The following relations can easily be verified.

$$V(a,c;\beta) = L(1,2)R(a,c;\beta)$$

$$R(a,c,\beta) = L(2,1)V(a,c,\beta)$$

$$V(\beta) = V(a,a;\beta) = L(1,2)R(a,a;\beta)$$

and

$$R(\beta) = R(a,a;\beta) = L(2,1)V(a,a;\beta)$$

Also

$$T^*(a,c;\alpha,\beta) = L(1,2)T(a,c;\alpha,\beta)$$

$$T(a,c;\alpha,\beta) = L(2,1)T^*(a,c;\alpha,\beta)$$

$$T^*(\alpha,\beta) = T^*(a,a;\alpha,\beta) = L(1,2)T(a,a;\alpha,\beta)$$

and

$$T(\alpha,\beta) = T(a,a;\alpha,\beta) = L(2,1)T^*(a,a;\alpha,\beta)$$

We shall now connect these classes with the univalent functions. A single-valued function f is said to be univalent in a domain D if it never takes on the same value twice. By S, K, S^*, C and C^* denote the subclasses of A which are respectively univalent, close-to-convex, starlike, convex and quasi-convex in E. In [8], Robertson defined the subclasses of C and S^* by using the order of the class as follows. A function $f \varepsilon S$ is called a convex function of order $\beta_1, 0 \le \beta_1 < 1$, if and only if $Re \frac{(zf'(z))'}{f'(z)} > \beta_1$, $z \varepsilon E$. We denote this class as $C(\beta_1)$. Also a function $f \varepsilon S$ is called starlike function of order $\beta_1, 0 \le \beta_1 < 1$ if and only if $Re \frac{zf'(z)}{f(z)} > \beta_1$, $z \varepsilon E$. We call this class $S^*(\beta_1)$. Obviously

$$f \varepsilon C(\beta_1) \leftrightarrow zf' \varepsilon S^*(\beta_1)$$

Libera [3] introduced the terminology of order and type together in the class $K(\alpha_1,\beta_1)$ of close-to-convex functions. A function $f \varepsilon a$ is said to be close-to-convex of order α_1 type β_1, $0 \le \alpha_1 < 1$; $0 \le \beta_1 < 1$, if and only if there exists a function $g \varepsilon S^*(\beta_1)$ such that $Re \frac{zf'(z)}{g(z)} > \alpha_1$, $z \varepsilon E$. Further $f \varepsilon C^*(\alpha_1,\beta_1) \leftrightarrow zf' \varepsilon K(\alpha_1,\beta_1)$ we refer to [7].

Indeed from the above definitions of the various subclasses of the various subclasses of A, we deduce readily the following:

$$S^*(\beta_1) \subset S^* \subset R(\beta) \subset A,$$

$$C(\beta_1) \subset C \subset V(\beta) \subset R(\beta) \subset A$$

and

$$C^*(\alpha_1,\beta_1) \subset C^* \subset T^*(\alpha,\beta) \subset T(\alpha,\beta) \subset A,$$

$$K(\alpha_1,\beta_1) \subset K \subset T(\alpha,\beta) \subset A,$$

where

$$0 \le \alpha_1 < 1, \quad 0 \le \beta_1 < 1 \quad \text{and} \quad -1 < -\alpha_1 \le \alpha; \quad -1 < -\beta_1 \le \beta.$$

2. MAIN RESULTS

We first state certain results which will be needed in proving our main theorems.

__Lemma 2.1__. [6] Let $\phi(u,v)$ be the complex function, $\phi: D \to C, D \subset C \times C$ (C-complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function ϕ satisfies the conditions:

(i) $\phi(u,v)$ is continuous in D;

(ii) $(1,0) \varepsilon D$ and $Re\{\phi(0,1)\} > 0$;

(iii) $Re\{\phi(iu_2, v_1)\} < 0$ for all $(iu_2, v_1) \varepsilon D$ and such that $v_1 \le (1 + u_2^2)/2$.

Let $h(z) = 1 + c_1 z + \ldots$ be analytic in E, such that $(h(z), zh'(z)) \varepsilon E$ for all $z \varepsilon E$. If $Re\{\phi(h(z), zh'(z))\} > 0 (z \varepsilon E)$, then $Re h(z) > 0$ for $z \varepsilon E$.

Let $I_\lambda(f)$ denote a functional defined by

$$I_\lambda(f) = \frac{\lambda+1}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt \tag{2.1}$$

for $f \varepsilon A$ and for a real number $\lambda > 1$. The functional $I_\lambda(f)$, when $\lambda \varepsilon N$, was studied by Bernardi [1], and in particular, $I_1(f)$ was considered earlier by Libera [4] and Livingston [5]. We note that $I_\lambda(f)$ is a particular solution of the ordinary first order differential equation

$$tg'(t) + \lambda g(t)(+ + 1)f(t)$$

at the point $t = z$. Also by comparing (1.9) and (2.1), we have $I_\lambda(f) = L(\lambda+2, \lambda+1)f$. For our next results we refer to [9].

__Theorem 2.1__. Let $g \varepsilon R(a,c;\beta)$ and let, for $\lambda \ge \beta > -1$, $I_\lambda(g)$ be defined by (2.1). The $I_\lambda(g)$ is also in the class $R(a,c;\beta)$.

We shall now prove the following.

__Theorem 2.2__. Let $f \varepsilon T(a,c;\alpha,\beta)$ and let, for $\lambda \ge \alpha, \beta > -1$, $I_\lambda(f)$ be defined by (2.1). Then $I_\lambda(f) \varepsilon T(a,c;\alpha,\beta)$.

__Proof__: Since $f \varepsilon T(a,c;\alpha,\beta)$, there exists $g \varepsilon R(a,c,\beta)$ such that

$$Re\left\{\frac{z[L(a,c)f(z)]'}{L(a,c)g(z)}\right\} > -\alpha$$

Now, from Theorem 2.1, we know that $I_\lambda(g) \varepsilon R(a,c;\beta)$. Let

$$\frac{z[L(a,c)I_\lambda(f)]'}{L(a,c)I_\lambda(g)} = (1+a)h(z) - \alpha, \tag{2.2}$$

where $$h(z) = 1 + c_1 z + c_2 z^2 + \ldots..$$

Note that

$$z[L(a,c)I_\lambda(f)]' = (\lambda+1)L(a,c)f(z) - \lambda L(a,c)I_\lambda(f) \tag{2.3}$$

which readily yields

$$z^2[L(a,c)I_\lambda(f)]'' = (\lambda+1)z[L(a,c)f(z)]' - (\lambda+1)z[L(a,c)I_\lambda(f)] \tag{2.4}$$

Now, differentiating both sides of (2.2) logarithmically and using (2.3) and (2.4), we obtain

$$\frac{(\lambda+1)z[L(a,c)f(z))]'}{z[L(a,c)I_\lambda(f)]'} - \frac{(\lambda+1)L(a,c)g(z)}{L(a,c)I_\lambda(g)} = \frac{(1+\alpha)zh'(z)}{(1+da)h(z)-\alpha}$$

or, equivalently,

$$\frac{(\lambda+1)L(a,c)g(z)}{z[L(a,c)I_\lambda(f)]'}\left\{\frac{z[L(a,c)f(z)]'}{L(a,c)g(z)}-\frac{z[L(a,c)I_\lambda(f)]'}{L(a,c)I_\lambda(g)}\right\}=\frac{(1+\alpha)zh'(z)}{(1+\alpha)h(z)-\alpha} \tag{2.5}$$

After simplification, and taking

$$\frac{z[L(a,c)I_\lambda(g)]'}{L(a,c)I_\lambda(g)}=(1+\beta)H(z)-\beta,$$

where $ReH(z)=h_1>0$ and $\beta>-1$, we have, from (2.5),

$$\frac{z[L(a,c)f(z)]'}{L(a,c)g(z)}=(1+\alpha)h(z)-\alpha+\frac{(1+\alpha)zh'(z)}{(1+\beta)H(z)-\beta+\lambda}$$

or

$$\frac{z[L(a,c)f(z)]'}{L(a,c)g(z)}+\alpha=(1+\alpha)h(z)+\frac{(1+\alpha)zh'(z)}{(1+\beta)H(z)-\beta+\lambda} \tag{2.6}$$

We form the function $\phi(u,v)$ by taking

$$u=h(z) \quad \text{and} \quad v=zh'(z)$$

in (2.6) as

$$\phi(u,v)=(1+\alpha)u+\frac{(1+\alpha)v}{(1+\beta)H(z)-\beta+\lambda} \tag{2.7}$$

It is clear that the function $\phi(u,v)$ defined by (2.7) satisfies conditions (i) and (ii) of Lemma 2.1 easily. To verify condition (iii), we proceed as follows.

$$Re\phi(iu_2,v_1)=\frac{(1+\alpha)v_1\{(1+\beta)h_1-\beta+\lambda\}}{[(1+\beta)h_1-\beta+\lambda]^2+[(1+\beta)h_2]^2}$$

where $H(z)=h_1+ih_2$, h_1 and h_2 being the functions of x and y and $ReH(z)=h_1>0$.

By putting $v_1\le-\frac{1}{2}(1+u_2^2)$, we obtain

$$Re\phi(iu_2,v_1)\le-\frac{(1+da)(1+u_2^2)\{(1+\beta)h_1-\beta+\lambda\}}{[(1+\beta)h_1-\beta+\lambda]^2+[(1+\beta)h_2]^2}\le 0$$

Hence, by Lemma 2.1, $Reh(z)>0$ and this implies that $I_\lambda(f)\in T(a,c;\alpha,\beta)$. This proves our theorem.

Corollary 2.1. Let $f\in T(a,c;\alpha,\beta)$. Then, for $\lambda\ge\alpha,\beta>-1$ $L(a,c)I_\lambda(f)\in K$

Proof: From Theorem 2.2, we clearly see that $L(a,c)I_\lambda(f)\in K$. The second assertion follows easily from the fact that

$$L(a,c)I_\lambda(f)=I_\lambda(L(a,c)f(z)).$$

Next we have:

Theorem 2.3. Let $f\in T^*(a,c;\alpha,\beta)$. Then for $\lambda\ge\alpha,\beta>-1$, $I_\lambda(f)$ also belongs to $T^*(a,c;\alpha,\beta)$.

Proof: Since

$$f\in T^*(a,c;\alpha,\beta)\leftrightarrow zf'\in T(a,c;\alpha,\beta),$$

we observe, using Theorem 2.2, that

$$I_\lambda(zf')\in T(a,c;\alpha,\beta).$$

and this implies that

$$z(I_\lambda(f))' \varepsilon T(a,c;\alpha,\beta) .$$

Hence $I_\lambda(f) \varepsilon T^*(a,c;\alpha,\beta)$. This completes the proof.

Corollary 2.2. Let $f \varepsilon T^*(a,c;\alpha,\beta)$. Then, for $\lambda \geq \alpha, \beta > -1$, $L(a,c)I_\lambda(f) \varepsilon C^*$ and $I_\lambda(L(a,c)f(z)) \varepsilon C^*$.

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