Internat. J. Math. & Math. Sci.
VOL. 15 NO. 2 (1992) 397-398

RESEARCH NOTES

NOTE ON MODULATIONAL INSTABILITY OF BOUSSINESQ WAVETRAINS

S. ROY CHOUDHURY

Department of Mathematics
University of Central Florida
Orlando, FL 32816

(Received September 3, 1991)

ABSTRACT. It is demonstrated that modulational instability may occur in Boussinesq wavetrains for wavenumbers $k^2 > 1$, in contrast to the result found by Shivamoggi and Debnath. Both Whitham's averaged Lagrangian formulation and the technique of Benjamin and Feir are used.

Key Words and Phrases: Modulational stability, Boussinesq wavetrains, nonlinear waves.
1991 AMS Subject Classification: 76E30, 76B15, 76B25

1. INTRODUCTION. The weakly nonlinear stability analysis of periodic wavetrains in fluids and plasmas has been considered by several authors [1]. Shivamoggi and Debnath [2] considered the modulational instability of both the Korteweg-De Vries (KdV) and Boussinesq wavetrains using the variational formulation of Whitham. In this note, we demonstrate that modulational instability may indeed exist for Boussinesq wavetrains, in contrast to the result stated in Ref. [2].

2. MAIN RESULTS. Following the notation of [2], we obtain the averaged Lagrangian (Ref 2, Eq. (4.14)) for the Boussinesq wavetrains:

$$\bar{L} = \left[\frac{1}{2}k^2(\omega^2 - k^2) + \frac{k^6}{2}\right]\frac{a_1^2}{2} + \left[\frac{k^2}{144}(\omega^2 - k^2) + \frac{k^6}{36} - \frac{k^6}{24}\right]a_1^4. \tag{2.1}$$

Here, ω and k are the angular frequency and wavenumber, respectively. From the Whitham theory of averaged Lagrangians, the wavetrain is modulationally unstable if $\omega_2 < 0$ for a dispersion relation (for nonlinear dispersive wavetrains) of the form

$$\omega = \omega_0(k) + \omega_2(k)a_1^2. \tag{2.2}$$

From (2.1), the dispersion relation $\partial\bar{L}/\partial a_1 = 0$ is:

$$\omega^2 = \frac{2k^4(a_1^2 - 9)}{a_1^2 + 18} + k^2. \tag{2.3}$$

Notice that Eq. (3) differs from the corresponding equation (4.15) of [2]. Treating the amplitude a_1 as small in the weakly nonlinear theory yields:

$$\omega = (k^2 - k^4)^{1/2} + \frac{k^2 a_1^2}{12(1 - k^2)}. \tag{2.4}$$

Hence, from (2) and (4)

$$\omega_2 = -\frac{k^2}{12(1-k^2)}. \tag{2.5}$$

Thus, $\omega_2 < 0$ and modulational instability exists in the Boussinesq equation for the range of wavenumbers $k^2 > 1$, unlike the absolute modulational stability found in Ref. 2.

Note that the existence of modulational instability for the Boussinesq equation

$$\phi_{tt} - \phi_{xx} - \phi_{xxxx} - (\phi^2)_{xx} = 0, \tag{2.6}$$

may also be seen by carrying out an analysis similar to Benjamin and Feir [3]. Substituting the modulated solution

$$\phi(x,t) = b_0[1 + \varepsilon(x,t)] \tag{2.7}$$

around the spatially independent Stokes solution b_0 of Eq. (6), and writing

$$\varepsilon(x,t) = \hat{\varepsilon}_{-n}(0)e^{-i(k_n x + \omega t)} + \hat{\varepsilon}_n(0)e^{i(k_n x + \omega t)}, \tag{2.8}$$

We obtain from (6):

$$(-\omega^2 + k_n^2 - k_n^4 + 2b_0 k_n^2)\hat{\varepsilon}_{-n}(0) = 0$$

Clearly, for $\omega = \omega_r + i\omega_i$ and $\omega_r = 0$, the growth rate ω_i is given by:

$$\omega_i^2 = k_n^4 - (1 + 2b_0)k_n^2.$$

Hence, $\omega_i^2 > 0$ and modulational instability may occur for $k_n^2 > (1 + 2b_0)$.

REFERENCES

1. WHITHAM, G.B., Nonlinear Dispersion of Water Waves, J. Fluid Mech. 27 (1967) 399-412; LIGHTHILL, M.J., Contributions to the Theory of Waves in Nonlinear Dispersive Systems, J. Inst. Math. Appl. 1 (1965) 269-306; BENJAMIN, T.B., Instability of Periodic Wavetrains in Nonlinear Dispersive Systems, Proc. Roy. Soc. London 299A (1967) 59-75.
2. SHIVAMOGGI, B.K. and DEBNATH, L., Modulational Stability of Korteweg-De Vries and Boussinesq Wavetrains, Int. J. Math and Math Sci. 6 (1983) 811-817.
3. BENJAMIN, T.B. and FEIR, J.E., The Disintegration of Wavetrains in Deep Water, Part I, J. Fluid Mech. 27 (1967) 417-430.