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A CLASS OF UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS

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ABSTRACT. $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is said to be in $V(\theta_n)$ if the analytic and univalent function f in the unit disc E is normalised by $f(0) = 0$, $f'(0) = 1$ and $\arg a_n = \theta_n$ for all n. If further there exists a real number β such that $\theta_n+(n-1)\beta \equiv \pi(\mathrm{mod}\ 2\pi)$ then f is said to be in $V(\theta_n,\beta)$. The union of $V(\theta_n,\beta)$ taken over all possible sequence $\{\theta_n\}$ and all possible real number β is denoted by V. $V_n(A,B)$ consists of functions $f \in V$ such that

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+Aw(z)}{1+Bw(z)},$$

$-1 \leq A < B \leq 1$, where $n \in N \cup \{0\}$ and $w(z)$ is analytic, $w(0) = 0$ and $|w(z)| < 1$, $z \in E$. In this paper we find the coefficient inequalities, and prove distortion theorems.

KEY WORDS AND PHRASES. Varying arguments, Ruscheweyh derivative, Distortion theorems, Coefficient estimates.

MATHEMATICS SUBJECT CLASSIFICATION No. 30C45

1. INTRODUCTION.

Let A denote the class of functions f(z) analytic in the unit disc $E = \{z : |z| < 1\}$. Let S denote the subclass of A consisting functions normalised by $f(0) = 0$ and $f'(0) = 1$ which are univalent in E. The Hadamard product $(f*g)(z)$ of two functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ in A is given by,

$$(f*g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m.$$

Let $D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z)$, $n \in N \cup \{0\}$ where $N = \{1,2,3,\ldots\}$. Ruscheweyh [2] observed that $D^n f(z) = z(z^{n-1}f(z))^{(n)}/n!$. $D^n f(z)$ is called the n^{th} Ruscheweyh derivative of $f(z)$ by Al-Amiri [1].

DEFINITION 1. (Silverman [3]). $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is said to be in $V(\theta_n)$ if $f \in S$ and $\arg a_n = \theta_n$ for all n. If further there exists a real number β such that $\theta_n + (n-1)\beta \equiv \pi (\mathrm{mod}\ 2\pi)$, then f is said to be in $V(\theta_n, \beta)$. The union of $V(\theta_n, \beta)$ taken overall possible sequences $\{\theta_n\}$ and all possible real number β is denoted by V.

Now we define the class $V_n(A,B)$ consisting of functions $f \in V$ such that $\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+Aw(z)}{1+Bw(z)}$, $-1 \leq A < B \leq 1$, where $n \in N \cup \{0\}$ and $w(z)$ is analytic, $w(0) = 0$ and $|w(z)| < 1$, $z \in E$. Let $K_n(A,B)$ denote the class of functions $f \in V$ such that $zf'(z) \in V_n(A,B)$.

2. COEFFICIENT INEQUALITIES.

THEOREM 1. Let $f \in V$. Then $f \in V_n(A,B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\ (m-1)!} C_m |a_m| < (B-A). \qquad (2.1)$$

where $C_m = (B+1)(n+m)-(1+A)(n+1)$.

PROOF. Suppose $f \in V_n(A,B)$. Then

$$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \quad -1 \leq A < B \leq 1$$

$w(z)$ is analytic, $w(0) = 0$ and $|w(z)| < 1$, $z \in E$. We get

$$w(z) = \frac{D^n f(z) - D^{n+1}f(z)}{BD^{n+1}f(z) - AD^n f(z)}.$$

Since $\mathrm{Re}\, w(z) < |w(z)| < 1$, we obtain on simplification,

$$\mathrm{Re}\left\{\frac{\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\ (m-1)!} [(n+1)-(n+m)] a_m z^{m-1}}{(B-A) + \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!\ (m-1)!} [B(n+m)-A(n+1)] a_m z^{m-1}}\right\} < 1. \qquad (2.2)$$

Since $f \in V$, f lies in $V(\theta_m, \beta)$ for some sequence $\{\theta_m\}$ and a real number β such that

$$\theta_m + (m-1)\beta \equiv \pi (\mathrm{mod}\ 2\pi). \quad \text{Set } z = re^{i\beta}.$$

Then we get,

$$\mathrm{Re}\left\{\frac{\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[(n+1)-(n+m)]|a_m|r^{m-1}e^{i(\theta_m+\overline{m-1}\,\beta)}}{(B-A)+\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[B(n+m)-A(n+1)]|a_m|r^{m-1}e^{i(\theta_m+\overline{m-1}\,\beta)}}\right\} < 1. \tag{2.3}$$

$$\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[(n+m)-(n+1)]|a_m|\,r^{m-1}$$

$$< (B-A)-\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[B(n+m)-A(n+1)]|a_m|r^{m-1}$$

$$\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[(B+1)(n+m)-(1-A)(n+1)]|a_m|\,r^{m-1} < (B-A)$$

Hence,

$$\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}C_m|a_m|\,r^{m-1} < (B-A). \tag{2.4}$$

Letting $r \to 1$ we get (2.1).

Conversely, suppose $f \in V$ and satisfies (2.1). In view of (2.4) which is implied by (2.1), since $r^{m-1} < 1$, we have,

$$\left|\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[(n+1)-(n+m)]a_m z^{m-1}\right|$$

$$\leq \sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[(n+m)-(n+1)]|a_m|\,r^{m-1}$$

$$< (B-A)-\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[B(n+m)-A(n+1)]|a_m|r^{m-1}$$

$$\leq \left|(B-A)-\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}[A(n+1)-B(n+m)]a_m z^{m-1}\right|$$

which gives (2.2) and hence follows that $f \in V_n(A,B)$.

COROLLARY 1. If $f \in V$ is in $V_n(A,B)$ then,

$$|a_m| \leq \frac{(n+1)!(m-1)!(B-A)}{(n+m-1)!\,C_m}$$

for $m \geq 2$. The equality holds for the function f given by,

$$f(z) = z + \frac{(n+1)!(m-1)!(B-A)}{(n+m-1)!\,C_m}e^{i\theta_m}z^m,\ z \in E.$$

THEOREM 2. Let $f \in V$. Then $f(z) = z+\sum_{m=2}^{\infty}a_m z^m$ is in $K_n(A,B)$ if and only if

$$\sum_{m=2}^{\infty}\frac{(n+m-1)!}{(n+1)!(m-1)!}mC_m|a_m| < (B-A).$$

THEOREM 3. Let $f(z) = z+\sum_{m=2}^{\infty} a_m z^m \in V_n(A,B)$, with $\arg a_m = \Theta_m$ where $[\Theta_m+(m-1)\beta] \equiv \pi (\mathrm{mod}\ 2\pi)$. Define $f_1(z) = z$ and

$$f_m(z) = z + \frac{(n+1)!(m-1)!(B-A)e^{i\Theta_m} z^m}{(n+m-1)!\ C_m},\ m = 2,3,\ldots,\ z \in E.$$

$f \in V_n(A,B)$ if and only if f can be expressed as $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ where $\mu_m \geq 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

PROOF. If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1$, $\mu_m \geq 0$, then,

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!C_m\mu_m}{(n+1)!(m-1)!} \cdot \frac{(n+1)!(m-1)!(B-A)}{(n+m-1)!\ C_m}$$

$$= \sum_{m=2}^{\infty} \mu_m(B-A) = (1-\mu_1)(B-A) \leq (B-A).$$

Hence $f \in V_n(A,B)$.

Conversely, let

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in V_n(A,B),$$

define, $\mu_m = \dfrac{(n+m-1)!|a_m|C_m}{(n+1)!(m-1)!(B-A)}$, $m = 2,3,\ldots$ and define

$\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. From Theorem 1, $\sum_{m=2}^{\infty} \mu_m \leq 1$ and so $\mu_1 \geq 0$.

Since, $\mu_m f_m(z) = \mu_m z+a_m z^m$,

$$\sum_{m=1}^{\infty} \mu_m f_m(z) = z + \sum_{m=2}^{\infty} a_m z^m = f(z).$$

THEOREM 4. Define $f_1(z) = z$ and

$$f_m(z) = z + \frac{e^{i\Theta_m}(n+1)!(m-1)!(B-A)z^m}{(n+m-1)!\ mC_m},\ m = 2,3,\ldots\ z \in E.$$

Then $f \in K_n(A,B)$ if and only if f can be expressed as

$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ where $\mu_m \geq 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

3. DISTORTION THEOREMS.

THEOREM 5. Let the function $f(z) = z+\sum_{m=2}^{\infty} a_m z^m$ be in the class $V_n(A,B)$. Then,

$$|z|-(B-A)|z|^2/C_2 \leq |f(z)| \leq |z|+(B-A)|z|^2/C_2 \tag{3.1}$$

$$1-2(B-A)|z|/C_2 \leq |f'(z)| \leq 1+2(B-A)|z|/C_2. \tag{3.2}$$

PROOF. $|f(z)| = |z+\sum_{m=2}^{\infty} a_m z^m| \leq |z|+|z|^2 \sum_{m=2}^{\infty} |a_m|$

and $|f(z)| \geq |z|-|z|^2 \sum_{m=2}^{\infty} |a_m|$. Since $\frac{(n+m-1)!C_m}{(n+1)!(m-1)!}$ is an increasing function of $m \geq 2$ and $f(z) \in V_n(A,B)$, by Theorem 1, we have

$$\frac{(n+1)!}{(n+1)!\ 1!} C_2 \sum_{m=2}^{\infty} |a_m| \leq \sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} C_m|a_m| \leq (B-A)$$

that is,

$$\sum_{m=2}^{\infty} |a_m| \leq \frac{B-A}{C_2} \tag{3.3}$$

From (3.3) we get (3.1)

$$|f'(z)| = |1+\sum_{m=2}^{\infty} ma_m z^{m-1}| \leq 1+|z| \sum_{m=2}^{\infty} m|a_m|$$

and

$$|f'(z)| \geq 1 - |z| \sum_{m=2}^{\infty} m|a_m|.$$

Since $\frac{(n+m-1)!\ C_m}{(n+1)!\ m!}$ is an increasing function of $m \geq 2$ and

$\frac{(n+m-1)!\ mC_m}{(n+1)!\ (m+1)!} < \frac{(n+m-1)!\ mC_m}{(n+1)!\ m!}$ by Theorem 1, we have,

$$\frac{(n+1)!\ C_2}{(n+1)!\ 2} \sum_{m=2}^{\infty} m|a_m| \leq \sum_{m=2}^{\infty} \frac{(n+m-1)!\ C_m|a_m|}{(n+1)!\ (m-1)!} \leq (B-A)$$

that is,

$$\sum_{m=2}^{\infty} m|a_m| \leq \frac{2(B-A)}{C_2}. \tag{3.4}$$

From (3.4) we get (3.2). Further for the function $f(z) = z+\frac{(B-A)}{C_2} z^2$, we can see that the results of the Theorem are sharp.

COROLLARY 2. Let $f(z) = z+\sum_{m=2}^{\infty} a_m z^m$ be in the class $V_n(A,B)$. Then $f(z)$ is included in a disc with its center at the origin and radius r given by $r = (C_2+B-A)/C_2$ and $f'(z)$ is included in a disc with its center at the origin and radius r_1 given by $r_1 = [C_2+2(B-A)]/C_2$.

THEOREM 6. Let the function $f(z) = z+\sum_{m=2}^{\infty} a_m z^m$ be in the class $K_n(A,B)$, then,

$$|z|-(B-A)|z|^2/2\ C_2 \leq |f(z)| \leq |z|+(B-A)|z|^2/2\ C_2$$

and

$$1-(B-A)|z|/C_2 \leq |f'(z)| \leq 1+(B-A)|z|/C_2$$

for $z \in E$. The results are sharp for the function $f(z) = z+(B-A)z^2/2\ C_2$.

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