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RESEARCH NOTES

COMPLETE LIFT OF A STRUCTURE SATISFYING $F^K - (-)^{K+1} F = 0$

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ABSTRACT. The idea of f-structure manifold on a differentiable manifold was initiated and developed by Yano [1], Ishihara and Yano [2], Goldberg [3] and among others. The horizontal and complete lifts from a differentiable manifold M^n of class C^∞ to its cotangent bundles have been studied by Yano and Patterson [4,5]. Yano and Ishihara [6] have studied lifts of an f-structure in the tangent and cotangent bundles. The purpose of this paper is to obtain integrability conditions of a structure satisfying $F^K - (-)^{K+1}F = 0$ and $F^W - (-)^{W+1}F \neq 0$ for $1 < W < K$, in the tangent bundle.

KEY WORDS AND PHRASES. Tangent bundle, Complete lift, F-structure, Integrability, Distributions.

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1. INTRODUCTION.

Let F be a nonzero tensor field of the type (1,1) and of class C^∞ on an n dimensional manifold M^n such that [7]

$$F^K - (-)^{K+1} F = 0 \quad \text{and} \quad F^W - (-)^{W+1} F \neq 0 \text{ for } 1 < W < K, \tag{1.1}$$

where K is a fixed positive integer greater than 2. The degreé of the manifold being $K(K\geq 3)$. Such a structure on M^n has been called $F(K, -(-)^{K+1})$ - structure of rank r, where the rank $(F) = r$ and is constant on M^n. The case when K is odd and $K(\geq 3)$ has been considered in this paper.

Let the operators on M^n be defined as follows [7]

$$l = (-)^{K+1} F^{K+1} \quad \text{and} \quad m = I - (-)^{K+1} F^{K+1}, \tag{1.2}$$

where I denotes the identity operator on M^n. We will state the following two theorems. [7]

THEOREM (1.1). Let M^n be an $F(K,-(-)^{K+1})$ manifold then,

$$l + m = I, \quad l^2 = l \quad \text{and} \quad m^2 = m \tag{1.3}$$

For F satisfying (1.1), there exist complementary distributions L and M, corresponding to the projection operators l and m respectively. If the rank of F is constant and is equal to $r = r(F)$ them dim L = r and dim M = (n-r).

THEOREM (1.2). We have

a) $Fl = lF = F$ and $Fm = mF = 0$ (1.4)a

b) $F^{K-1}\, l = l$ and $F^{K-1}\, m = 0$ (1.4)b

Then $F^{\frac{K-1}{2}}$ acts on L as an almost product structure and on M as a null operator.

2. COMPLETE LIFT ON $F(K, -(-)^{K+1})$ - STRUCTURE IN TANGENT BUNDLE.

Let M be an n-dimensional differentiable manifold of class C^{∞} and $T_p(M^n)$ the tangent space at a point p of M_n and

$$T(M^n) = \bigcup_{p\in M^n} T_p(M^n) \text{ is the tangent bundle over the manifold } M^n.$$

Let us denote by $T_s^{\ r}(M^n)$, the set of all tensor fields of class C^{∞} and of type (r,s) in M^n and $T(M^n)$ be the tangent bundle over M^n. The complete lift F^C of an element of $T_1^{\ 1}(M^n)$ with local components $F_i^{\ h}$ has components of the form [5]

$$F^{C} : \begin{pmatrix} F_i^{\ h} & 0 \\ \delta_i^{\ h} & F_i^{\ h} \end{pmatrix}$$

Now we obtain the following results on the complete lift of F satisfying (1.1).

THEOREM (2.1). For $F \in T_1^{\ 1}(M^n)$, the complete lift F^C of F is an $F(K,-(-)^{K+1})$ - structure iff it is for F also. Then F is of rank r iff F^C is of rank 2r.

PROOF. Let F, $G \in T_1^{\ 1}(M^n)$. Then we have [5]

$$(FG)^C = F^C G^C \qquad (2.2)$$

Replacing G by F in (2.2) we obtain

$$(FF)^C = F^C F^C$$

$$\text{or, } (F^2)^C = (F^C)^2 \qquad (2.3)$$

Now putting $G = F^{K-1}$ in (2.2) since G is (1,1) tensor field therefore F^{K-1} is also (1,1) so we obtain $(FF^{K-1})^C = F^C(F^{K-1})^C$ which in view of (2.3) becomes

$$(F^K)^C = (F^C)^K \qquad (2.4)$$

Taking complete lift on both sides of equation (1.1) we get

$$(F^K)^C - ((-)^{K+1}F)^C = 0$$

which in consequence of equation (2.4) gives

$$(F^C)^K - (-)^{K+1}\, F^C = 0 \qquad (2.5)$$

Thus equation (1.1) and (2.5) are equivalent. The second part of the theorem follows in view of equation (2.1).

Let F satisfying (1.1) be an F-structure of rank r in M^n. Then the complete lifts l^C of l and m^C of m are complementary projection tensors in $T(M^n)$. Thus there exist in $T(M^n)$ two complementary distributions L^C and M^C determined by l^C and m^C respectively.

3. INTEGRABILITY CONDITIONS OF $F(K,-(-)^{K+1})$ STRUCTURE IN TANGENT BUNDLE.

Let $F \in T_1^{\ 1}(M^n)$, then the Nijenhuis tensor N_F of F satisfying (1.1) is a tensor field of the type (1,2) given by [6]

$$N_F(X,Y) = [FX,FY] - F[FX,Y] - F[X,FY] + F^2[X,Y]. \tag{3.1)a}$$

Let N^C be the Nijenhuis tensor of F^C in $T(M^n)$ of F in M^n, then we have

$$N^C(X^C,Y^C) = [F^CX^C,F^CY^C] - F^C[F^CX^C,Y^C] - F^C[X^C,F^CY^C] + (F^2)^C[X^C,Y^C] \tag{3.1)b}$$

For any $X,Y \in T_0^{\ 1}(M^n)$ and $F \in T_1^{\ 1}(M^n)$ we have [5]

$$[X^C,Y^C] = [X,Y]^C \text{ and } (X+Y)^C = X^C+Y^C \tag{3.2)a}$$

$$F^CX^C = (FX)^C \tag{3.2)b}$$

From (1.4)a and (3.2)b we have

$$F^Cm^C = (Fm)^C = 0 \tag{3.3}$$

THEOREM (3.1). The following identities hold

$$N^C(m^CX^C,m^CY^C) = (F^C)^2\ [m^CX^C,m^CY^C], \tag{3.4}$$

$$m^CN^C(X^CY^C) = m^C[F^CX^C,F^CY^C] \tag{3.5}$$

$$m^CN^C(1^CX^C,1^CY^C) = m^C[F^CX^C,F^CY^C] \tag{3.6}$$

$$m^CN^C((F^C)^{K-2}X^C,\ (F^C)^{K-2}Y^C) = m^C[1^CX^C,1^CY^C] \tag{3.7}$$

PROOF. The proofs of (3.4) to (3.7) follow in view of equations (1.4), (3.1)a, and (3.3)

THEOREM (3.2). For any X, Y $\in T_0^{\ 1}(M^n)$, the following conditions are equivalent.

$$m^CN^C(X^C,Y^C) = 0, \tag{i}$$

$$m^CN^C(1^CX^C,1^CY^C) = 0, \tag{ii}$$

$$m^CN^C((F^{K-2})^CX^C,(F^{K-2})^CY^C) = 0 \tag{iii}$$

PROOF. In consequence of equations (3.1)b and (1.4),a,b it can be easily proved that

$N^C(1^CX^C,1^CY^C) = 0$ iff $N^C((F^{K-2})^CX^C,(F^{K-2})^CY^C) = 0$ for all X and $Y \in T_0^{\ 1}(M^n)$

Now due to the fact that equations (3.5) and (3.6) are equal, the conditions (i), (ii) and (iii) are equivalent to each other.

THEOREM (3.3). The complete lift of M^C of the distribution M in $T(M^n)$ is integrable iff M is integrable in M^n.

PROOF. It is known that the distribution M is integrable in M^n iff [2]

$$1[mX,mY] = 0 \tag{3.8}$$

for any $X,Y \in T_0^{\ 1}(M^n)$.

Taking complete lift on both sides of equations (3.8) we get

$$1^C[m^CX^C,m^CY^C] = 0 \tag{3.9}$$

where $1^C = (I - m)^C = I - m^C$, is the projection tensor complementary to m^C. Thus the conditions (3.8) and (3.9) are equivalent.

THEOREM (3.4). For any $X, Y \in T_0^1(M^n)$, let the distribution M be integrable in M^n iff $N(mX,mY) = 0$.

Then the distribution m^C is integrable in $T(M^n)$ iff $l^C N^C(m^C X^C, m^C Y^C) = 0$ or equivalently, $N^C(m^C X^C, m^C Y^C) = 0$.

PROOF. By virtue of condition (3.4) we have

$$N^C(m^C X^C, m^C Y^C) = (F^C)^2 [m^C X^C, m^C Y^C].$$

Multiplying throughout by l^C we get

$$l^C N^C(m^C X^C, m^C Y^C) = (F^C)^2 l^C [m^C X^C, m^C Y^C].$$

which in view of equation (3.9) becomes

$$l^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.10}$$

Making use of equation (3.3), we get

$$m^C N^C(m^C X^C, m^C Y^C) = 0 \tag{3.11}$$

Adding (3.10) and (3.11) we obtain

$$(l^C + m^C) N^C(m^C X^C, m^C Y^C) = 0$$

Since $l^C + m^C = I^C = I$ we have $N^C(m^C X^C, m^C Y^C) = 0$.

THEOREM (3.5). For any $X, Y \in T_0^{\ 1}(M^n)$ let the distribution L be integrable in M^n that is $mN(X,Y) = 0$ then the distribution L^C is integrable in $T(M^n)$ iff any one of the conditions of theorem (3.2) is satisfied.

PROOF. The distribution L is integrable in M^n iff [2] holds i.e. $m[lX,lY] = 0$. Thus the distribution L^C is integrable in $T(M^n)$ iff $m^C[l^C X^C, l^C Y^C] = 0$.
On making use of equation (3.7), the theorem follows.

We now define the following:

(i) distribution L is integrable,

(ii) an arbitrary vector field Z tangent to an integral manifold of L,

(iii) the operator $\overset{*}{F}$, such that $\overset{*}{F}Z = FZ$.

Hence by virtue of theorem (1.2) the induced structure $\overset{*}{F}$ is an almost product structure on each integral manifold of L and $\overset{*}{F}$ makes tangent spaces invariant of every integral manifold of L. Let us denote the vector valued 2-form $\overset{*}{N}(Z,W)$, the Nijenhuis tensor corresponding to the Nijenhuis tensor of the almost product structure induced form $F(K,-(-)^{K+1})$ structure, on each integral manifold of L and for any two $Z,W \in T_0^{\ 1}(M^n)$ tangent to an integral manifold of L, then we have

$$\overset{*}{N}(Z,W) = [\overset{*}{F}Z,\overset{*}{W}] - \overset{*}{F}[\overset{*}{F}Z,W] - \overset{*}{F}[Z,\overset{*}{F}W] + \overset{*}{F}{}^2 [Z,W], \tag{3.12}$$

which in view of (3.1)b and (3.12) yields

$$N^C[l^C X^C, l^C Y^C] = \overset{*}{N}{}^C(l^C X^C, l^C Y^C) \tag{3.13}$$

DEFINITION (3.1). We say that $F(K,-(-)^{K+1})$ - structure is partially integrable if the distribution L is integrable and the almost product structure F induced from $\overset{*}{F}$ on each integral manifold of L is also integrable.

THEOREM (3.6). For any $X, Y \in T_0^1(M^n)$ let the $F(K,-(-)^{K+1})$ - structure be partially integrable in M^n i.e. $N(lX,lY) = 0$. Then the necessary and sufficient condition for $F(K,-(-)^{K+1})$ - structure to be partially integrable in $T(M^n)$ is that $N^C(l^CX^C, l^CY^C) = 0$ or equivalently $N^C((F^{K-2})^CX^C, (F^{K-2})^CY^C) = 0$.

PROOF. In view of equation (1.4) and equation (3.1)b we can prove easily that $N^C(l^CX^C, l^CY^C) = 0$ iff $N^C((F^{K-2})^CX^C, (F^{K-2})^CY^C) = 0$, for any $X, Y \in T_0^1(M^n)$.

Now in view of equation (3.13) and theorem (3.5), the result follows immediately. When both distributions L and M are integrable we can choose a local coordinate system such that all L and M are respectively represented by putting (n - r) local coordinates constant and r-coordinates constant. We call such a coordinate system an adapted coordinate system. It can be supposed that in an adapted coordinate system the projection operators l and m have the components of the form

$$l = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad m = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

respectively where I_r denotes the unit matrix of order 'r' and I_{n-r} is of order (n - r).

Since F satisfies equation (1.4)a, the tensor has components of the form

$$F = \begin{pmatrix} F_r & 0 \\ 0 & 0 \end{pmatrix}$$

is an adapted coordinate system where F_r denotes rxr square matrix.

DEFINITION (3.2). We say that on $F(K,-(-)^{K+1})$ structure is integrable if

(i) The structure $F(K,-(-)^{K+1})$ is partially integrable.

(ii) The distribution M is integrable i.e. $N(mX,mY) = 0$.

(iii) The components of the $F(K,-(-)^{K+1})$ structure are independent of the coordinates which are constant along the integral manifold of L in an adapted coordinate system.

THEOREM (3.7). For any $X,Y \in T_0^1(M^n)$ let $F(K,-(-)^{K+1})$ - structure be integrable in M^n iff $N(X,Y) = 0$. Then the $F(K,-(-)^{K+1})$ - structure is integrable in $T(M^n)$ iff $N^C(X^C,Y^C) = 0$.

PROOF. In view of equations (3.1)a and (3.1)b we get

$$N^C(X^C,Y^C) = (N(X,Y))^C.$$

Since $F(K,-(-)^{K+1})$ is integrable in M^n thus the theorem follows.

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