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A CHARACTERIZATION OF THE GENERALIZED MEIJER TRANSFORM

E.Y. DEEBA

Department of Applied Mathematical Sciences
University of Houston-Downtown
Houston, Texas 77002

E.L. KOH

Department of Mathematics & Statistics
University of Regina
Regina, Canada S4S 0A2

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ABSTRACT.

The purpose of the note is to prove a representation theorem for the generalized Meijer transform defined in [2]. In particular, we shall state and prove necessary and sufficient conditions for a function $F(p)$ to be the generalized Meijer transform of a generalized function.

KEYWORDS AND PHRASES. Generalized Meijer transform, generalized function, Bessel differential operator, Representation theorem.

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1. INTRODUCTION.

The generalized Meijer transform has been defined and studied in [2] and [3] and is given by

$$(\bar{M}_\mu f)(p) = \frac{2p}{\Gamma(1-\mu)} < f(t),\ (pt)^{\mu/2} K_\mu(2\sqrt{pt}) > \tag{1.1}$$

where $\mu > -1, K_\mu$ is the modified Bessel function of third kind and order μ, p belongs to a region of the complex plane and f belongs to the dual $M'_{\mu,\gamma}$ of the space $M_{\mu,\gamma}$ defined by

$$M_{\mu,\gamma} = \{\phi \epsilon C^\infty(I) | \lambda^\mu_{\gamma,k}(\phi) < \infty\}$$

and

$$\lambda^\mu_{\gamma,k}(\phi) = \sup_{t \epsilon I} |e^{\gamma\sqrt{t}} t^{1-\mu} B^k_{-\mu}(\phi(t))|, k = 0, 1, 2, \ldots$$

γ being any real number and $B_{-\mu} = t^\mu D t^{1-\mu} D (D = \frac{d}{dt})$ is the Bessel differential operator. The properties of the space $M_{\mu,\gamma}$ and its dual have been studied in [2]. Furthermore, in [2] the transform (1.1) has been shown to be analytic and an inversion theorem, in the distributional sense, has been established. We note here that if $f(t)$ is locally integrable on $I = (0, \infty)$ and $f(t)e^{-r\sqrt{t}} t^{-1+\mu}$ is absolutely integrable on I, then we obtain the classical Meijer transform

$$(M_\mu f)(p) = \frac{2p}{\Gamma(1-\mu)} \int_0^\infty f(t)(pt)^{\mu/2} K_\mu(2\sqrt{pt}) dt. \tag{1.2}$$

In [3], we applied the generalized Meijer transform to a boundary value problem with distributional conditions. To arrive at the solution, it was necessary to use a characterization of the Meijer transform. In this note we shall state and prove necessary and sufficient conditions for a function $F(p)$ to be the generalized Meijer transform of a generalized function f in $M'_{\mu,\gamma}$. This will be the content of Section 3 while Section 2 will be devoted to preliminary results and background material.

2. PRELIMINARIES.

For the sake of completeness, we shall collect in this section the background material that will be needed in proving the representation theorem.

Throughout we shall denote the interval $(0,\infty)$ by I and $B^k_{-\mu}$ the k-iterate of the Bessel differential operator. It can be shown that for $B^k_{-\mu}$

$$B^k_{-\mu}((pt)^{\mu/2}K_\mu(2\sqrt{pt})) = p^k(pt)^{\mu/2}K_\mu(2\sqrt{pt}) \tag{2.1}$$

$$B^k_{-\mu}((pt)^{\mu/2}I_\mu(2\sqrt{pt})) = p^k(pt)^{\mu/2}I_\mu(2\sqrt{pt}) \tag{2.2}$$

where I_μ is the modified Bessel function of the first kind given by

$$I_\mu(2\sqrt{pt}) = \sum_{k=0}^{\infty} \frac{(pt)^{k+\mu/2}}{k!\Gamma(k+1+\mu)}, \mu \quad \text{any real number} \tag{2.3}$$

and K_μ is the modified Bessel function of the third kind given by

$$K_\mu(2\sqrt{pt}) = \begin{cases} \frac{\pi}{2\sin\mu\pi}\left(\sum_{k=0}^{\infty}\frac{(pt)^{k-\mu/2}}{k!\Gamma(k+1-\mu)} - \sum_{k=0}^{\infty}\frac{(pt)^{\mu/2+k}}{k!\Gamma(k+1+\mu)}\right), \mu \text{ not integer} \\ \frac{1}{2}\sum_{k=0}^{-\mu-1}\frac{(-1)^k(-\mu-k-1)!}{k!}(pt)^{\mu/2+k} + (-1)^\mu\sum_{k=0}^{\infty}\frac{(pt)^{k-\mu/2}}{k!(-\mu+k)!} \\ \quad \times[-\log C(\sqrt{pt}) + \frac{1}{2}(\sum_{i=1}^{k}\frac{1}{i} + \sum_{i=1}^{-\mu+k}\frac{1}{i})], \mu \text{ integer.} \end{cases} \tag{2.4}$$

$C = e^\gamma$ (γ is Euler's constant, Watson [5]). Again, from [5], the asymptotic expansions of I_μ and K_μ are given by

$$K_\mu(2\sqrt{pt}) = \frac{\sqrt{\pi}}{2}(pt)^{\frac{-1}{4}}e^{-2\sqrt{pt}}[1 + 0(|pt|^{\frac{-1}{2}})], -\pi < \arg p < \pi \tag{2.5}$$

and

$$I_\mu(2\sqrt{pt}) = \begin{cases} \frac{1}{2\sqrt{\pi}}(pt)^{\frac{-1}{4}}(e^{2\sqrt{pt}} + ie^{-2\sqrt{pt}+i\mu\pi})[1 + 0(|pt|^{\frac{-1}{2}})], -\frac{\pi}{2} < \arg p < \frac{3\pi}{2} \\ \frac{1}{2\sqrt{\pi}}(pt)^{\frac{-1}{4}}(e^{2\sqrt{pt}} - ie^{-2\sqrt{pt}-i\mu\pi})[1 + 0(|pt|^{\frac{-1}{2}})], -\frac{3\pi}{2} < \arg p < \frac{\pi}{2} \end{cases} \tag{2.6}$$

It was shown in [2] that the space $M_{\mu,\gamma}$ is a complete Fréchet space and that

$$D_t[(pt)^{\mu/2}K_\mu(2\sqrt{pt})] \text{ and } D_p[(pt)^{\mu/2}K_\mu(2\sqrt{pt})]$$

both belong to $M_{\mu,\gamma}$. Further, if γ and α are real numbers such that $\gamma < \alpha$, then $M_{\mu,\alpha}$ is a subspace of $M_{\mu,\gamma}$ and the restriction of $f \epsilon M'_{\mu,\gamma}$ to $M_{\mu,\alpha}$ is in $M'_{\mu,\gamma}$. This implies that there is a real number σ_f, called the abscissa of definition of f, such that the restriction of f to $M_{\mu,\gamma}$ is in $M'_{\mu,\gamma}$ if $\gamma > \sigma_f$ and is not in $M'_{\mu,\gamma}$ if $\gamma < \sigma_f$. the operator $B^k_{-\mu}$ and its adjoint are respectively continuous linear operators on $M_{\mu,\gamma}$ and $M'_{\mu,\gamma}$. Also, the adjoint $B^{*k}_{-\mu}$ can be shown to be B^k_μ.

For any $f \epsilon M'_{\mu,\gamma}$ and $p \epsilon \Omega_f = \{p \epsilon C | Re 2\sqrt{p} > \gamma > \sigma_f,\ p \neq 0, |\arg p| < \pi\}$ the following have been established in [2]:

$$\text{(i)} \quad \bar{M}_\mu(B^k_{-\mu}f)(p) = p^k(\bar{M}_\mu f)(p) \tag{2.7}$$

which is a basis for an operational calculus of the transform $\bar{M}_\mu$

(ii) $\bar{M}_\mu f$ is analytic in Ω_f and

$$D_p(\bar{M}_\mu f)(p) = \frac{2}{\Gamma(1-\mu)} < f(t),\ D_p p(pt)^{\mu/2} K_\mu(2\sqrt{pt}) > \tag{2.8}$$

and

(iii) the inversion formula is

$$f(t) = \lim_{\theta_1 \to \pi} \frac{\Gamma(1-\mu)}{2\pi i} \int_{-\theta_1}^{\theta_1} (\bar{M}_\mu f)(p) p^{-1} (pt)^{-\mu/2} I_\mu(2\sqrt{pt}) dp(\theta) \tag{2.9}$$

where $p(\theta) = \gamma^2/4e^{i\theta} \sec^2 \theta/2$, γ is a fixed real number in Ω_f and the limit is to be understood in the sense of convergence in $D'(I)$, the dual of the space $D(I)$ of all smooth functions on I whose support is contained in a compact subset K of I equipped with the semi-norms

$$\rho_n(\phi) = \sup_{t \in I} |D_t^n(\phi)|.$$

Finally, we remark that if $f(t)$ is locally integrable on I and $f(t)e^{-\gamma\sqrt{t}} t^{-1+\mu}$ is absolutely integrable on I, then $f(t)$ generates a regular member f of $M'_{\mu,\gamma}$ via

$$< f, \phi > = \int_0^\infty f(t)\phi(t) dt. \tag{2.10}$$

As noted earlier for such functions the transform in (1.1) reduces to the classical Meijer transform given in (1.2).

A result that will be needed in our proof of the representation theorem is

THEOREM A. [Theorem 4 [1]]. If $Re\mu \geq -\frac{1}{2}$, $Re\sqrt{p} > \gamma_0 \geq 0$ and $F(p)$ is analytic and bounded according to $|F(p)| < M|p|^{-q}$ where $q < \frac{3}{2} Re\mu + 2$, then for real $c > \gamma_0$ and $Re\sqrt{p} > c$, $F(p) = M_\mu(f)$ where

$$f(t) = \frac{\Gamma(1+\mu) t^{-\mu/2}}{2\pi i} \int_{Re\sqrt{p}=c} F(p) p^{-1-\mu/2} I_\mu(2\sqrt{pt}) dp.$$

3. MAIN RESULT.

In this section we shall give a necessary and sufficient condition for a function $F(p)$ to be the generalized Meijer transform of a function f in $M'_{\mu,\gamma}$. As we shall see later in the proof of the necessary part, the real number μ must be restricted to $-\frac{1}{2} \leq \mu < 1$.

Before we state the result, we need the following lemma stated in our context (see [4], p. 18).

LEMMA 3.1. For any function $f \epsilon M'_{\mu,\gamma}$, there exist a positive constant c and a non-negative integer r such that for all $\phi \epsilon M_{\mu,\gamma}$

$$| < f, \phi > | \leq c\rho_r(\phi) \tag{3.1}$$

where $\rho_r = \max_{0 \leq k \leq r} \{\lambda^\mu_{\gamma,1}, \lambda^\mu_{\gamma,2}, \ldots, \lambda^\mu_{\gamma,k}\}$.

THEOREM 3.1. A necessary and sufficient condition for a function $F(p)$ to be the generalized Meijer transform $\bar{M}_\mu(f)$ of a generalized function f in $M'_{\mu,\gamma}$ is that

(i) there exists a region $\Omega_f = \{p \epsilon C | Re 2\sqrt{p} > \gamma, p \neq 0, |\arg p| < \pi\}$ on which $F(p)$ is analytic,

(ii) $F(p)$ is bounded by a polynomial in $|p|$.

PROOF. Assume that $F(p) = (\bar{M}_\mu f)(p)$ for $f \epsilon M'_{\mu,\gamma}$. The analyticity of $F(p)$ follows from Theorem 3.2 of [2]. We only need to prove that $F(p)$ is bounded by a polynomial in $|p|$. Since $(pt)^{\mu/2} K_\mu(2\sqrt{pt}) \epsilon M_{\mu,\gamma}$, Lemma 3.1 implies that there exist a positive constant c and a non-negative integer r such that

$$\begin{aligned} |F(p)| = |(\bar{M}_\mu f)(p)| &= \left|\frac{2p}{\Gamma(1-\mu)} < f(t),(pt)^{\mu/2}K_\mu(2\sqrt{pt}) > \right| \\ &\leq c|p|\rho_r((pt)^{\mu/2}K_\mu(2\sqrt{pt})) \\ &= c|p| \max_{0\leq k\leq r} \sup_{t\epsilon I} |e^{\gamma\sqrt{t}}t^{1-\mu}B^k_{-\mu}((pt)^{\mu/2}K_\mu(2\sqrt{pt})| \\ &= c|p| \max_{0\leq k\leq r} |p|^k \sup_{t\epsilon I} |e^{\gamma\sqrt{t}}t^{1-\mu}(pt)^{\mu/2}K_\mu(2\sqrt{pt})| \\ &\leq c|p|^{\gamma+1} \end{aligned}$$

by virtue of the series expansion (2.4) and the asymptotic properties (2.5) of $K_\mu(2\sqrt{pt})$. Hence $F(p)$ is bounded by a polynomial in $|p|$.

Assume that $F(p)$ satisfies (i) and (ii) of Theorem 3.1. Let $|F(p)| \leq P_n(|p|)$ where $P_n(|p|)$ is a polynomial in $|p|$ of degree n. Let $q\epsilon R$ be such that $q > \frac{3}{2}Re\mu + 2$ and m be an integer such that $m \geq q+n$. Then, for some $M > 0$, $|p|^{-m}|F(p)| \leq Mp^{-m}|p|^n \leq M|p|^{-q}$. Thus $p^{-m}F(p)$ satisfies the hypothesis of Theorem A stated in Section 2. Therefore, for $Re\sqrt{p} > c > \gamma_0$,

$$p^{-m}F(p) = M_\mu(g)(p) = \frac{2p}{\Gamma(1+\mu)}\int_0^\infty (pt)^{\mu/2}K_\mu(2\sqrt{pt})g(t)dt$$

where

$$g(t) = \frac{\Gamma(1+\mu)t^{-\mu/2}}{2\pi i}\int_{Re\sqrt{p}=c} p^{-m}F(p)p^{-1-\mu/2}I_\mu(2\sqrt{pt})dp.$$

We will show next that $e^{-\gamma\sqrt{t}}t^{-1+\mu}g(t)$ is absolutely integrable on I and conclude from (2.10) that $g(t)$ generates a regular member g of $M'_{\mu,\gamma}$.

We consider two cases

(i) for $|pt| \leq 1$, $0 < t < \infty$, (2.3) implies that

$$|e^{-\gamma\sqrt{t}}(pt)^{-1+\mu/2}I_\mu(2\sqrt{pt})| \leq Me^{-\gamma\sqrt{t}}|pt|^{-1+\mu}$$

which is integrable on I for $Re\mu > 0$.

(ii) for $|pt| \geq 1$, $Re\sqrt{p} = c$, $Re\mu < 1$, (2.6) implies that

$$|e^{-\gamma\sqrt{t}}(pt)^{-1+\mu/2}I_\mu(2\sqrt{pt})| \leq \frac{1}{2\sqrt{\pi}}|e^{(-\gamma+2\sqrt{p})\sqrt{t}}||pt|^{\mu/2-5/4}$$

which is of order $e^{-\alpha\sqrt{t}}t^{-1/2}(\alpha > 0)$. Thus

$$e^{-\gamma\sqrt{t}}(pt)^{\mu/2}I_\mu(2\sqrt{pt}) \quad \text{is absolutely integrable on } 0 \leq t < \infty.$$

That is,

$$\int_0^\infty |e^{-\gamma\sqrt{t}}(pt)^{\mu/2-1}I_\mu(2\sqrt{pt})|dt \leq c_\mu|p|^{Re\mu}$$

where c_μ is sufficiently large depending upon μ. To show that $g(t)e^{-r\sqrt{t}}t^{-1+m}$ is absolutely integrable, we invoke Fubini's theorem and the fact

$$\int_{Re\sqrt{p}=c>\gamma\geq 0} p^{-m-\mu+\mu}F(p)dp \tag{3.2}$$

is finite because $|p|^{-m}F(p) \leq |p|^{-m}P_n(|p|) \leq M|p|^q$ since $q > \frac{3}{2}Re\mu + 2$, $-q < -(2+\epsilon)$ for our choice of $0 < Re\mu < 1$. Thus (3.2) is at least quadratically decreasing for large p along $Re\sqrt{p} = c$ in both directions.

We have shown that $g(t)e^{-\gamma\sqrt{t}}t^{-1+\mu}$ is absolutely integrable on $0 < t < \infty$. Thus $g(t)$ generates a regular number in $M'_{\mu,\gamma}$ for $0 < Re\mu < 1$. Thus the function $p^{-m}F(p)$ is a generalized Meijer transform whose region of definition is $\{p|Re\sqrt{p} > c > \gamma \geq 0\}$.

We finally note that to find $f \epsilon M'_{\mu,\gamma}$ explicitly, we set $f = B_\mu^m g$. Then $\bar{M}_\mu f = \bar{M}_\mu(B_\mu^m g) = p^m \bar{M}_\mu(g) = F(p)$.

References

[1] Conlan, J. and E.L. Koh, On the Meijer transformation, Internat. J. Math and Math. Sci., Vol. 1 (1978), 145-159.

[2] Koh, E.L., E.Y. Deeba, and M.A. Ali, The Meijer Transformation of Generalized Functions, Internat. J. Math and Math. Sciences, Vol. 16, No. 2 (1987), 267-286.

[3] Koh, E.L., E.Y. Deeba, and M.A. Ali, On the generalized Meijer transformation, *Generalized Functions, Convergence Structure and Their Application*, Plenum Press, New York, 1988 (pages 219-226).

[4] Zemanian, A.H., *Generalized Integral Transformations*, Inter-Science, New York, 1968.

[5] Watson, G.N., A Treatise on the Theory of Bessel Functions, Univ. Press, Cambridge, 1966.